Matrix Lie Groups - 2022

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These notes are based mostly on the books of Hall and Rossmann.

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1 The Exponential

- If $f\colon \mathbb{R}\to\mathbb{R}$ satisfies $f'(x)=af(x)$, then $f(x)=f(0)\cdot e^{ax}$ follows from

$$ax=\int_0^x a\,dt=\int_0^x\frac{f'(t)}{f(t)}dt=\int_0^x(\ln f(t))'\,dt=\ln f(x)-\ln f(0)\,.$$

  We could also try to find a power-series solution $f(x)=\sum_{k=0}^{\infty}c_kx^k$, and solve: $c_k=\frac{a\cdot c_{k-1}}{k}$ (and $c_0=f(0)$), hence $f(x)=f(0)\sum_{k=0}^{\infty}\frac{(ax)^k}{k!}$.

- $e^x\colon\mathbb{R}^+\to\mathbb{R}^\times$ turns addition into multiplication.

- What if $p\colon \mathbb{R}\to\mathbb{R}^n$ satisfies $p'(t)=X\cdot p(t)$ for a fixed $X\in M_n(\mathbb{R})$? We can try to find a power series solution of the form $p(t)=\sum_{k=0}^{\infty}t^k p_k$ (with $p_k\in\mathbb{R}^n$). We get again $p_k=\frac{1}{k}Xp_{k-1}$, so $p_k=\frac{X^k}{k!}p_0$, and $p(t)=\sum_{k=0}^{\infty}\frac{t^kX^kp_0}{k!}=\left(\sum_{k=0}^{\infty}\frac{(tX)^k}{k!}\right)p_0=e^{tX}p_0$, and we can check this is a solution (inside the circle of convergence we can differentiate term by term).

- Differentiation: for $A,B\colon\mathbb{R}\to M_n(\mathbb{R})$ (or $\mathbb{C}$), $(A(t)B(t))'=A'(t)B(t)+A(t)B'(t)$, which also gives $A^{-1}(t)'=-A^{-1}(t)A'(t)A^{-1}(t)$ for invertible A (compare this with $f^{-1}(t)'=\frac{f'(t)}{f^2(t)}$ in the commutative case!)

- Frobenius norm: $\|A\|_F=\sqrt{\sum_{i,j}|a_{ij}|^2}=\mathrm{tr}AA^*=\sum_{\sigma\in Sing(A)}\sigma^2$. Like any norm satisfies $\|A+B\|\le\|A\|+\|B\|$, $\|\alpha A\|=|\alpha|\,\|A\|$, $\|A\|\ge 0$ with $\|A\|=0$ only for $A=0$, and is also submultiplicative: $\|AB\|\le\|A\|\,\|B\|$ (using Cauchy-Schwartz).

- Thus, $e^X=\sum_{k=0}^{\infty}\frac{X^k}{k!}$ converges for any X, and is continuous.

- Properties:
  - $e^0=I$, $e^{X^t}=\left(e^X\right)^t$, $e^{YXY^{-1}}=Ye^XY^{-1}$ (by distributivity).
  - $\left(e^{tX}\right)'=\left(\sum_{k=0}^{\infty}\frac{(tX)^k}{k!}\right)'=\sum_{k=1}^{\infty}\frac{t^{k-1}X^k}{(k-1)!}=Xe^{tX}=e^{tX}X$
  - $f(t)=e^{tX}$ is the unique solution of $f\colon\mathbb{R}\to M_n(\mathbb{R})$, $f(0)=I$, $f'(t)=Xf(t)$.Pf: Check that $\left(e^{-tX}f(t)\right)'\equiv 0$.
  - If $XY=YX$ then $e^{X+Y}=e^Xe^Y$. (not in general, i.e. $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$).
    Exercise: X,Y commute $\Leftrightarrow$ e^{tX},e^{tY} commute $\forall t$ $\Leftrightarrow$ $e^{sX+tY}=e^{sX}e^{tY}$ $\forall s,t$.
  - $X\mapsto e^X: M_n(\mathbb{R})\to GL_n(\mathbb{R})$, and $\left(e^X\right)^{-1}=e^{-X}$.
  - $t\mapsto e^{tX}:\mathbb{R}\to GL_n(\mathbb{R})$ is a homomorphism.
  - $f(t)=e^{tX}$ is the unique differentiable homomorphism $\mathbb{R}\to GL_n(\mathbb{R})$ with $f'(0)=X$. Pf: we have $f'(t)=Xf(t)$ since $\frac{f(t+h)-f(t)}{h}=\frac{f(h)-f(0)}{h}f(t)$.

- Example: For $X=\left(\begin{smallmatrix}a&b\\-b&a\end{smallmatrix}\right)$, $e^{tX}=e^{at}\left(\begin{smallmatrix}\cos bt&\sin bt\\-\sin bt&\cos bt\end{smallmatrix}\right)$.

- If $X=P^{-1}\mathrm{diag}(a_1,\ldots,a_n)P$, then

$$e^X=e^{P^{-1}\mathrm{diag}(a_1,\ldots,a_n)P}=P^{-1}e^{\mathrm{diag}(a_1,\ldots,a_n)}P=P^{-1}\mathrm{diag}\left(e^{a_1},\ldots,e^{a_n}\right)P.$$

  Exercise: compute e^X for a Jordan block $X=\begin{pmatrix}a&1&\\&\ddots&\ddots\\&&a\end{pmatrix}$. This settles computations...

- $\det e^X=e^{\mathrm{trace}\,X}$: clear for diagonalizable, which are dense in $M_n(\mathbb{C})$.

- For $\|A-I\|<1$,

$$\log A:=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(A-I)^k$$

  converges and satisfies

$$\begin{aligned}\|A-I\|<1 &\quad\Rightarrow\quad e^{\log A}=A\\ \|X\|<\log 2 &\quad\Rightarrow\quad \log e^X=X.\end{aligned}$$

  Pf: The diagonalizable matrices are dense in GL_n, and if A is diagonalizable then $\|A-I\|_2\le\|A-I\|_F<1$ implies $|1-\lambda|<1$ for every $\lambda\in Spec(A)$, and we use conjugation.

- Thus, exp bijects a neighborhood of $0\in M_n$ with a neighborhood of $I\in GL_n$: if $U=\exp\left(\mathring{B}_{\log 2}(0)\right)$ then $\mathring{B}_{\log 2}(0)\underset{\exp}{\overset{\log}{\leftrightarrows}}U$.

- exp is not locally injective everywhere: We have $e^X = I$ for all $X = \begin{pmatrix} 2ai & 2b \\ -2b & -2ai \end{pmatrix}$ with $a^2+b^2=\pi^2$, as they are all conjugate to $\begin{pmatrix} 2\pi i & \\ & -2\pi i \end{pmatrix}$ (and $\mathbb{C} \to M_2(\mathbb{R})$ turns this to an example in $M_4(\mathbb{R})$).

- Applications:
  - Roots: for $A \in U$, $\sqrt[n]{A} := e^{\frac{1}{n}\log A}$ is an n-th root of A in U, and it is the unique n-th root in $\exp\left(\mathring{B}_{\frac{\log 2}{n}}(0)\right)$.
  - Every continuous homomorphism $\mathbb{R} \to GL_n(\mathbb{C})$ is of the form $t \mapsto e^{tX}$.

- Trotter-Lie Formula (First take on failure of exp to be homomorphism): $e^{X+Y} = \lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m$.
  - $\log(I+A) = A + O\left(\|A\|^2\right)$ as $\|A\| \to 0$. More concretely, for $\|A\| \le \frac{1}{2}$

$$\|\log(I+A) - A\| \le \|A\|^2\left(\frac{1}{2} + \frac{\|A\|}{3} + \frac{\|A\|^2}{4} + \dots\right) \le \|A\|^2 \log\frac{16}{e^2}.$$

  - For fixed X, $e^{\frac{X}{m}} = I + \frac{X}{m} + O\left(\frac{1}{m^2}\right)$ (since $\left\|e^{\frac{X}{m}} - I - \frac{X}{m}\right\| \le \frac{e^{\|X\|}}{m^2}$), hence $e^{\frac{X}{m}}e^{\frac{Y}{m}} = I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$.
  - As $e^{\frac{X}{m}}e^{\frac{Y}{m}} \overset{m\to\infty}{\longrightarrow} I$, for m large enough we have $\left\|e^{\frac{X}{m}}e^{\frac{Y}{m}} - I\right\| \le \frac{1}{2}$, hence

$$\begin{aligned}\log e^{\frac{X}{m}}e^{\frac{Y}{m}} &= \log\left(I + \left(e^{\frac{X}{m}}e^{\frac{Y}{m}} - I\right)\right) = e^{\frac{X}{m}}e^{\frac{Y}{m}} - I + O\left(\left\|e^{\frac{X}{m}}e^{\frac{Y}{m}} - I\right\|^2\right) \\ &= \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right) + O\left(\left\|\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right\|^2\right) = \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right),\end{aligned}$$

  so

$$e^{\frac{X}{m}}e^{\frac{Y}{m}} = e^{\frac{X}{m}+\frac{Y}{m}+O\left(\frac{1}{m^2}\right)} \Longrightarrow \left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m = e^{X+Y+O\left(\frac{1}{m}\right)} \overset{m\to\infty}{\longrightarrow} e^{X+Y}.$$

2 Matrix Lie Groups

- A *Matrix Lie Groups* is a closed subgroup of $GL_n(\mathbb{C})$ (which itself embeds as a closed subgroup of $GL_{2n}(\mathbb{R})$).
- Nonexamples: $GL_n(\mathbb{Q})$, $\left\{\begin{pmatrix} e^{2\pi i t} & \\ & e^{2\pi i\alpha t}\end{pmatrix} \middle| t \in \mathbb{R}\right\}$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.
- Examples: GL_n, SL_n, (unipotent) upper-triangular, $\mathbb{R}^+$, $\mathbb{C}^+$, S^1, $\mathbb{H}^\times$, $\mathbb{H}^1$.
- For $G \le GL_n, H \le GL_m$ we have $G \times H \le GL_{m+n}$.

2.1 Classical groups

- Let V be a fin.dim. v.s. over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *bilinear form* on V is $b\colon V \times V \to \mathbb{F}$ (if b is fixed we sometimes write $\langle v, w\rangle$ for $b(v,w)$) which is linear in each entry. We denote

$$Aut(b) = \{A \in GL(V) \,|\, \forall v, w \in V : b(Av, Aw) = b(v,w)\}$$

  where $GL(V)$ are invertible linear transformations from V to itself. Choosing a basis for V identifies it with $\mathbb{F}^n$. Under this identification, $b(v,w) = v^t B w$ for some $B \in M_n(\mathbb{F})$ (specifically, $B_{ij} = b(v_i, v_j)$ for a basis $\{v_i\}$), and $Aut(b)$ corresponds to $Aut(B) := \{A \in GL_n(\mathbb{F}) \,|\, A^t B A = B\}$. This is a matrix Lie group in $GL_n(\mathbb{F})$.

- We can also write $Aut(B) = \left\{A \in GL_n(\mathbb{F}) \,\middle|\, A = B^{-1}\left(A^t\right)^{-1} B\right\}$, which presents $Aut(B)$ as the fixed points of the involutary automorphism $A \mapsto B^{-1}\left(A^t\right)^{-1} B$ of $GL_n(\mathbb{F})$.

- Change of basis: If we want to change our identification of $V \cong \mathbb{F}^n$ to another one, this is given by $v \mapsto Pv$ (where $P \in GL_n(\mathbb{F})$ is the "change of basis" matrix), and then B becomes $P^t B P$. We say B and C are *congruent* if they represent the same bilinear form, namely, $\exists P \in GL_n(\mathbb{F})$ with $C = P^t B P$. In this case it is clear that $Aut(B) \cong Aut(C)$ (in fact, they are conjugate subgroups of $GL_n(\mathbb{F})$).

- Note that also $Aut\left(B\right)=Aut\left(\alpha B\right)$ for $\alpha\in\mathbb{F}^{\times}$, even if B and αB are not congruent.
- b is called
  - *non-degenerate* if $\forall v\neq0\ \exists w$ such that $b\left(v,w\right)\neq0$. This is equivalent to $\det B\neq0$.
  - *symmetric* if $b\left(v,w\right)=b\left(w,v\right)$ (equivalent to $B^{t}=B$).
  - *anti-symmetric* (or skew-symmetric) if $b\left(v,w\right)=-b\left(w,v\right)$ (equivalent to $B^{t}=-B$). Over char $\neq2$ this is equivalent to $b\left(v,v\right)=0\ \forall v$.
- One can also consider the affine isometry group of (V,b), which is $Aff\left(V,b\right)=Aut\left(b\right)\ltimes V$, and can be considered as a matrix Lie group in $GL_{n+1}\left(\mathbb{F}\right)$ via $(A,v)\mapsto\left(\begin{smallmatrix}A & v\\ 0 & 1\end{smallmatrix}\right)$.

2.1.1 Orthogonal groups

- Sylvester: A symmetric regular $B\in M_{n}\left(\mathbb{R}\right)$ is congruent to $\operatorname{diag}\left(1^{\times p},-1^{\times q}\right)$ for (a unique) $0\leq p\leq n$, called the *signature* of B.
  Proof (without uniqueness): Let $\langle v,w\rangle=v^{t}Bw$. There exists v with $\langle v,v\rangle\neq0$, since $\langle v,w\rangle\neq0$ for some v,w, which implies that one of $\langle v,v\rangle$, $\langle w,w\rangle$ or $\langle v+w,v+w\rangle$ must be nonzero. Now replace v with $\frac{v}{\sqrt{|\langle v,v\rangle|}}$, so that we have $\langle v,v\rangle=1$ or -1. Observe that $\dim v^{\perp}=n-1$, since $w\mapsto\langle v,w\rangle$ has rank one, and that $\langle v\rangle\cap v^{\perp}=0$, so that $\mathbb{R}^{n}=\langle v\rangle\oplus v^{\perp}$. Now check that $\langle\cdot,\cdot\rangle\big|_{v^{\perp}\times v^{\perp}}$ is symmetric and non-degenerate, and repeat everything again. At the end, you will have found vectors $v_{1},\ldots,v_{n}$ with $\langle v_{i},v_{j}\rangle=\pm\delta_{i,j}$, as desired.
- We denote $Aut\left(\operatorname{diag}\left(1^{\times p},-1^{\times q}\right)\right)$ by $O\left(p,q\right)$. By Sylvester, for every symmetric non-degenerate bilinear form b on $\mathbb{R}^{n}$, $Aut\left(b\right)\cong O\left(p,q\right)$ for some $0\leq p\leq n$ (and we denote $O\left(n,0\right)=O\left(n\right)$). Note that $O\left(p,q\right)\cong O\left(q,p\right)$ (by $Aut\left(B\right)\cong Aut\left(-B\right)$) and one can check that $O\left(p,q\right)\ncong O\left(p',q'\right)$ if $\{p,q\}\neq\{p',q'\}$.
- If $A\in Aut\left(B\right)$ then $\det\left(A^{t}BA\right)=\det B$ (and $\det B\neq0$) forces $\det A=\pm1$, and we define $SO\left(p,q\right)=\{A\in O\left(p,q\right)\,|\,\det A=1\}$. This is a subgroup of index 2 in $O\left(p,q\right)$ (since $\operatorname{diag}\left(-1,1^{\times n-1}\right)\in O\left(p,q\right)$, for example).
- Over $\mathbb{C}$, every regular symmetric matrix is congruent to I, since in the proof of Sylvester we can even replace v by $\frac{v}{\sqrt{\langle v,v\rangle}}$ when $\langle v,v\rangle<0$, and thus obtain $\langle v,v\rangle=1$. There is thus a unique complex orthogonal group which we denote by $O_{n}\left(\mathbb{C}\right)$, and again $SO_{n}\left(\mathbb{C}\right)$ is of index 2.

2.1.2 Symplectic groups

- If B is non-degenerate anti-symmetric, then it is always congruent to $\Omega=\left(\begin{smallmatrix}0 & I\\ -I & 0\end{smallmatrix}\right)$ (in fact over any $\mathbb{F}$ with char $\neq2$), and we denote $Aut\left(\Omega\right)$ by $Sp_{2n}\left(\mathbb{F}\right)$.
- Again $\det A=\pm1$ for $A\in Sp_{2n}\left(\mathbb{F}\right)$ is easy, but it turns out that actually $\det A=1$. One way: prove that the Pfaffian $pf\left(A\right)=\frac{1}{2^{n}n!}\sum_{\sigma\in S_{2n}}\operatorname{sgn}\sigma\prod_{i=1}^{n}a_{\sigma(2i-1),\sigma(2i)}$ satisfies $pf\left(\Omega\right)=pf\left(A^{T}\Omega A\right)=\det\left(A\right)pf\left(\Omega\right)$ and $pf\left(\Omega\right)\neq0$.

2.1.3 General bilinear

- Why did we restrict to non-degenerate and (anti-)symmetric forms? In general we can always write $b=b_{s}+b_{a}$ where $b_{s}\left(v,w\right)=\frac{b(v,w)+b(w,v)}{2}$ is symmetric and $b_{a}\left(v,w\right)=\frac{b(v,w)-b(w,v)}{2}$ is anti-symmetric. Obviously $Aut\left(b\right)\supseteq Aut\left(b_{s}\right)\cap Aut\left(b_{a}\right)$, but it turns out that this is an equality (at least in the non-degenerate case).
- b is called *reflexive* if $b\left(v,w\right)=0$ implies $b\left(w,v\right)$ (so $\perp$ is symmetric).
  Exercise: b is reflexive iff it is either symmetric or anti-symmetric.

2.1.4 Unitary groups

For a complex vector space V, a *sesquilinear* (latin: one and a half linear) form on V is a form $b : V \times V \to \mathbb{C}$ which is additive in both entries and satisfies $b(\alpha v, \beta w) = \overline{\alpha}\beta b(v, w)$.

- By $V \cong \mathbb{C}^n$, it corresponds to $v^* Bw$ for some $B \in M_n(\mathbb{C})$, and B, C are congruent if $C = P^* BP$ for some $P \in GL_n(\mathbb{C})$. Now $Aut(b)$ corresponds to $\{A \in GL_n(\mathbb{C}) \mid A^* BA = B\}$, which is a matrix Lie group in $GL_n(\mathbb{C})$.
- Again non-degenerate corresponds to $\det B \neq 0$. Assuming b is non-degenerate, it cannot be symmetric, nor anti-symmetric, as in both cases $\langle iv, w\rangle = \pm \langle w, iv\rangle$ forces $\langle v, w\rangle = 0$ for all v, w.
- But it can be *hermitian*: $\langle v, w\rangle = \overline{\langle w, v\rangle}$, which is equivalent to $B^* = B$. Any non-degenerate hermitian $B \in M_n(\mathbb{C})$ is congruent to $\operatorname{diag}(1^{\times p}, -1^{\times q})$ for a unique $0 \leq p \leq n$: the proof is like in Sylvester, once $\langle v, v\rangle \neq 0$ we know $\langle v, v\rangle \in \mathbb{R}$ by hermiticity, and we can replace v by $\frac{v}{\sqrt{|\langle v,v\rangle|}}$ to get $\langle v, v\rangle = \pm 1$. We cannot force $\langle v, v\rangle = 1$ because $\langle \alpha v, \alpha v\rangle = |\alpha|^2 \langle v, v\rangle$ always has the same sign.
- We denote $Aut(\operatorname{diag}(1^{\times p}, -1^{\times q}))$ by $U(p, q)$, and $U(n, 0) = U(n)$.
- $A^* BA = B$ gives $|\det A| = 1$ (when $\det B \neq 0$), and $SU(p, q) = U(p, q) \cap SL_n(\mathbb{C})$ is of infinite index in $U(p, q)$.
- We don't bother with anti-hermitian forms ($\langle v, w\rangle = -\overline{\langle w, v\rangle}$, or $B^* = -B$) because if B is anti-hermitian then iB is Hermitian and $Aut(B) = Aut(iB)$.
- We could stay with $\mathbb{R}$ using $\mathbb{C} \hookrightarrow M_2(\mathbb{R})$, which would embed $M_n(\mathbb{C}) \hookrightarrow M_n(M_2(\mathbb{R})) \cong M_{2n}(\mathbb{R})$. In fact $U(n)$ is precisely $O(2n) \cap Sp_{2n}(\mathbb{R})$, since i acts on $\mathbb{R}^2$ by $\Omega = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, so being complex-linear is the same as commuting with Ω, which for $A \in O(n)$ is the same as $A\Omega A^t = \Omega$.
  In fact $U(n)$ equals the intersection of any two out of three (and all three) among $O(2n), Sp_{2n}(\mathbb{R})$ and the image of $GL_n(\mathbb{C})$ in $GL_{2n}(\mathbb{R})$. (Is $U(p, q) = O(2p, 2q) \cap Sp_{2n}(\mathbb{R})$?)
- Exercise: $Sp_2(\mathbb{R}) \cong SU(1, 1) \cong SL_2(\mathbb{R}) \cong SO(2, 1)_0$ (G_0 is the connected component of the identity of a Lie group G).

2.1.5 Some topology

- Compactness: $(S)O(n), (S)U(n)$ are compact, and also $Sp(n)$ below.
- In a non-degenerate symmetric/anti-symmetric bilinear/hermitian space, if v, w satisfy $v \perp v$, $w \perp w$, $\langle v, w\rangle = 1$ (hence $\langle w, v\rangle = \pm 1$), then v, w are called a *hyperbolic pair* and the subspace $\operatorname{Span}(v, w)$ is called a *hyperbolic plane*. They are very useful, for example:
  - They allow to perform "Gram-Schmidt" - to get u which is orthogonal to v, rather then taking $u \mapsto u - \langle u, v\rangle v$, one takes $u \mapsto u - \langle u, v\rangle w$.
  - If such a plane exists $Aut(b)$ is non-compact since $(v, w) \mapsto \left(\alpha v, \frac{1}{\alpha} w\right)$ (and identity on $\operatorname{Span}(v, w)^{\perp}$) is in $Aut(b)$ for any $\alpha \neq 0$.
- $O(p, q), U(p, q)$ with $pq \neq 0$, $O_n(\mathbb{C})$, $Sp_n(\mathbb{F})$ all contain hyperbolic plane (note $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ are congruent even over $\mathbb{R}$), hence non-compact.
- $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$, $U(n), SO(n)$ are connected: in $U(n)$ every matrix is diagonalizable (even normal) and conjugate to $\operatorname{diag}\left(e^{it_1}, \ldots, e^{it_n}\right)$, and we can build a path to it. In $SO(n)$ we can do this with 2×2 rotation blocks matrices (diagonalize over $\mathbb{C}$, then take $\left(\begin{smallmatrix} a+bi & \\ & a-bi \end{smallmatrix}\right)$ back to $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$).
- For $pq \neq 0$, $SO(p, q)$ has two connected components and $O(p, q)$ four.

2.2 Quaternions

- Quaternions: $\mathbb{H} = \{r + xi + yj + zk \,|\, r, x, y, z \in \mathbb{R}\}$ with multiplication defined by $i^2 = j^2 = k^2 = ijk = -1$ (and $\mathbb{R} \subseteq \mathbb{H}$ in the center).
- Can be written as $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ with $\alpha j = j\overline{\alpha}$. Embeds in $M_2(\mathbb{C})$ by $\mathbb{H} \cong \left(\begin{smallmatrix}\alpha & \beta \\ -\overline{\beta} & \overline{\alpha}\end{smallmatrix}\right) \leq M_2(\mathbb{C})$, which shows $\mathbb{H}$ is associative.
- For $\alpha = r + xi + yj + zk$, its *conjugate* is $\overline{\alpha} = r - xi - yj - zk$. Exercise: $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$.
- Norm: $N(\alpha) = \alpha\overline{\alpha} = r^2 + x^2 + y^2 + z^2$. It is multiplicative since $\alpha\beta\overline{\alpha\beta} = \alpha\beta\overline{\beta}\,\overline{\alpha} = \alpha\overline{\alpha}\beta\overline{\beta}$, and it follows that $\mathbb{H}$ is a division ring ($\alpha^{-1} = \frac{\overline{\alpha}}{N(\alpha)}$). Under $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$, N corresponds to det.
- $\mathbb{H}^1 := \{\alpha \in \mathbb{H} \,|\, N(\alpha) = 1\}$ is a matrix Lie group, homeomorphic to S^3 (by definition of the latter).
- Exercise: The embedding $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ restricts to an isomorphism $\mathbb{H}^1 \cong SU(2)$.

2.2.1 Quaternions and rotations

- The *pure quaternions* are $\mathbb{P} = \mathrm{Span}_{\mathbb{R}}\{i, j, k\}$. We think of $(\mathbb{P}, N)$ as Euclidean three-space (N coincides with $\|\cdot\|^2$), and observe that for $\mathfrak{p} \in \mathbb{P}$, $\|\mathfrak{p}\|^2 = N(\mathfrak{p}) = \mathfrak{p}\overline{\mathfrak{p}} = -\mathfrak{p}^2$. Thus, $\mathbb{P}^1 := \{\mathfrak{p} \in \mathbb{P} \,|\, N(\mathfrak{p}) = 1\}$ (which is geometrically a two-sphere) consists entirely of square roots of -1.
- We can express the Euclidean inner product in $\mathbb{P}$ by polarization:

$$\langle \mathfrak{p}, \mathfrak{q} \rangle = \frac{\|\mathfrak{p} + \mathfrak{q}\|^2 - \|\mathfrak{p}\|^2 - \|\mathfrak{q}\|^2}{2} = \frac{-(\mathfrak{p} + \mathfrak{q})^2 + \mathfrak{p}^2 + \mathfrak{q}^2}{2} = \frac{-\mathfrak{p}\mathfrak{q} - \mathfrak{q}\mathfrak{p}}{2},$$

  so in particular $\mathfrak{p} \perp \mathfrak{q}$ (for $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}$) iff $\mathfrak{p}\mathfrak{q} = -\mathfrak{q}\mathfrak{p}$.
- $\mathbb{H}^1$ acts by conjugation on $\mathbb{P}$ (if $\alpha \in \mathbb{H}^1$ and $\mathfrak{p} \in \mathbb{P}$ then $\alpha\mathfrak{p}\alpha^{-1} = \alpha\mathfrak{p}\overline{\alpha}$ is pure since $\overline{\alpha\mathfrak{p}\overline{\alpha}} = -\alpha\mathfrak{p}\overline{\alpha}$). This action is by orthogonal transformations (since $N(\alpha\mathfrak{p}) = N(\mathfrak{p})$), so we obtain a homomorphism $\mathbb{H}^1 \to O(3)$, and in fact $\mathbb{H}^1 \to SO(3)$ (for example, since $\mathbb{H}^1 \cong S^3$ is connected).
- If $\mathfrak{p} \in \mathbb{P}^1$ (pure of norm one) and $\vartheta \in \mathbb{R}$, then $\mathfrak{p}^2 = -1$ implies $e^{\vartheta\mathfrak{p}} = \cos\vartheta + \mathfrak{p}\sin\vartheta$, which shows that $\exp : \mathbb{P} \twoheadrightarrow \mathbb{H}^1$ (similarly to $\exp : i\mathbb{R} \twoheadrightarrow S^1$ in $\mathbb{C}$). It turns out that $e^{\vartheta\mathfrak{p}}$ (acting via conjugation) rotates $\mathbb{P}$ by 2ϑ around the axis $\mathfrak{p}$ (whereas in $\mathbb{C}$, $e^{\vartheta i}$ acts via multiplication, and rotates by ϑ).
  - Proof: first verify that if $\mathfrak{q} \in \mathbb{P}^1$ with $\mathfrak{q} \perp \mathfrak{p}$ then $\mathfrak{p}\mathfrak{q} \in \mathbb{P}^1$ as well[1], and that $\{\mathfrak{p}, \mathfrak{q}, \mathfrak{p}\mathfrak{q}\}$ is an orthonormal basis for $\mathbb{P}$. Now, $e^{\vartheta\mathfrak{p}}\mathfrak{p}e^{-\vartheta\mathfrak{p}} = \mathfrak{p}$ as always so $\mathfrak{p}$ is fixed, and $e^{\vartheta\mathfrak{p}}\mathfrak{q}e^{-\vartheta\mathfrak{p}} = \cos 2\vartheta \cdot \mathfrak{q} + \sin 2\vartheta \cdot \mathfrak{q}\mathfrak{p}$, so we got a 2ϑ-rotation in the $(\mathfrak{q}, \mathfrak{p}\mathfrak{q})$-plane.
- By Euler's theorem, every element in $SO(3)$ is a rotation around some axis, so that the homomorphism $\mathbb{H}^1 \to SO(3)$ is onto. Furthermore, we see that if $e^{\vartheta\mathfrak{p}} \in \mathbb{H}^1$ acts trivially on $\mathbb{P}$ then $\vartheta \in \pi\mathbb{Z}$, so $e^{\vartheta\mathfrak{p}} = \pm 1$. This shows that $1 \to \{\pm 1\} \to \mathbb{H}^1 \to SO(3) \to 1$ is exact (in fact, $\mathbb{H}^1$ is the universal cover of $SO(3)$, as $\mathbb{H}^1 \cong S^3$ is simply-connected). In general, $SO(p,q)_0$ has a double universal cover for $p + q \geq 3$, called $\mathrm{Spin}(p,q)$. Thus, $\mathbb{H}^1 \cong \mathrm{Spin}(3)$.
- $\mathbb{H}^1 \times \mathbb{H}^1$ acts on $\mathbb{H} \cong \mathbb{R}^4$ via orthogonal transformations, by $(\alpha, \beta) \cdot \gamma = \alpha\gamma\beta^{-1}$. Thus we get $\mathbb{H}^1 \times \mathbb{H}^1 \to SO(4)$. Exercise: this is onto, and two-to-one, so that $\mathbb{H}^1 \times \mathbb{H}^1 \cong \mathrm{Spin}(4)$.

2.2.2 Quaternionic Classical groups

- The theory of vector spaces (bases, dimension, linear transformations/matrices) is pretty much the same over division rings as over fields. One difference though: the scalars $\mathbb{H}$ act on $\mathbb{H}^n$ from the right (in order to commute with the action of $GL_n(\mathbb{H})$ from the left). A binary form on $V \cong \mathbb{H}^n$ is called bilinear if $b(\alpha v, \beta w) = \alpha b(v,w)\beta$ and sesquilinear if $b(\alpha v, \beta w) = \overline{\alpha} b(v,w)\beta$.

[1] but unlike $\mathbb{H}^1$ and $S^1 \subseteq \mathbb{C}$, $\mathbb{P}^1$ is not a group (in fact, there is no topological group structure on S^2).

- $U^*(2n) = GL_n(\mathbb{H})$. Note it is not enough to have nonzero determinant! e.g. $\det\begin{pmatrix} i & j \\ i & j \end{pmatrix} = ij - ji = 2k$. In fact, the determinant is not well defined for non-commutative rings.

- $SU^*(2n) = SL_n(\mathbb{H})$. What is det here? We can observe the Dieudonné determinant (which is a homomorphism $GL_n(\mathbb{H}) \to \mathbb{H}^\times/[\mathbb{H}^\times,\mathbb{H}^\times] \cong \mathbb{R}^\times_{>0}$), or use $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ and take determinant there: for example $\det((\alpha + j\beta)) = \det\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \alpha\overline{\alpha} + \beta\overline{\beta}$.

- As over $\mathbb{C}$, there are no nondegenerate (anti-)symmetric bilinear forms over $\mathbb{H}$.

- The quaternionic-unitary group $Sp(p,q)$:[2] Any nondegenerate sesquilinear Hermitian form on $\mathbb{H}^n$ is congruent to $b(v,w) = v^* I_{p,q} w$ with $I_{p,q} = \mathrm{diag}(1^{\times p}, -1^{\times q})$, and we define $Sp(p,q) = \{A \in GL_{p+q}(\mathbb{H}) \mid A^* I_{p,q} A = I_{p,q}\}$. Under $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ we have $Sp(p,q) = U(2p,2q) \cap Sp_{2p+2q}(\mathbb{C})$, hence the name. In particular $Sp(n)$ is compact since $U(2n)$ is.

- The quaternionic orthogonal group $O^*(2n)$: Any nondegenerate sesquilinear anti-Hermitian form ($\langle v,w\rangle = -\overline{\langle w,v\rangle}$) on $\mathbb{H}^n$ is congruent to $b(v,w) = v^* j w$, and $O^*(2n) = O_n(\mathbb{H}) = \{A \in GL_n(\mathbb{H}) \mid A^* j A = jI\}$. It equals $O_{2n}(\mathbb{C}) \cap U(n,n)$, hence the name. Pf: find $\langle v,v\rangle \neq 0$. We have $\langle v,v\rangle \in \mathbb{P}$, and we can normalize (by $\sqrt{N(\langle v,v\rangle)} \in \mathbb{R}$) to get $\langle v,v\rangle \in \mathbb{P}^1$. Since $\mathbb{H}^1 \cong SO(3)$ acts transitively on $\mathbb{P}^1 \cong S^2$, there is $\alpha \in \mathbb{H}^1$ with $\langle \alpha v, \alpha v\rangle = \alpha \langle v,v\rangle \overline{\alpha} = j$, so we replace v with αv and continue as before.

3 Lie Algebras

- Abstract Lie Algebra: a (non-associative, non-unital) vector space $\mathfrak{g}$ over $\mathbb{F}$, with a $\mathbb{F}$-bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is denoted by $[\cdot,\cdot]$ (and called *bracket*), satisfying
  - $[X,X] = 0$ (which implies $[X,Y] = -[Y,X]$).
  - the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

- Example: if R is any associative algebra, then $[X,Y] := XY - YX$ defines a structure of a Lie algebra (the old multiplication is forgotten).
  Even for Lie algebras which don't arise in this way, the terminology pretends that it did: X, Y are called *commuting* if $[X,Y] = 0$, the *center* of $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{X \mid [X,\mathfrak{g}] = 0\}$, and $\mathfrak{g}$ is *commutative* if $[\mathfrak{g},\mathfrak{g}] = 0$.

- $\mathfrak{h} \leq \mathfrak{g}$ is a *subalgebra* if it is a linear subspace (so dimension considerations are helpful!) and $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. It is an *ideal* ($\mathfrak{h} \trianglelefteq \mathfrak{g}$) if furthermore $[\mathfrak{h},\mathfrak{g}] \subseteq \mathfrak{h}$. $\mathfrak{g}$ is called *simple* if it has no non-trivial ideals, and $\dim \mathfrak{g} \geq 2$ (equivalently, is non-commutative - in a commutative Lie algebra every subspace is an ideal).

- Direct sum: $\mathfrak{g} \oplus \mathfrak{g}'$ is defined by $[(X,X'),(Y,Y')] = \left([X,Y]_{\mathfrak{g}}, [X',Y']_{\mathfrak{g}'}\right)$. Inner direct sum (exercise): if $\mathfrak{h},\mathfrak{h}' \leq \mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ as vector spaces and $[\mathfrak{h},\mathfrak{h}'] = 0$, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$.

3.1 Lie algebras of matrix Lie groups

- A *matrix Lie algebra* is a subspace $V \leq M_n(\mathbb{F})$ which is closed under bracket.

- The (matrix) Lie algebra $\mathfrak{g} = \mathit{Lie}(G)$ of a matrix Lie group $G \leq GL_n(\mathbb{F})$ is $\mathfrak{g} = \left\{X \in M_n(\mathbb{F}) \,\middle|\, \forall t \in \mathbb{R} : e^{tX} \in G\right\}$.

[2] Also called: the unitary symplectic, compact symplectic, or hyperunitary group.

Examples:

$$\begin{aligned}
\mathfrak{gl}_n(\mathbb{F}) &= M_n(\mathbb{F}) \text{ (also for } \mathbb{F}=\mathbb{H}) \\
\mathfrak{sl}_n(\mathbb{F}) &= \{X \in M_n(\mathbb{F}) \mid \operatorname{trace}(X) = 0\} \\
\mathfrak{o}_n(\mathbb{F}) = \mathfrak{so}_n(\mathbb{F}) &= \left\{X \in M_n(\mathbb{F}) \,\middle|\, X^T = -X\right\} \\
\mathfrak{u}(n) &= \{X \in M_n(\mathbb{C}) \mid X^* = -X\} \\
\mathfrak{su}(n) &= \mathfrak{u}(n) \cap \mathfrak{sl}(n) \\
\mathfrak{so}(p,q) &= \left\{X \in M_n(\mathbb{R}) \,\middle|\, I_{p,q} X^T I_{p,q} = -X\right\} \\
\mathfrak{sp}_{2n}(\mathbb{F}) &= \left\{X \in M_n(\mathbb{F}) \,\middle|\, \Omega X^T \Omega = X\right\} \\
\mathfrak{sp}(n) &= \{X \in M_n(\mathbb{H}) \mid X^* = -X\} \overset{\mathbb{H}\cong\mathbb{C}^2}{=} \mathfrak{u}(2n) \cap \mathfrak{sp}_{2n}(\mathbb{C}) \\
\mathfrak{sp}(1) = Lie\left(\mathbb{H}^1\right) &= \mathbb{P} \subseteq \mathbb{H}
\end{aligned}$$

e.g., if $X^T = -X$ then $\left(e^{tX}\right)^T = \left(e^{tX^T}\right) = e^{-tX}$ implies $e^{tX} \in O(n)$, and if $e^{tX} \in O(n)$ for all t then

$$0 = (I)'(0) = \left(e^{tX} \cdot e^{tX^T}\right)'(0) = Xe^{tX}e^{tX^T} + e^{tX}X^T e^{tX^T}\big|_{t=0} = X + X^T.$$

- Example: a basis for $\mathfrak{so}(3)$ is

$$X_1 = \begin{pmatrix} 0 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \quad X_2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}, \quad X_3 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix},$$

  and the bracket is given by $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$. We have $e^{tX_1} = \begin{pmatrix} 1 & & \\ & \cos t & -\sin t \\ & \sin t & \cos t \end{pmatrix}$, and similarly for the other two, but $e^{tX_1 + sX_2}$ is already complicated.

- The following basis for $\mathfrak{su}(2)$:

$$X_1 = \tfrac{1}{2}\begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad X_2 = \tfrac{1}{2}\begin{pmatrix} & i \\ i & \end{pmatrix}, \quad X_3 = \tfrac{1}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \tag{3.1}$$

  satisfies the same brackets relations as in the previous example, which means that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ (recall that $SU(2) \not\cong SO(3)$ as topological groups - we saw that $SU(2) \cong \mathbb{H}^1$ is a double cover of $SO(3)$).

- $\mathfrak{g}$ is indeed a matrix Lie algebra: $\mathbb{R}\mathfrak{g} \subseteq \mathfrak{g}$ by the definition, $\mathfrak{g} + \mathfrak{g} \subseteq \mathfrak{g}$ by Trotter-Lie formula, and $[X,Y] \in \mathfrak{g}$ as follows: $\left(e^{tX}Ye^{-tX}\right)' = e^{tX}(XY - YX)e^{-tX}$, so that $[X,Y] = \left(e^{tX}Ye^{-tX}\right)'(0)$. But $e^{tX}Ye^{-tX} \in \mathfrak{g}$, since for any $s \in \mathbb{R}$ we have $e^{se^{tX}Ye^{-tX}} = e^{tX}e^{sY}e^{-tX} \in G$; Thus $\frac{e^{tX}Ye^{-tX} - Y}{t} \in \mathfrak{g}$ for any $t > 0$, hence $[X,Y] \in \mathfrak{g}$ ($\mathfrak{g}$ is closed, being a linear subspace of M_n).

- A way to get the commutator back from the group is:

$$[X,Y] = \left(e^{tX}Ye^{-tX}\right)'(0) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial s}\left(e^{tX}e^{sY}e^{-tX}\right)\big|_{s=0}\right]\Bigg|_{t=0}. \tag{3.2}$$

  - This shows for example that if G is commutative then so is $\mathfrak{g}$.
  - It also shows that if two Lie groups G, H have neighborhood of their identities which are isomorphic (e.g. $SO(3)$ and $U(2)$) then $\mathfrak{g} \cong \mathfrak{h}$.
  - Also, it gives a hint on how to define the Lie bracket of an abstract Lie group...

- Note that $\mathfrak{g}$ is only an algebra over $\mathbb{R}$, even when $G \leq GL_n(\mathbb{C} \text{ or } \mathbb{H})$, e.g. $\mathfrak{u}(n)$. We call G a *complex Lie group* if $\mathfrak{g}$ is also a Lie algebra over $\mathbb{C}$ (e.g. $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $O_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ - I think these are the only ones in the list above).

- Denote by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the *complexification* of $\mathfrak{g}$, which is the complex Lie algebra with the same basis and structure coefficients as $\mathfrak{g}$ has over $\mathbb{R}$. If $\mathfrak{g} \leq M_n(\mathbb{C})$ and $\mathfrak{g} \cap i\mathfrak{g} = 0$ then we have $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus i\mathfrak{g} \leq M_n(\mathbb{C})$ (including the Lie algebra structure), which gives for example

$$\mathfrak{sl}_n(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sl}_n(\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}}, \tag{3.3}$$

as $\mathfrak{su}(2) \cap i\,\mathfrak{su}(2) = 0$ and $\mathfrak{su}(2) + i\,\mathfrak{su}(2) = \mathfrak{sl}_2(\mathbb{C})$.

3.2 Exponential again

- Fix $G \leq GL_d(\mathbb{F})$, $\mathfrak{g} = Lie(G)$. By definition, we have $\exp : \mathfrak{g} \to G$. In general it is neither injective nor onto. The image $e^{\mathfrak{g}}$ is connected, so it is contained in the identity component of G, but it is not always onto this component (e.g. $\left(\begin{smallmatrix} -1 & 1 \\ & -1 \end{smallmatrix}\right)$ is not in $e^{\mathfrak{sl}_2(\mathbb{C})}$ and not in $e^{\mathfrak{gl}_2(\mathbb{R})}$).[3]
- Theorem: There is a neighborhood of $I \in GL_n(\mathbb{F})$ in which $A \in G \Leftrightarrow \log A \in \mathfrak{g}$.
  - Corollary: There are nbds. U of $0 \in \mathfrak{g}$ and V of $I \in G$ such that $\exp\big|_U : U \to V$ is a homeomorphism. Pf: take $0 < \varepsilon < \log 2$ for which $e^{B_\varepsilon(0)}$ is as in the theorem, and then let $U = B_\varepsilon(0) \cap \mathfrak{g}$, $V = e^{B_\varepsilon(0)} \cap G$.
  - Lemma: For $X_n \in M_m(\mathbb{F})$ with $e^{X_n} \in G$ and $X_n \to 0$, if $\frac{X_n}{\|X_n\|} \to X$ then $X \in \mathfrak{g}$. Pf: Since $\|X_n\| \to 0$, for any $t \in \mathbb{R}$ taking $k_n = \left\lfloor \frac{t}{\|X_n\|} \right\rfloor$ we have $k_n\|X_n\| \to t$, so that $e^{tX} = \lim e^{k_n\|X_n\|\frac{X_n}{\|X_n\|}} = \lim \left(e^{X_n}\right)^{k_n} \in G$.
  - Proof: $\Leftarrow$ if $\log A \in \mathfrak{g}$ then $A = e^{\log A} \in G$ for $\|A - I\| < 1$. $\Rightarrow$ assume to the contrary than we have $A_n \to I$ with $A_n \in G$ and $\log A_n \notin \mathfrak{g}$. We identify $V = M_n(\mathbb{F}) \cong \mathbb{R}^{cn^2}$ with $c = 1/2/4$ and the standard inner product, and write $V = \mathfrak{g} \oplus \mathfrak{g}^\perp$. Define $\Phi \colon V \to V$ by $\Phi(X + Y) = e^X e^Y$ for $X \in \mathfrak{g}, Y \perp \mathfrak{g}$. Then Φ is differentiable, and its Jacobian[4] satisfies $\mathbf{J}_\Phi(0) = I$ by checking on a basis composed of $\mathfrak{g}$ and $\mathfrak{g}^\perp$ elements. By the inverse function theorem, Φ has a continuous inverse around $\Phi(0) = I$, so we can write $A_m = \Phi(X_m + Y_m) = e^{X_m}e^{Y_m}$ with $X_m \in \mathfrak{g}$, $Y_m \perp \mathfrak{g}$, and $X_m, Y_m \to 0$. Since $\log A_n \notin \mathfrak{g}$ we have $Y_n \neq 0$, and after passing to a subsequence, $\frac{Y_n}{\|Y_n\|}$ converges to some $Y \in \mathfrak{g}^\perp$ of norm 1. As $e^{Y_n} = e^{-X_n}A_m \in G$ and $Y_n \to 0$, by the Lemma $Y \in \mathfrak{g}$, which is a contradiction.
- More corollaries:
  - As G acts transitively on G, it is a manifold, of dimension $\dim \mathfrak{g}$.[5]
  - If G is connected then $e^{\mathfrak{g}}$ generates G (every $A \in G$ equals $e^{X_1} \ldots e^{X_n}$ for some $X_i \in \mathfrak{g}$): for $g \in G$ pick a path $\gamma : I \to g$. We know $e^{\mathfrak{g}}$ contains some nbd. V of I, and the translates $\gamma(t)V$ cover γ, which is compact, so there is a finite subcover.
  - If G is connected and $\mathfrak{g}$ is commutative so is G (since elements in $e^{\mathfrak{g}}$ commute).
  - G_0 is closed (hence also a matrix Lie group): if $A_n \in G_0$ and $A_n \to A$, then $A_nA^{-1} \to I$, so $A_nA^{-1} = e^X$ for some n and $X \in \mathfrak{g}$, hence $A = e^{-X}A_n$ is path-connected to A_n (by $e^{-tX}A_n$).
  - $\mathfrak{g} = T_IG := \left\{\gamma'(0) \,\middle|\, \gamma \colon C^1(\mathbb{R}, G), \gamma(0) = I\right\}$: $\subseteq$ by $\gamma(t) = e^{tX}$, $\supseteq$: for $|t|$ small enough, $\gamma(t) = e^{\log\gamma(t)}$, and $\gamma'(0) = \left(e^{\log\gamma}\right)'(0) = (\log\gamma)'(0)\left(e^{\log\gamma(0)}\right) = (\log\gamma)'(0)$. Since $\log\gamma(t) \in \mathfrak{g}$ for t small enough, also $(\log\gamma)'(0) \in \mathfrak{g}$.

3.3 Homomoprhisms

- A matrix Lie groups hom. is a continuous group hom., e.g.: $\det \colon G \to \mathbb{F}^\times$, $e^{tX} : \mathbb{R} \to G$ (we saw these are the only (continuous) homs from $\mathbb{R}$ to GL_n), $SU(2) \cong \mathbb{H}^1 \to SO(3)$, $\mathbb{H}^1 \times \mathbb{H}^1 \to SO(4)$.
- G acts on its Lie algebra $\mathfrak{g}$ by conjugation: $A \cdot X = AXA^{-1}$. This gives the "adjoint representation" which is a Lie group hom. $\mathrm{Ad} \colon X \mapsto \mathrm{Ad}_X \colon G \to GL(\mathfrak{g})$. In fact, we even have $\mathrm{Ad} : G \to Aut_{LieAlg}(\mathfrak{g}) \leq GL(\mathfrak{g})$ as $\mathrm{Ad}_A[X, Y] = [\mathrm{Ad}_A X, \mathrm{Ad}_A Y]$.

[3] Maybe we will later show that if G is connected and compact then $e^{\mathfrak{g}} = G$.
[4] a.k.a. differential $D\Phi$ or total derivative $\nabla\Phi$.
[5] Even a smooth manifold, for those who know the term.

- For $\Phi\colon G\to H$ there is a unique Lie algebra hom. $Lie\,(\Phi)=d\Phi=\phi\colon\mathfrak{g}\to\mathfrak{h}$ (called the *differential* of Φ) such that $\Phi\left(e^{X}\right)=e^{\phi(X)}$ ($\forall X\in\mathfrak{g}$). In addition, Φ,ϕ intertwine Ad of G and H: $\phi\left(\mathrm{Ad}_{A}\left(X\right)\right)=\mathrm{Ad}_{\Phi(A)}(\phi\left(X\right))$ for $A\in G,X\in\mathfrak{g}$.
  - Example: $d\det=\mathrm{trace}$ since $\det e^{X}=e^{\mathrm{trace}\,X}$.
  - Since $e^{\psi(\phi(X))}=\Psi\left(e^{\phi(X)}\right)=\Psi\left(\Phi\left(X\right)\right)$, *Lie* is a functor from matrix Lie groups to Lie algebras. For connected groups *Lie* is faithful: If $\Phi,\Psi\colon G\to H$ and $\phi=\psi$ then $\Phi=\Psi$ since $e^{\mathfrak{g}}$ generates G.
  - Pf: Let $\gamma\colon t\mapsto\Phi\left(e^{tX}\right):\mathbb{R}\to H$. As γ is a (continuous) hom., there is $Z\in\mathfrak{h}$ with $\gamma\left(t\right)=e^{tZ}$ (see "Applications" in §1), hence γ is smooth, ϕ is homogeneous, and it equals

$$\phi\left(X\right)=\left(e^{t\phi(X)}\right)'\left(0\right)=\Phi(e^{tX})'\left(0\right),$$

implying uniqueness. For the rest:

$$\begin{aligned}\phi\left(X+Y\right)&=\Phi\left(e^{t(X+Y)}\right)'\left(0\right)=\Phi\left(\lim_{m}\left(e^{\frac{t}{m}X}e^{\frac{t}{m}Y}\right)^{m}\right)'\left(0\right)=\left[\lim_{m}\left(\Phi\left(e^{\frac{t}{m}X}\right)\Phi\left(e^{\frac{t}{m}Y}\right)\right)^{m}\right]'\left(0\right)\\&=\left[\lim_{m}\left(e^{\phi\left(\frac{t}{m}X\right)}e^{\phi\left(\frac{t}{m}Y\right)}\right)^{m}\right]'\left(0\right)=\left[\lim_{m}\left(e^{\frac{t}{m}\phi(X)}e^{\frac{t}{m}\phi(Y)}\right)^{m}\right]'\left(0\right)=\left(e^{t(\phi(X)+\phi(Y))}\right)'\left(0\right)=\phi\left(X\right)+\phi\left(Y\right).\\\phi\left(\mathrm{Ad}_{A}\left(X\right)\right)&=\Phi\left(e^{tAXA^{-1}}\right)'\left(0\right)=\Phi\left(A\right)\Phi\left(e^{tX}\right)\Phi\left(A^{-1}\right)'\left(0\right)=\Phi\left(A\right)\phi\left(X\right)\Phi\left(A^{-1}\right)=\mathrm{Ad}_{\Phi(A)}(\phi\left(X\right)).\\\phi\left([X,Y]\right)&=\phi\left(\left(e^{tX}Ye^{-tX}\right)'\left(0\right)\right)=\phi\left(\mathrm{Ad}_{e^{tX}}\left(Y\right)'\left(0\right)\right)\overset{\star}{=}\phi\left(\mathrm{Ad}_{e^{tX}}\left(Y\right)\right)'\left(0\right)=\left(\Phi(e^{tX})\phi(Y)\Phi(e^{-tX})\right)'\left(0\right)\\&=\left(e^{t\phi(X)}\phi(Y)e^{-t\phi(X)}\right)'\left(0\right)=\left[\phi\left(X\right),\phi\left(Y\right)\right].\end{aligned}$$

where $\star$ is from ϕ being linear (and thus also continuous).

- Another example: $\mathrm{ad}=d\,\mathrm{Ad}:X\mapsto\mathrm{ad}_{X}:\mathfrak{g}\to\mathfrak{gl}\left(\mathfrak{g}\right)$. For $X,Y\in\mathfrak{g}$ we find that

$$\mathrm{ad}_{X}\left(Y\right)=\left(\mathrm{Ad}_{e^{tX}}\right)'\left(0\right)\left(Y\right)=\left(\mathrm{Ad}_{e^{tX}}\left(Y\right)\right)'\left(0\right)=\left[X,Y\right].$$

Spelling this out, we get a bracket-description of conjugation by exponents:

$$e^{X}Ye^{-X}=\mathrm{Ad}_{e^{X}}\left(Y\right)=e^{\mathrm{ad}_{X}}\left(Y\right)=\sum\frac{\mathrm{ad}_{X}^{k}(Y)}{k!}=Y+\left[X,Y\right]+\frac{\left[X,\left[X,Y\right]\right]}{2}+\frac{\left[X,\left[X,\left[X,Y\right]\right]\right]}{3!}+\dots$$

- $Lie\,(\ker\Phi)=\ker\phi$ (for $\Phi\colon G\to H$): $\phi\left(X\right)=0\Leftrightarrow\forall t:\Phi\left(e^{tX}\right)=e^{t\phi(X)}=1\Leftrightarrow X\in Lie\,(\ker\Phi)$.

4 From Algebra to Groups

- Our goals: Does every (matrix) Lie algebra arise from a Lie group? Does every homomorphism between Lie algebras arise from a Lie group homomorphism? If $\mathfrak{g}$ does come from G, can we recover G from $\mathfrak{g}$ (without the exponent, using $\mathfrak{g}$ only as an abstract Lie algebra)?

- Preparation: $d\exp$. Theorem (Poincare?): For $X,Y\in M_{n}\left(\mathbb{F}\right)$,

$$\left(e^{X+tY}\right)'\left(0\right)=e^{X}\xi\left(\mathrm{ad}_{X}\right)\left(Y\right),\qquad\xi\left(z\right)=\frac{1-e^{-z}}{z}=\sum_{k=0}^{\infty}\frac{\left(-z\right)^{k}}{\left(k+1\right)!}.\tag{4.1}$$

Note $\mathrm{ad}_{X},\xi\left(\mathrm{ad}_{X}\right)\in End_{\mathbb{F}}\left(M_{n}\left(\mathbb{F}\right)\right)\cong\mathbb{F}^{n^{4}},e^{\mathrm{ad}_{X}}\in GL\left(M_{n}\left(\mathbb{F}\right)\right)$.

  - Lemma: for a bounded operator M, $\lim_{m\to\infty}\frac{1}{m}\sum_{j=0}^{m-1}e^{-\frac{j}{m}M}=\xi\left(M\right)$. Pf: the l.h.s. is a Darboux sum of $\int_{0}^{1}e^{-tM}dx=\frac{-e^{-tM}}{M}\Big|_{0}^{1}=\xi\left(M\right)$. On diagonalizable this is enough. Conclude by continuity?

– Pf: note $\left(e^{X+tY}\right)'(0)$ is continuous in X,Y.

$$\left(e^{X+tY}\right)'(0)=\left[\left(e^{\frac{X}{m}+\frac{tY}{m}}\right)^{m}\right]'(0)=\frac{1}{m}\left[\left(e^{\frac{X}{m}+tY}\right)^{m}\right]'(0)=\frac{1}{m}\sum_{j=0}^{m-1}\left(e^{\frac{X}{m}+tY}\right)^{m-j-1}\left(e^{\frac{X}{m}+tY}\right)'\left(e^{\frac{X}{m}+tY}\right)^{j}\Big|_{t=0}$$
$$=\frac{1}{m}\sum_{j=0}^{m-1}\left(e^{\frac{X}{m}}\right)^{m-j-1}\left(e^{\frac{X}{m}+tY}\right)'(0)\left(e^{\frac{X}{m}}\right)^{j}=e^{\left(\frac{m-1}{m}X\right)}\frac{1}{m}\sum_{j=0}^{m-1}\mathrm{Ad}_{e^{-\frac{j}{m}X}}\left(\left(e^{\frac{X}{m}+tY}\right)'(0)\right)$$
$$=e^{\left(\frac{m-1}{m}X\right)}\sum_{j=0}^{m-1}e^{\mathrm{ad}_{-\frac{j}{m}X}}\left(\left(e^{\frac{X}{m}+tY}\right)'(0)\right)=e^{\left(\frac{m-1}{m}X\right)}\left[\frac{1}{m}\sum_{j=0}^{m-1}e^{-\frac{j}{m}\mathrm{ad}_X}\right]\left(\left(e^{\frac{X}{m}+tY}\right)'(0)\right)$$
$$\overset{m\to\infty}{\longrightarrow}e^{X}\xi\left(\mathrm{ad}_X\right)\left(\left(e^{tY}\right)'(0)\right)=e^{X}\xi\left(\mathrm{ad}_X\right)(Y).$$

- Recovering G from $\mathfrak{g}$: in some neighborhood of the identity we have $e^Xe^Y=e^{\log\left(e^Xe^Y\right)}$, so we want to express $\log\left(e^Xe^Y\right)$ using the Lie structure alone. This is achieved by the ***BCH formula***: for $\|X\|,\|Y\|\le\log\sqrt{2}$,

$$\text{(Poincaré)}\quad\log\left(e^Xe^Y\right)=X+\left(\int_0^1\psi\left(e^{\mathrm{ad}_X}e^{t\,\mathrm{ad}_Y}\right)dt\right)(Y)\quad\left(\psi(z)=\frac{z\log z}{z-1}=1-\sum_{k=1}^{\infty}\frac{(1-z)^k}{k(k+1)}\right)$$
$$\text{(Dynkin)}\quad\log\left(e^Xe^Y\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{\substack{\forall i\in[k]\\ r_i+s_i>0}}\frac{\left[X^{(\times r_1)},Y^{(\times s_1)},\ldots,X^{(\times r_k)},Y^{(\times s_k)}\right]}{r_1!s_1!\ldots r_k!s_k!\left(r_1+s_1+\ldots+r_k+s_n\right)}$$

where e.g. $\left[X^{(\times2)},Y^{(\times2)}\right]=[X,[X,[Y,Y]]]$. Note $\psi(z)=\xi(\log z)^{-1}$ converges at $|z-1|<1$.
Proof: Take $Z(t)=\log\left(e^Xe^{tY}\right)$ so that

$$\log\left(e^Xe^Y\right)=Z(1)=Z(0)+\int_0^1Z'(t)\,dt=X+\int_0^1Z'(t)\,dt.$$

Using (4.1) we have

$$\left(e^{Z(t)}\right)'(t)=\left(e^Xe^{tY}\right)'(t)=e^Xe^{tY}Y=e^{Z(t)}Y,\qquad\text{and also}$$
$$\left(e^{Z(t)}\right)'(t)=\frac{d}{dh}\left(e^{Z(t)+hZ'(t)}\right)\Big|_{h=0}=e^{Z(t)}\xi\left(\mathrm{ad}_{Z(t)}\right)\left(Z'(t)\right),$$

so that
$$Z'(t)=\xi\left(\mathrm{ad}_{Z(t)}\right)^{-1}(Y)\overset{\bigstar}{=}\xi\left(\log\left(e^{\mathrm{ad}_X}e^{t\,\mathrm{ad}_Y}\right)\right)^{-1}(Y)=\psi\left(e^{\mathrm{ad}_X}e^{t\,\mathrm{ad}_Y}\right)(Y),$$

where $\bigstar$ is by $e^{\mathrm{ad}_{Z(t)}}=\mathrm{Ad}_{e^{Z(t)}}=\mathrm{Ad}_{e^Xe^{tY}}=\mathrm{Ad}_{e^X}\mathrm{Ad}_{e^{tY}}=e^{\mathrm{ad}_X}e^{t\,\mathrm{ad}_Y}$.

- From BCH, $\log\left(e^Xe^Y\right)=X+Y+\frac{[X,Y]}{2}+\frac{[X,[X,Y]]-[Y,[X,Y]]}{12}-\frac{[X,[Y,[X,Y]]]}{24}+$ terms with more than four X,Y.

- Given $\phi\colon\mathfrak{g}\to\mathfrak{h}$, there is $f\colon U\to H$ for some nbd. $I\in U\subseteq G$ which satisfies $f\left(e^X\right)=e^{\phi(X)}$, and $f(AB)=f(A)f(B)$ for $A,B,AB\in U$. Pf: defining $f\left(e^X\right)=e^{\phi(X)}$, for $\|X\|<\log2$ we obtain $f\left(e^X\right)=e^{\phi(X)}$. We choose U such that $\overset{\exp}{\underset{\log}{}}:U\rightleftarrows e^U$, and that BCH holds on $\log U$ and on $\log\phi(U)$, and we need to show that (for X,Y small) $e^{\phi\left(\log e^Xe^Y\right)}=f\left(e^Xe^Y\right)=f\left(e^X\right)f\left(e^Y\right)=e^{\phi(X)}e^{\phi(Y)}$, namely that $\phi\left(\log e^Xe^Y\right)=\log e^{\phi(X)}e^{\phi(Y)}$. Using BCH for $\mathfrak{g}$ and $\mathfrak{h}$, it is enough to show that

$$\phi\left(\psi\left(e^{\mathrm{ad}_X}e^{t\,\mathrm{ad}_Y}\right)(Y)\right)=\psi\left(e^{\mathrm{ad}_{\phi(X)}}e^{t\,\mathrm{ad}_{\phi(Y)}}\right)(\phi(Y)),$$

which follows from the intertwining of ad: $\phi\left(\mathrm{ad}_X(Y)\right)=\mathrm{ad}_{\phi(X)}\phi(Y)$ (due to $\phi\in\mathrm{Hom}_{LieAl}(\mathfrak{g},\mathfrak{h})$).

- If G is simply-connected, any hom. $f\colon U\to H$ where U is a nbd. of $I\in G$ extends to a hom. $\Phi\colon G\to H$. Idea: for $A\in G$ construct a path $\gamma:I\rightsquigarrow A$, and define $\Phi(A)=\prod_{j=m}^{1}f\left(\gamma(t_j)\gamma(t_{j-1})^{-1}\right)$ where $(t_0,\ldots,t_m)$ is a partition of I fine enough that $\gamma(t_j)\gamma(t_{j-1})^{-1}\in U$. This is well defined since passing to a finer partition does not change $\Phi(A)$. A small enough homotopic change of γ also does not change $\Phi(A)$, by taking a partition whose points are not affected by the change. As G is simply-connected, this means Φ is well defined globally. Φ agrees with f on U, and is a homomorphism by concatenation of paths.

- Combining the last two, we have that if G is simply connected then $\phi\colon \mathfrak{g}\to\mathfrak{h}$ is obtained from (a unique) $\Phi\colon G\to H$: if we construct f on U by BCH and extend it to Φ, for any X we have $\frac{X}{m}\in U$ for some m, hence $\Phi\left(e^{X}\right)=\Phi\left(e^{\frac{X}{m}}\right)^{m}=e^{m\phi\left(\frac{X}{m}\right)}=e^{\phi(X)}$.

- Decomposing: if G is simply-connected and $\mathfrak{g}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ then $G=H_1\times H_2$ for simply connected H_1,H_2 with $Lie\left(H_i\right)=\mathfrak{h}_i$: The projection $\pi_2\colon\mathfrak{g}\twoheadrightarrow\mathfrak{h}_2$ (w.r.t. $\mathfrak{h}_1\oplus\mathfrak{h}_2$) comes from some $\Pi_2\colon G\to G$, and for $H_1:=\ker\Pi_2$ we have $Lie\left(H_1\right)=\ker\pi_2=\mathfrak{h}_1$. Similarly, $H_2=\ker\Pi_1$ has $Lie(H_2)=\mathfrak{h}_2$. From $\Pi_2\big|_{e^{\mathfrak{h}_1}}=0$, $\Pi_2\big|_{e^{\mathfrak{h}_2}}=id$, and $G=\left\langle e^{\mathfrak{h}_1+\mathfrak{h}_2}\right\rangle=\left\langle e^{\mathfrak{h}_1}e^{\mathfrak{h}_2}\right\rangle$ (using $[\mathfrak{h}_1,\mathfrak{h}_2]=0$), it follows that $\Pi_2: G\twoheadrightarrow H_2$ is also a projection, which implies that H_2 is simply connected, and likewise for H_1. Thus, $H_1\times H_2$ (externally) is simply-connected, so $\mathfrak{h}_1\oplus_{ext}\mathfrak{h}_2\xrightarrow{\cong}\mathfrak{g}$ corresponds to some $\Phi\colon H_1\times H_2\to G$, which is an isomoprhism (its inverse is $\Pi_1\times\Pi_2$).

- Does every matrix Lie algebra $\mathfrak{g}$ corresponds to a matrix Lie group? No: if $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ then $\mathfrak{g}=\mathbb{R}\left(\begin{smallmatrix} i & \\ & \alpha i\end{smallmatrix}\right)$ is a matrix Lie algebra, whose exponent is not closed (it is a dense irrationally sloped line in the torus $\left(\begin{smallmatrix} U(1) & \\ & U(1)\end{smallmatrix}\right)$). However, $\mathfrak{g}\mapsto\langle e^{\mathfrak{g}}\rangle$ and $G\mapsto Lie\left(G\right)$ do constitute a correspondence between matrix Lie algebras in $GL_n(\mathbb{C})$ and connected subgroups of $GL_n(\mathbb{C})$ (not necessarily closed). Pf: Rossmann §2.5 and Hall §5.9.

4.1 Covers

- What happens if G is not simply-connected? A *universal cover* for G is a simply-connected Lie group $\widetilde{G}$ equipped with a map $\widetilde{G}\to G$ which induces an isomorphism $\widetilde{\mathfrak{g}}=\mathfrak{g}$. examples: $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)\mapsto\left(e^{it}\right):\mathbb{R}\to U\left(1\right)$, and $\mathbb{H}^1\to SO\left(3\right)$.

- Given $\phi\colon\mathfrak{g}\to\mathfrak{h}$, there exists $\Phi\colon\widetilde{G}\to H$ with $d\Phi=\phi$ (assuming $\widetilde{G}$ exists).

- Any Lie group G has a universal cover $\widetilde{G}$ (by alg. top. arguments), but it is not always a matrix Lie group. For example, $\widetilde{SL_2(\mathbb{R})}$ is not: First, $SL_2(\mathbb{R})\simeq\left(\begin{smallmatrix}t & a\\ 0 & t^{-1}\end{smallmatrix}\right)\times SO(2)$ (by Gram-Schmidt), which shows that $\pi_1\left(SL_2\left(\mathbb{R}\right)\right)=\mathbb{Z}$. Any hom. $\phi:\mathfrak{sl}_2(\mathbb{R})\to\mathfrak{gl}_n(\mathbb{R})$ corresponds to some $\Phi\colon SL_2(\mathbb{R})\to GL_n(\mathbb{C})$ by noting that ϕ extends to $\phi_{\mathbb{C}}\colon\mathfrak{sl}_2\left(\mathbb{C}\right)\to\mathfrak{gl}_n\left(\mathbb{C}\right)$, and $SL_2\left(\mathbb{C}\right)\simeq\left(\begin{smallmatrix}t & a\\ 0 & t^{-1}\end{smallmatrix}\right)\times SU(2)$ is simply-connected, so there is $\Phi_{\mathbb{C}}\colon SL_2\left(\mathbb{C}\right)\to GL_n(\mathbb{C})$, and we can take $\Phi=\Phi_{\mathbb{C}}\big|_{SL_2(\mathbb{R})}$. Now, if we had $G\subseteq GL_n\left(\mathbb{R}\right)$ with $\Phi\colon G\to SL_2(\mathbb{R})$ and $d\Phi$ an isomorphism, then $\psi=\left(d\Phi\right)^{-1}$ would induce some $\Psi\colon SL_2\left(\mathbb{R}\right)\to GL_n\left(\mathbb{C}\right)$, such that Φ,Ψ are inverse to each other on a nbd. of $I\in G$, hence $G\cong SL_2\left(\mathbb{R}\right)$, and G is not simply-connected.[6]

4.2 Representations

Assume throughout that G is connected.

- A (Lie group) rep. $\Pi\colon G\to GL\left(V\right)$ gives rise to (Lie algebra) representation $\pi\colon\mathfrak{g}\to\mathfrak{gl}\left(V\right)$ with $\Pi\left(e^{X}\right)=e^{\pi(X)}$, $\pi(X)(v)=\Pi(e^{tX})(v)'(0)$, and $\pi\left(\mathrm{Ad}_A X\right)=\mathrm{Ad}_{\Pi(A)}\,\pi\left(X\right)$. Every (L.A.) representation $\pi\colon\mathfrak{g}\to\mathfrak{gl}\left(V\right)$ is obtained from a (L.G.) representation $\Pi\colon\widetilde{G}\to GL\left(V\right)$ (and if G is simply-connected then $G=\widetilde{G}$).
  - Π is irreducible iff π is: if $\pi(X)W\subseteq W$ then $e^{\pi(X)}W\subseteq W$, hence $\Pi(e^{\mathfrak{g}})W\subseteq W$, and if $\Pi(G)W\subseteq W$ then $\pi(X)W=\Pi\left(e^{tX}\right)'(0)\,W\subseteq W$.
  - $\Pi\cong\Pi'$ iff $\pi\cong\pi'$ (similar proof).
  - If V has an inner-product, Π is unitary ($\Pi(A)^*=\Pi\left(A\right)^{-1}$) iff π is unitary ($\pi(X)^*=-\pi(X)$).

- Every $G\leq GL_n\left(\mathbb{C}\right)$ has the standard representation $id\colon G\to GL_n(\mathbb{C})$, and the adjoint representation $\mathrm{Ad}\colon G\to GL(\mathfrak{g})$. Likewise for $\mathfrak{g}$.

- We have $Lie\left(\Pi\oplus\Psi\right)=\pi\oplus\psi$ but $Lie\left(\Pi\otimes\Psi\right)=\pi\otimes id+id\otimes\psi$ and $Lie\left(\Pi^*\right)=-\pi^T$ (for the contragradient $\Pi^*(A)=\left(\Pi(A^{-1})\right)^T$).

[6] In fact, even the double cover of $SL_2\left(\mathbb{R}\right)$, which is called the *metaplectic group*, is not a matrix Lie group - perhaps the same argument works - complexification would then yield a double cover of $SL_2\left(\mathbb{C}\right)$ which is impossible.

- A complex representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a real Lie algebra admits $\pi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{gl}(V)$, and $\mathfrak{g}, \mathfrak{g} + i\mathfrak{g}$ have the same invariant subspaces in V, so π is irreducible iff $\pi_{\mathbb{C}}$ is.

4.3 Example - $SU(2)$

- $SU(2)$ acts on $V_m = \left\{ \sum_{k=0}^{m} \alpha_k z^{m-k} w^k \right\}$ (homogeneous polynomials of degree m in z, w), by $(Af)(z,w) = f\left(A^{-1}\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\right)$.

- V_m is irreducible. Pf: Compute for the basis in (3.1)

$$\pi(X_3)f(z,w) = \left[\Pi\left(e^{tX_3}\right) f(z,w)\right]'(0) = \begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix} f(z,w)'(0) = f\left(\begin{pmatrix} \cos t/2 & \sin t/2 \\ -\sin t/2 & \cos t/2 \end{pmatrix} \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\right)'(0)$$
$$= \left(\tfrac{\partial f(z,w)}{\partial z} \;\; \tfrac{\partial f(z,w)}{\partial w} \right) \left(\frac{1}{2} \begin{pmatrix} -\sin t/2 & \cos t/2 \\ -\cos t/2 & -\sin t/2 \end{pmatrix} \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \right)_{t=0} = \left(\tfrac{\partial f(z,w)}{\partial z} \;\; \tfrac{\partial f(z,w)}{\partial w} \right) \begin{pmatrix} \frac{w}{2} \\ -\frac{z}{2} \end{pmatrix} = \left(\frac{w}{2}\frac{\partial}{\partial z} - \frac{z}{2}\frac{\partial}{\partial w} \right) f(z,w),$$

and similarly $\pi(X_1) = -\frac{iz}{2}\frac{\partial}{\partial z} + \frac{iw}{2}\frac{\partial}{\partial w}$, $\pi(X_2) = -\frac{iw}{2}\frac{\partial}{\partial z} - \frac{iz}{2}\frac{\partial}{\partial w}$. We complexify: $\mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{su}(2) \oplus i\,\mathfrak{su}(2) = \mathfrak{sl}_2(\mathbb{C})$ (see (3.3)). Taking the basis $H = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right), X = \left(\begin{smallmatrix} & 1 \\ 0 & \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} & 0 \\ 1 & \end{smallmatrix}\right)$ for $\mathfrak{sl}_2(\mathbb{C})$, we have $\pi(X) = \pi(-iX_2 - X_3) = -i\pi(X_2) - \pi(X_3)$ and so on, giving

$$\pi(X) = -w\frac{\partial}{\partial z}, \qquad \pi(Y) = -z\frac{\partial}{\partial w}, \qquad \pi(H) = -z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}.$$

Applying this to the natural basis of V_m we get

$$\begin{aligned} \pi(X)\left(z^{m-k}w^k\right) &= (k-m)z^{m-k-1}w^{k+1} \\ \pi(Y)\left(z^{m-k}w^k\right) &= -kz^{m-k+1}w^{k-1} \\ \pi(H)\left(z^{m-k}w^k\right) &= (2k-m)z^{m-k}w^k, \end{aligned} \tag{4.2}$$

which shows in particular that $\pi(Y)^{\mathbb{N}}\pi(X)^{\mathbb{N}}(f)$ span V_m for any $f \neq 0$.

- Remark: the eigenvalues of H (namely $-m, -m+2, \ldots, m$) are called the *weights* of the representation (so V_m has *highest weight* m).

- V_m are exhaustive: If V is an irrep of $SU(2)$, it is fin. dim. by Weyl's unitarity trick, and by complexification V is an irrep of $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$.[7] Let $v \in V$ be an H-eigenvector with $Hv = \alpha v$. Using $[X,Y] = H$, $[H,X] = 2X$, and $[H,Y] = -2Y$ we see that

$$HXv = ([H,X] + XH)\,v = (2X + XH)\,v = (2+\alpha)\,Xv, \tag{4.3}$$

so that Xv has H-eigenvalue $\alpha + 2$ (but possibly $Xv = 0$).[8] Similarly, $HYv = (\alpha - 2)Yv$. As $HY^{\mathbb{N}}v$ have different H-eigenvalues, eventually $Y^{n+1}v = 0$, and we take $v_0 = Y^n v$, getting $Yv_0 = 0$ and $Hv_0 = \lambda v_0$ (for $\lambda = \alpha - 2n$). Set $v_k := X^k v_0$, so that $Hv_k = (\lambda + 2k)\,v_k$, and $Yv_k = -k\,(\lambda + k - 1)\,v_{k-1}$ by induction:

$$\begin{aligned} Yv_{k+1} = YXv_k &= (XY - [X,Y])\,v_k = XYv_k - Hv_k \\ \text{(ind. hyp.)} &= -k\,(\lambda + k - 1)\,v_k - (\lambda + 2k)\,v_k = -(k+1)(k+\lambda)v_k. \end{aligned}$$

We have $v_{m+1} = 0$ for some $v_m \neq 0$ (as they have different H-eigenvalues), hence $0 = Yv_{m+1} = -\,(m+1)\,(m+\lambda)\,v_m$ implies $\lambda = -m$, and taking $b_k = \frac{(-1)^k}{(m)_k} v_k$ we obtain

$$Xb_k = (k-m)b_{k+1}, \qquad Yb_k = -kb_{k-1}, \qquad Hb_k = (2k-m)b_k. \tag{4.4}$$

[7] Be warned however that $SL_2(\mathbb{C})$ and thus $\mathfrak{sl}_2(\mathbb{C})$ have infinite-dimension irreps as well.

[8] An important point: in (4.3) we look at products, but $\mathfrak{sl}_2(\mathbb{C})$ is not closed under multiplication (e.g. $XY \notin \mathfrak{sl}_2(\mathbb{C})$). A wrong solution is to say it is ok because we can work in $\mathfrak{gl}_2(\mathbb{C})$ (it is wrong because the representation V is only of $\mathfrak{sl}_2$). A right solution is to note that in (4.3) we never have XH without v after it, and XHv is well defined by $X(Hv)$. The way to make this formal without v, is to take XH as an element of the *universal enveloping algebra of* $\mathfrak{sl}_2(\mathbb{C})$, which we won't cover.

From (4.2) we see that $z^{m-k}w^k \mapsto b_k$ embeds V_m in V, and as V is irreducible they are isomorphic (as $\mathfrak{sl}_2(\mathbb{C})$-reps, hence as $\mathfrak{su}(2)$-reps, hence as $SU(2)$-reps).

- We could also start from studying $\mathfrak{su}(2)$-reps, taking 4.4 as a definition of an action of $\mathfrak{su}(2)_\mathbb{C}$ on $\mathbb{C}^{m+1}$ with basis $b_0, \ldots, b_m$, and verifying it respects the bracket relations. Then, we could restrict it to a representation of $\mathfrak{su}(2)$ (still on $\mathbb{C}^{m+1}$), which itself comes from a representation of $SU(2)$, since the latter is simply-connected. However, this goes through BCH, and does not give us an explicit global description as we had in $V_m = \left\{ \sum_{k=0}^{m} \alpha_k z^{m-k} w^k \right\}$.
- Since $SO(3) \cong SU(2)/\{\pm I\}$ and $((-I)f)(z,w) = (-1)^m f(z,w)$, the irreps of $SO(3)$ are $\{V_{2m}\}_{m\in\mathbb{N}}$ (every irrep of $SO(3)$ can be pulled back to an irrep of $SU(2)$).

4.4 Roots and weights in $SU(3)$

- $SU(3)$ acts irreducibly on homogeneous polynomials in three variables, but now these are not exhaustive. And in any case, we want to conduct a study that will generalize to all classical groups.
- We now start from $\mathfrak{g} = \mathfrak{su}(3)_\mathbb{C} = \mathfrak{sl}_3(\mathbb{C})$, rather than from $SU(3)$. We define H_1, X_1, Y_1 via the top-left copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_3(\mathbb{C})$, and H_2, X_2, Y_2 via the bottom-right copy. We also define $X_3 = [X_1, X_2] = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ and $Y_3 = [Y_2, Y_1] = \left(\begin{smallmatrix} 0&0&0\\0&0&0\\1&0&0 \end{smallmatrix}\right)$ and together X_*, Y_*, H_* are a basis of $\mathfrak{sl}_3(\mathbb{C})$.
- As H_1, H_2 commute, so do $\pi(H_1), \pi(H_2)$ for any representation (π, V) of $\mathfrak{g}$. For $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$, we define $V_\mu = \{v \in V \mid H_i v = \mu_i v\}$, and if $V_\mu \neq 0$ we say μ is a *weight* for V.
  - Since H_1, H_2 commute, V has at least one weight.
  - The weights are in $\mathbb{Z}^2$, since if $v \in V_\mu$ we can restrict it to a $\mathfrak{sl}_2(\mathbb{C})$-rep via $\langle H_1, X_1, Y_1\rangle$ or $\langle H_2, X_2, Y_2\rangle$.
  - Another (basis independent!) way to think of weights is as linear functionals $\mu\colon \mathfrak{h} \to \mathbb{C}$, where $\mathfrak{h}$ is the diagonal subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Now $v \in V$ is of weight μ if $Hv = \mu(H)v$ for every $H \in \mathfrak{h}$.
- The **non-zero** weights $(\alpha_1, \alpha_2) \in \mathbb{Z}^2 \backslash (0,0)$ of the representation $(\mathrm{ad}, \mathfrak{g})$ are called the *roots* of $\mathfrak{g}$, and the corresponding $v \in \mathfrak{g}_\alpha$ are called *root-vectors*. In turns out that $\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2} \in M_8(\mathbb{C})$ are diagonal w.r.t. X_*, Y_*, H_*. In our case, X_*, Y_* are root-vectors with roots

$$X_1 : (2,-1)\,,\ X_2 : (-1,2)\,,\ X_3 : (1,1)\,,\ Y_1 : (-2,1)\,,\ Y_2 : (1,-2)\,,\ Y_3 : (-1,-1)\,,$$

  but H_1, H_2 are not root-vectors, since they have weights $(0,0)$.
- If $v \in V_\mu$ and $Z \in \mathfrak{g}_\alpha$ (i.e. v is of weight μ, and Z is a root-vector with root α), then $Zv \in V_{\mu+\alpha}$ since

$$H_i Z v = ([H_i, Z] + Z H_i)\, v = \alpha_i Z v + Z \mu_i v = (\alpha_i + \mu_i) Z v.$$

  This also implies that if V is irreducible then $V = \bigoplus_{\mu\in\mathbb{Z}^2} V_\mu$, since the latter is $\mathfrak{g}$-stable (and nonempty).
- For general $\mathfrak{g}$, the *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian $\mathfrak{h} \leq \mathfrak{g}$ such that ad_H is diagonalizable for every $H \in \mathfrak{h}$. This implies that $\{\mathrm{ad}_H \mid H \in \mathfrak{h}\}$ are diagonalizable simultaneously, and we can define weights and roots again, and see they shift weights as before.
- We call $\alpha_1 = (2,-1), \alpha_2 = (-1,2)$ the *positive simple roots*, and observe that every root is either in $\mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\alpha_1, \alpha_2\}$ (the *positive* roots), or in $\mathrm{Span}_{\mathbb{Z}_{\leq 0}}\{\alpha_1, \alpha_2\}$ (the *negative* roots). We order the weights by $\mu \preceq \mu'$ when $\mu' - \mu \in \mathrm{Span}_{\mathbb{Q}_{\geq 0}}\{\alpha_1, \alpha_2\}$. Even though this is only a partial order, we will see later that there is a unique highest weight (as we had m in $\mathfrak{sl}_2(\mathbb{C})$).
- The Highest Weight Theorem: Every (fin. dim.) irrep has a unique highest weight $\mu \in \mathbb{N}^2_{\geq 0}$, and every $\mu \in \mathbb{N}^2_{\geq 0}$ is the highest weight of a unique irrep.

- Lemma: If $Z_1, \dots, Z_n$ is a basis for $\mathfrak{g}$, then $Z_{i_1} \cdot \dots \cdot Z_{i_r} v$ can be expressed as a linear combinations of terms of the form $Z_1^{k_1} \cdot \dots \cdot Z_n^{k_n} v$, with $\sum k_i \leq r$. Idea: using induction on r, apply $Z_i Z_j v = Z_j Z_i v + [Z_i, Z_j] v$, and $[Z_i, Z_j] = \sum c_{ijk} Z_k$ to move the Z_i to the desired position.
- Lemma: If v_0 has weight μ and $X_* v_0 = 0$, then μ is the unique highest weight for $\langle v_0 \rangle$, and $\langle v_0 \rangle_\mu = \mathbb{C} v_0$. Proof: Since $X_* v_0 = 0$ and $H_i v_0 \in \mathbb{C} v_0$, taking $Z_1, \dots, Z_8 = Y_*, H_*, X_*$ we have

$$\langle v_0 \rangle = \mathrm{Span}_{\mathbb{C}} \left\{ \left(\prod_{j=1}^{n} Z_{i_j} \right) v_0 \,\middle|\, \substack{n \in \mathbb{N} \\ i_j \in [8]} \right\} = \mathrm{Span}_{\mathbb{C}} \left\{ Y_1^{k_1} Y_2^{k_2} Y_3^{k_3} v_0 \,\middle|\, k_* \in \mathbb{N} \right\},$$

  and $Y_1^{k_1} Y_2^{k_2} Y_3^{k_3} v_0$ (with $k_* \neq (0,0,0)$) all have weight strictly lower than μ.
- If V is fin. dim. irrep it has some $\preceq$-highest weight μ, and for $0 \neq v_0 \in V_\mu$ this implies $X_* v_0 = 0$ (X_* being the positive roots), so by the lemma μ is the unique highest weight for $\langle v_0 \rangle = V$. In addition, $\mu \in \mathbb{N}_{\geq 0}^2$ since $X_i v_0 = 0$ implies that $H_i v_0 \in \mathbb{N}_{\geq 0} v_0$, by our analysis of $\mathfrak{sl}_2(\mathbb{C})$.
- Given $\mu \in \mathbb{N}_{\geq 0}^2$, let $S = \mathbb{C}^3$ with $\mathfrak{g}$ acting by multiplication (which is induced from $SU(3)$ acting by multiplication), and S^* the contragradient, which is $Z(v) = -Z^T v$. Then S, S^* are irreducible, and $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \in S$ and $e_3' = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \in S^*$ have highest weights $(1,0)$ and $(0,1)$ respectively. In particular $X_* e_1, X_* e_3' = 0$, from which follows that $v_0 = e_1^{\otimes \mu_1} \otimes {e_3'}^{\otimes \mu_2} \in S^{\otimes \mu_1} \otimes S^{* \otimes \mu_2}$ satisfies $X_* v_0 = 0$ and has weight μ (for this recall that $Z(v \otimes w) = Z(v) \otimes w + v \otimes Z(w)$. By the lemma, μ is the unique highest weight for $V = \langle v_0 \rangle$ and $\mathbb{C} v_0 = V_\mu$. To show V is irreducible, we note that S and S^* (with the standard inner product on $\mathbb{C}^3$) are unitary representations of $\mathfrak{su}(3)$ (though not of $\mathfrak{g} = \mathfrak{su}(3)_{\mathbb{C}}$!), hence so is $S^{\otimes \mu_1} \otimes S^{* \otimes \mu_2}$, hence so is V. Thus, V decomposes into irreducibles: $V = \bigoplus V_i$, hence $\mathbb{C} v_0 = V_\mu = \bigoplus (V_i)_\mu$, showing $v_0 \in (V_i)_\mu$ for a unique i and in particular thus $V = \langle v_0 \rangle \subseteq V_i$.
- For irreps V, W with highest weight μ, we want to show $V \cong W$. By general representation theory nonsense, direct sums and subrepresentations preserve complete reducibility. For $\mathbb{C} v = V_\mu$ and $\mathbb{C} w = W_\mu$, we obtain that $U = \langle (v, w) \rangle \leq V \oplus W$ decomposes into irreps: $U = \bigoplus U_i$. But again $X_* (v, w) = 0$ and $(v, w) \in U_\lambda$ show that $\mathbb{C}(v, w) = U_\lambda = (U_i)_\lambda$ for a unique i, hence $U = \langle (v, w) \rangle = U_i$ is irreducible. Since the projections $V \oplus W \to V, W$ restrict to nonzero maps $U \to V, W$, by Schur Lemma the three are isomorphic.

Lie Groups - Exercise 1

1. Compute e^X for a X a Jordan block (i.e. $\begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & & a \end{pmatrix}$).
2. A matrix A is called *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$ and *unipotent* if $A - I$ is nilpotent. Note $\log A$ makes sense for all unipotent A (it is a polynomial in A). Show that exp and log give a complete bijection between the nilpotent and the unipotent matrices (over $\mathbb{R}$ or $\mathbb{C}$).
3. Show that $\exp\colon M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is onto.
4. $\exp\colon M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is not onto since $\det\left(e^X\right) = e^{\mathrm{tr}(X)} > 0$, but it is not even onto $GL_n^+(\mathbb{R})$ (matrices with positive determinant): Show that $A = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right) \notin \exp(M_2(\mathbb{R}))$. Hint: show that if $e^X = A$ then X is non-diagonalizable over $\mathbb{C}$, and $\mathrm{tr}(X) = 0$.
5. Every $X \in M_2(\mathbb{R})$ (over $\mathbb{R}$ or $\mathbb{C}$) can be written as $\frac{\mathrm{tr}(X)}{2} I + Y$ with $\mathrm{tr}(Y) = 0$. Show that $Y^2 = -\det(Y)\, I$, and that
$$e^X = e^{\frac{\mathrm{trace}(X)}{2}} \left(\cos\left(\sqrt{\det Y}\right) I + \frac{\sin\left(\sqrt{\det Y}\right)}{\sqrt{\det Y}} Y \right).$$
6. Optional: use the previous exercise to show that the image of $\exp\colon M_2(\mathbb{R}) \to GL_2(\mathbb{R})$ is
$$\left\{ A \in GL_2^+(\mathbb{R}) \,\middle|\, \mathrm{tr}(A) > -2\sqrt{\det A} \right\} \cup \{ cI \mid c < 0 \}$$
(I think, could have got this one wrong).
7. Show that the following are equivalent:
    (a) X, Y commute
    (b) e^{sX}, e^{tY} commute for all $s, t \in \mathbb{R}$
    (c) $e^{sX+tY} = e^{sX} e^{tY}$ for all $s, t \in \mathbb{R}$

Lie Groups - Exercise 2

May 4, 2022

1. Prove that $Z(\mathbb{H}) = \mathbb{R}$.

2. Prove that $\mathbb{H}^1 \cong SU(2)$.

3. (a) Show that the action of $\mathbb{H}^1 \times \mathbb{H}^1$ on $\mathbb{H}$ by $(\alpha, \beta) \cdot \gamma = \alpha\gamma\beta^{-1}$ gives a homomorphism $\Phi : \mathbb{H}^1 \times \mathbb{H}^1 \to SO(4)$.
   (b) Show that $\ker \Phi = \{(1, 1), (-1, -1)\}$.
   (c) Show that Φ is onto. Hint: identifying $\mathbb{H} \cong \mathbb{R}^4$ via the basis $1, i, j, k$, let $\alpha, \beta, \gamma, \delta \in \mathbb{H}$ be such that $(\alpha|\beta|\gamma|\delta) \in SO(4)$. Show that using $\mathbb{H}^1 \times \mathbb{H}^1$ one can take $(\alpha, \beta, \gamma, \delta)$ to $(1, i, j, k)$. It is helpful to remember that you already know well the action of the diagonal $\left\{(\alpha, \alpha) \in \mathbb{H}^1 \times \mathbb{H}^1\right\}$ on $\mathbb{P}$...

4. Let $\mathbb{F}$ be a field with $\operatorname{char} \mathbb{F} \neq 2$, and define the $\mathbb{F}$-quaternions $\mathbb{H}_{\mathbb{F}}$ as the algebra over $\mathbb{F}$ with basis $1, i, j, k$ and product defined by $i^2 = j^2 = k^2 = ijk = -1$, associativity and distributivity (and 1 being the multiplicative identity). Prove that if -1 is a square in $\mathbb{F}$ then $\mathbb{H}_{\mathbb{F}} \cong M_2(\mathbb{F})$.

5. Let $\mathbb{E}/\mathbb{F}$ be a quadratic field extension with $\operatorname{char} \mathbb{F} \neq 2$ and $\operatorname{Gal}(\mathbb{E}/\mathbb{F}) = \{id, \sigma\}$. We know that $\mathbb{E} = \mathbb{F}\left[\sqrt{\delta}\right]$ for some $\delta \in \mathbb{F}$, so that $\sigma\left(\sqrt{\delta}\right) = -\sqrt{\delta}$. Now, assume that the norm map

$$N\colon \mathbb{E}^\times \to \mathbb{F}^\times, \quad N\left(a + b\sqrt{\delta}\right) = \left(a + b\sqrt{\delta}\right) \cdot \sigma\left(a + b\sqrt{\delta}\right) = a^2 - \delta b^2$$

   is onto.

   (a) Show that this is always the case for finite fields.
   (b) Show that the following unitary groups are isomorphic:

$$U_n(\mathbb{E}/\mathbb{F}) = \{A \in GL_n(\mathbb{E}) \mid A^*A = I\}$$
$$U_n(\mathbb{E}/\mathbb{F}, J) = \{A \in GL_n(\mathbb{E}) \mid A^*JA = J\},$$

   where $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$, and $(A^*)_{ij} = \sigma(A_{ji})$. Hint: find hyperbolic pairs in the Hermitian space $\mathbb{E}^n$ (with the standard product $\langle v, w\rangle = v^*w$).
   (c) Optional: Show that $U_n(\mathbb{E}/\mathbb{F}, B) \cong U_n(\mathbb{E}/\mathbb{F}, J)$ for any hermitian matrix $B \in GL_n(\mathbb{E})$.[1]

[1]Since every finite field $\mathbb{F}$ has a unique quadratic extension, this implies there is a unique unitary group over $\mathbb{F}$ in each dimension!

Lie Groups - Exercise 3

June 22, 2022

1. Show that $\mathfrak{sl}_n(\mathbb{F}) = \{X \in M_n(\mathbb{F}) \,|\, \mathrm{trace}(X) = 0\}$.
2. Let G be the group of all upper-triangular matrices in $GL_n(\mathbb{F})$. Show that $\mathfrak{g} = Lie(G)$ is the algebra of all upper-triangular matrices in $M_n(\mathbb{F})$.
3. What is $\mathfrak{g}$ when G is the group of all unipotent upper-triangular matrices in $GL_n(\mathbb{F})$?
4. The *derivations* of an $\mathbb{F}$-algebra A are

$$\mathrm{Der}_{\mathbb{F}}(A) = \{f \in \mathrm{End}_{\mathbb{F}}(A) \,|\, f(ab) = f(a)b + af(b)\}.$$

   (a) Show that if $f \in \mathrm{Der}_{\mathbb{F}}(A)$ then $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} f^{n-k}(a) f^k(b)$.
   (b) Show that $\mathrm{Der}_{\mathbb{F}}(A)$ is a Lie algebra, w.r.t. $[f,g] = f \circ g - g \circ f$.
   (c) Show that if $\mathfrak{g}$ is a finite-dimensional Lie algebra[1] then $Lie(\mathrm{Aut}_{LieAl}(\mathfrak{g})) = \mathrm{Der}_{\mathbb{R}}(\mathfrak{g})$ (we consider $\mathrm{Aut}_{LieAl}(\mathfrak{g})$ as a Lie group by $\mathrm{Aut}_{LieAl}(\mathfrak{g}) \leq GL(\mathfrak{g}) \cong GL_{\dim \mathfrak{g}}(\mathbb{R})$). Conclude (or show directly) that $\mathrm{ad} \in \mathrm{Hom}_{LieAl}(\mathfrak{g}, \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}))$.
5. Let $H \subseteq G \subseteq GL_n(\mathbb{F})$ be matrix Lie groups. Show that:
   (a) If $H \trianglelefteq G$ (normal) then $\mathfrak{h} \trianglelefteq \mathfrak{g}$ (ideal).
   (b) If G and H are connected and $\mathfrak{h} \trianglelefteq \mathfrak{g}$ then $H \trianglelefteq G$.
6. Optional: If G is abelian and connected then $e^{\mathfrak{g}} = G$.

[1] if you want you can assume it is the Lie algebra of a Lie group G, but I don't think it helps

Lie Groups - Final Exercise

October 6, 2022

1. Recall that $Sp_{2n}(\mathbb{F}) = \{A \mid A^T \Omega A = \Omega\}$, where $\Omega = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$ (and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Show that $\mathfrak{sp}_{2n}(\mathbb{F}) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \middle| \begin{array}{l} A, B, C \in \mathfrak{gl}_n(\mathbb{F}) \\ B = B^T, C = C^T \end{array} \right\}$, and compute its dimension.

2. Recall the quaternionic-unitary group $Sp(n) = \{A \in GL_n(\mathbb{H}) \mid A^* A = I\}$ (where $(A^*)_{ij} = \overline{A_{ji}}$ with the quaternionic conjugate). Show that $\mathfrak{sp}(n)_{\mathbb{C}}$ (the complexification of its Lie algebra) is isomorphic to $\mathfrak{sp}_{2n}(\mathbb{C})$. [1]

3. From now on we focus on $\mathfrak{sp}_4(\mathbb{C})$, and replace Ω by $\Omega = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$ (they are congruent, and you'll soon see why this one serves us better). Work out what is $\mathfrak{sp}_4(\mathbb{C})$ using this Ω.

4. Show that $\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$ form two copies of $\mathfrak{sl}_2(\mathbb{C})$, and use them to define $H_1, X_1, Y_1, H_2, X_2, Y_2$ as we did in $\mathfrak{sl}_3(\mathbb{C})$. We define weights by H_1, H_2 (namely, (μ_1, μ_2) is a weight for V if there exist $0 \neq v \in V$ with $H_i v = \mu_i v$). Deduce that the weights of a fin. dim. representation of $\mathfrak{sp}_4(\mathbb{C})$ are in $\mathbb{Z}^2$.

5. Extend $H_1, X_1, Y_1, H_2, X_2, Y_2$ to a basis of $\mathfrak{sp}_4(\mathbb{C})$, and verify that in this basis ad_{H_1} and ad_{H_2} are diagonal (if they are not you probably tried to be unnecessarily creative in choosing your basis - remedy this).

6. Find the roots of $\mathfrak{sp}_4(\mathbb{C})$ and draw them in $\mathbb{R}^2$. Choose two roots α_1, α_2 such that every root is in either $\mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\alpha_1, \alpha_2\}$ or in $\mathrm{Span}_{\mathbb{Z}_{\leq 0}}\{\alpha_1, \alpha_2\}$ (we call α_1, α_2 the "positive simple roots"). Try to make it so that X_1, X_2 are root-vectors with positive roots (though this is not crucial).

7. Order the weights by $\mu \preceq \mu'$ when $\mu' - \mu \in \mathrm{Span}_{\mathbb{Q}_{\geq 0}}\{\alpha_1, \alpha_2\}$, and show that every fin. dim. irreducible representation of $\mathfrak{sp}_4(\mathbb{C})$ has a unique vector of highest weight (up to scalars), and this weight is in $\mathbb{N}^2_{\geq 0}$ (or in another quadrant of the plane, if you didn't bother to make the roots corresponding to X_1, X_2 positive).

8. Find all the weights of the standard representation $S = \mathbb{C}^4$ (with $\mathfrak{sp}_4(\mathbb{C})$ acting by multiplication). Which is the highest?

9. Show that the highest weight in an irreducible representation is actually in $\{(\mu_1, \mu_2) \mid \mu_1 \geq \mu_2 \geq 0\}$ (or in another eighth-plane, which one depends again on your choice of positive roots).
   Hint: one of your positive simple root-vectors was one of $\{X_1, X_2, Y_1, Y_2\}$; Find another copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sp}_4(\mathbb{C})$ whose "X" vector is your **other** positive simple root-vector.

For the next two questions you can assume total reducibility (namely, that every representation of $\mathfrak{sp}_4(\mathbb{C})$ decomposes as a some of irreducible ones). You are welcome to try to prove it, or you can simply decide we only care about representations of $\mathfrak{sp}(n)$, where the standard representation is unitary w.r.t. the standard inner product (this is what we did in class, where we studied representation of $\mathfrak{su}(3)$, whose complexifiction was $\mathfrak{sl}_3(\mathbb{C})$).

10. Find an irreducible representation V of $\mathfrak{sp}_4(\mathbb{C})$ with highest weight $(1,1)$. Hint: look at $S \otimes S$. [2]

11. Show that for any $\mu_1 \geq \mu_2 \geq 0$, $S^{\otimes(\mu_1 - \mu_2)} \otimes V^{\otimes \mu_2}$ has an irreducible subrepresentation of highest weight (μ_1, μ_2).

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[1] The triplet $Sp(n), Sp_n(\mathbb{R}), Sp_n(\mathbb{C})$ is analogue to $SU(n), SL_n(\mathbb{R}), SL_n(\mathbb{C})$ - the first group is compact, the second is not, and both have a Lie algebra whose complexification is isomorphic to that of the third (which is a complex Lie group). In general, every complex semisimple Lie algebra has a compact and a non-compact "real form". We saw those of $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$, can you work out such a triplet for $\mathfrak{so}_n(\mathbb{C})$?

[2] If you're not used to tensors: taking any basis $e_1, \ldots, e_4$ for S, we can define $S \otimes S$ as the vector space with basis $\{e_i \otimes e_j\}_{i,j \in [4]}$, and each $Z \in \mathfrak{sp}_4(\mathbb{C})$ acts on the basis elements by $Z(e_i \otimes e_j) = (Ze_i) \otimes e_j + e_i \otimes (Ze_j)$.