JÓNSSON AND MAGIDOR FILTERS

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ABSTRACT. We study the Jónsson and Magidor partition properties and their strengthened filter forms, primarily in the context of the Axiom of Determinacy. By applying methods of Kleinberg concerning strong partition cardinals κ and their associated cardinal sequence $(\kappa_n : n \in \omega)$, we show that the limit cardinal κ_ω is a Magidor cardinal if $\kappa > \aleph_1$. We also force over a model of AD and $V = L(\mathbb{R})$ to get the existence of a singular cardinal λ carrying a Magidor filter, addressing a question from [GHS17].

0. Introduction

In this work, we consider strong partition properties of Jónsson and Magidor, and the existence of associated filters, both under the axiom of choice and in the choiceless context, under the axiom of determinacy.

An infinite cardinal λ is Jónsson iff $\lambda \to [\lambda]^{<\omega}_\lambda$, that is for every coloring $c : [\lambda]^{<\omega} \to \lambda$ one can find $A \in [\lambda]^\lambda$ such that $c''[A]^{<\omega} \neq \lambda$, in which case we shall say that A is c-monochromatic. An infinite cardinal λ is ω-Jónsson iff $\lambda \to [\lambda]^\omega_\lambda$. Erdős and Hajnal proved in [EH66] that there are no ω-Jónsson cardinals in ZFC. In the absence of the axiom of choice, for example under AD, the stronger notion of ω-Jónsson cardinals may be consistent.

An intermediate concept between the notions of Jónsson and ω-Jónsson cardinals is the one of Magidor cardinal (Magidority), which was developed in [GH18]. A cardinal λ is Magidor iff $\lambda \to [\lambda]^{\aleph_0\text{-bd}}_\lambda$, where $\aleph_0$-bd refers to all bounded subsets of λ of cardinality $\aleph_0$. Many results concerning the possible strength and obstructions to Magidor cardinals appear when considering the ordertype restricted notion of ω-Magidority. We say that λ is ω-Magidor iff $\lambda \to [\lambda]^{\omega\text{-bd}}_\lambda$ where ω-bd refers to all bounded subsets of λ of ordertype ω. Magidor observed that if λ satisfies the large cardinal property I1 then λ is Magidor, see [Kan03, Question 24.1], hence Magidority is consistent with ZFC.

It is known that under AD, many cardinals below Θ satisfy many partition properties. Considering Jónssonicity, Kleinberg proved that the first uncountable cardinals up to and including $\aleph_\omega$ are Jónsson ([Kan03]). Jackson extended the result to all cardinals $\kappa < \aleph_{\omega_1}$, Steel (see [Ste10]) proved that every regular uncountable cardinal below Θ is measurable and thus Jónsson. Jackson, Ketchersid, Schlutzenberg, and Woodin, have independently proved that the results extend to all uncountable cardinals below Θ ([JKSW14]). In this work we incorporate Magidority into the analysis of large cardinals under determinacy. Strong results in ZF concerning ω-Jónsson cardinals appear in [AS04].

We consider here the question of whether AD implies that certain singular cardinals below Θ are Magidor. We show that under AD, $\aleph_\omega$ is ω-Magidor and that for every cardinal κ, satisfying $\aleph_1 < \kappa < \Theta$ and $\kappa \to (\kappa)^\kappa_2$, its associated singular cardinal κ_ω is Magidor. In particular $\aleph_{\omega+\omega}$ which is associated to $\kappa = \aleph_{\omega+1}$ is Magidor. Our results in the AD context build on the work of Kleinberg ([Kan03]) on cardinals $\kappa < \Theta$ satisfying the strong partition property, and their associated ultrapowers sequence $(\kappa_n : n < \omega)$.

A natural strengthening of square bracket relations is given by the following filter version of these notions.

Definition 0.1. (Square brackets filters)
Let $\mathscr{F}$ be a uniform filter over λ which contains all the end-segments of λ.

  ($\aleph$) We shall say that $\mathscr{F}$ is Jónsson iff for every $c : [\lambda]^{<\omega} \to \lambda$ one can find $A \in \mathscr{F}$ such that $c''[A]^{<\omega} \neq \lambda$.

(⊐) We shall say that $\mathscr{F}$ is Magidor iff for every $c : [\lambda]^{\aleph_0\text{-bd}} \to \lambda$ one can find $A \in \mathscr{F}$ such that $c''[A]^{\aleph_0\text{-bd}} \neq \lambda$.

The filter version of square brackets relations is usually stronger than the classical one. In the particular case of Magidority, it is known that Magidor filters do not exist in ZFC despite the fact that there might be Magidor cardinals. However, we shall force the existence of Magidor filters over a model of AD.

The paper is organized as follows:

In **Section 1**, we consider the notion of Jónsson filters $\mathscr{F}$ and their Jónsson degrees $\alpha_J(\mathscr{F})$ (see Lemma 1.4) in the ZFC context. We show that a singular limit of cardinals carrying Jónsson filters of bounded degree must also carry a Jónsson filter. This generalizes a well-known theorem of Prikry that a singular limit of measurable cardinals carries a Jónsson filter.

In **Section 2** we extend the analysis of Kleinberg sequences $\langle \kappa_n : n < \omega \rangle$ with respect to Magidority, and show that if $\kappa > \aleph_1$ is a strong partition cardinal, then the associated limit of its Kleinberg sequence κ_ω is Magidor, and that $\aleph_\omega$ is ω-Magidor. We also show that each κ_n, $2 < n < \omega$ in the Kleinberg sequence is not Magidor.

Finally, in **Section 3** we consider the Magidor filter property at Prikry generic extensions of AD models. This follows a long sequence of combinatorial and arithmetical results obtained by forcing with Prikry-like posets over models of AD: Apter [Apt81], Spector [Spe80], Henle [Hen83], and Apter-Henle-Jackson [AHJ00]. We show that a singular cardinal carrying a Magidor filter may exist in such extensions, providing a partial answer to [GHS17, Question 2.5] concerning the existence of Magidor filters in models of AD.

1. Sufficient conditions for Jónsson filters over singular cardinals

In this section we work in ZFC and we follow in the footsteps of Prikry ([Pri70]) in order to show that a singular cardinal which is a limit of cardinals carrying Jónsson filters (with some parameter) must also carry a Jónsson filter. In the next section we shall prove a similar theorem under AD.

Prikry used normal ultrafilters over the measurable cardinals along the way, but the proof needs only the partition theorem of Rowbottom for such ultrafilters. In particular, one can replace the normal ultrafilters by Rowbottom or Jónsson filters. This shift is quite meaningful since the existence of a normal ultrafilter is expensive in terms of consistency strength. The proof below follows the ideas of Prikry [Pri70] and the presentation of Kanamori [Kan03].

We phrase the theorem (and the proof) in the more general context of θ-Rowbottom cardinals. Rowbottom cardinals possess two equivalent definitions. The first one is model theoretic. We say that κ is Rowbottom iff for every structure $\mathfrak{A} = \langle A, Q, \ldots \rangle$ for a countable language such that $|A| = \kappa$, $|Q| < \kappa$, there is an elementary substructure $\mathfrak{B} = \langle B, Q \cap B, \ldots \rangle$ so that $|B| = \kappa$ and $|Q \cap B| \leq \aleph_0$. One may add another parameter, and define κ as a ν-Rowbottom cardinal exactly in the same way but with the demand $|B| = \kappa$ and $|Q \cap B| < \nu$.

The second natural definition of Rowbottom cardinals is combinatorial, and based on the square brackets notation. We say that κ is Rowbottom iff $\kappa \to [\kappa]^{<\omega}_{\gamma, <\omega_1}$ for all $\gamma < \kappa$. This means that for every $\gamma < \kappa$ and a coloring $f : [\kappa]^{<\omega} \to \gamma$, there exists a set $A \in [\kappa]^\kappa$ for which $f''[A]^{<\omega}$ is countable. Again, we can generalize the definition into ν-Rowbottomness upon replacing ω_1 by ν, i.e. $\kappa \to [\kappa]^{<\omega}_{\gamma, <\nu}$. We note that if $\kappa > \mathrm{cf}(\kappa)$ is such a cardinal then $\mathrm{cf}(\kappa) < \nu$ ([Kan03]). In particular, if κ is a singular Rowbottom cardinal then κ has countable cofinality.

Theorem 1.1. *(Rowbottom filters over singular cardinals)*
Assume that:

($\aleph$) $\mathrm{cf}(\kappa) = \nu \leq \theta < \kappa$.
($\beth$) $\langle \kappa_\alpha : \alpha < \nu \rangle$ *is a continuous increasing sequence of cardinals which tends to κ.*
($\gimel$) $\mathscr{F}_{\alpha+1}$ *is a $\kappa_{\alpha+1}$-complete θ^+-Rowbottom filter over $\kappa_{\alpha+1}$ for every $\alpha < \nu$.*
($\daleth$) $\gamma < \kappa \Rightarrow \gamma^\theta < \kappa$.

Then there exists a θ^+-Rowbottom filter over κ.

Proof.
We define a uniform filter D over κ as follows:

$$x \in D \Leftrightarrow \exists \beta \forall \alpha \in [\beta, \nu), x \cap \kappa_{\alpha+1} \in \mathscr{F}_{\alpha+1}.$$

Our objective is to show that D is a θ^+-Rowbottom filter. The degree of completeness of D is low, and in the typical case of Rowbottom cardinals (i.e. $\mathrm{cf}(\kappa) = \omega$) it is not even $\aleph_1$-complete. This fact is problematic, since the completeness of the filter is used in most proofs of constructing monochromatic sets. The solution of this problem is to choose our large sets in such a way that the ordinal β in the above definition of D is constant.

We begin by the following definition of type. For every $s \in [\kappa]^{<\omega}$ let type(s) be the set $\{(\alpha, \ell) \in \nu \times \omega : 0 < \ell = |s \cap (\kappa_{\alpha+1} - \kappa_\alpha)|\}$. Assume $\gamma < \kappa$ and $f : [\kappa]^{<\omega} \to \gamma$ is any function. Let T be a type (i.e., a set of ordered pairs from $\nu \times \omega$), let H be a subset of κ and let $W_H^f(T)$ be the set $\{\zeta < \gamma : \exists s \in [H]^{<\omega}, T = \text{type}(s), f(s) = \zeta\}$. We shall try to prove the following:

> For every $\gamma < \kappa$ and every $f : [\kappa]^{<\omega} \to \gamma$ there exists $H \in D$
> such that $|W_H^f(T)| \leq \theta$ for every type T.

If we succeed then the proof of the theorem will be accomplished. Indeed, the number of types is $|\nu \times \omega|$, and for every type there are at most θ values for f on H, so $|f''[H]^{<\omega}| \leq \nu \cdot \theta = \theta \cdot \theta < \kappa$. Since $H \in D$ we conclude that D is a θ^+-Rowbottom filter.

For proving the above claim, we decompose it into $\aleph_0$ local assertions, which we shall prove by induction. Let $(\star)_n$ be the assertion that for every $\gamma < \kappa$ and every $f : [\kappa]^{<\omega} \to \gamma$ there is a subset $H_n \subseteq \kappa$ such that:

($\aleph$)$_n$ $H_n \cap \kappa_{\alpha+1} \in \mathscr{F}_{\alpha+1}$ whenever $\kappa_{\alpha+1} > \gamma$.
($\beth$)$_n$ If T is any type such that $|T| = n$ then $|W_{H_n}^f(T)| \leq \theta$.

If we prove $(\star)_n$ for every $n \in \omega$ then we are done. We will choose, for every $n \in \omega$, a set $H_n \subseteq \kappa$ which satisfies $(\star)_n$. Then we will set $H = \bigcap\{H_n : n \in \omega\}$. Although D is not $\aleph_1$-complete in general, here we have the same end-segment for every H_n (it is determined by the fixed number of colors, γ). The completeness of each $\mathscr{F}_{\alpha+1}$ and item $(\aleph)_n$ for every $n \in \omega$ make sure that $H \in D$. From $(\beth)_n$ for every $n \in \omega$ it follows that $|W_H^f(T)| \leq \theta$ for every type T, since there must be some n which is the type of T.

The proof of $(\star)_n$ is by induction on $n \in \omega$. If $n = 0$ then $T = \emptyset$ and hence only $s = \emptyset$ is of type T. We can choose $H_0 = \kappa \in D$ and the requirements hold trivially. So we focus on $(\star)_{n+1}$, assuming that $(\star)_n$ has been shown.

Recall that $f : [\kappa]^{<\omega} \to \gamma$ for some $\gamma < \kappa$. For every $\alpha < \nu$ such that $\gamma < \kappa_{\alpha+1}$ and every $t \in [\kappa_\alpha]^{<\omega}$ we define a function $f_t : [\kappa_{\alpha+1} - \kappa_\alpha]^{<\omega} \to \gamma$ by $f_t(u) = f(t \cup u)$. Assumption ($\beth$) in the statement of the theorem provides us with $x_t^\alpha \in \mathscr{F}_{\alpha+1}$ such that $|f_t''[x_t^\alpha]^{<\omega}| \leq \theta$. We let $x_\alpha = \bigcap\{x_t^\alpha : t \in [\kappa_\alpha]^{<\omega}\} - \kappa_\alpha$. Notice that $x_\alpha \in \mathscr{F}_{\alpha+1}$ for every pertinent $\alpha < \nu$.

By ($\daleth$), we may assume that $\gamma^\theta = \gamma$. Fix a bijection $h : [\gamma]^{\leq\theta} \to \gamma$. For every $(\alpha, \ell) \in \nu \times \omega$ we define $f_{\alpha\ell} : [\kappa_\alpha]^{<\omega} \to [\gamma]^{\leq\theta}$ by

$$f_{\alpha\ell}(t) = f_t''[x_\alpha]^{<\omega} := \{\eta < \gamma \mid \exists u \in [x_\alpha]^{<\omega}, f_t(u) = \eta\}.$$

We convert $f_{\alpha\ell}$ which is a set-mapping into a function $g_{\alpha\ell} : [\kappa_\alpha]^{<\omega} \to \gamma$ by letting $g_{\alpha\ell}(t) = h(f_{\alpha\ell}(t))$.

For every $(\alpha, \ell) \in \nu \times \omega$ we use the induction hypothesis $(\star)_n$ with respect to $g_{\alpha\ell}$ in order to elicit $H_{\alpha\ell} \subseteq \kappa$ such that:

(a) $H_{\alpha\ell} \cap \kappa_{\alpha+1} \in \mathscr{F}_{\alpha+1}$ whenever $\gamma < \kappa_{\alpha+1}$.
(b) If T' is a type of size n then $|W_{H_{\alpha\ell}}^{g_{\alpha\ell}}(T')| \le \theta$.

Finally, we can define the following set:

$$H_{n+1} = \bigcup\{x_\alpha : \gamma < \kappa_{\alpha+1}\} \cap \bigcap\{H_{\alpha\ell} : (\alpha, \ell) \in \nu \times \omega\}.$$

First we observe that $H_{n+1} \in D$. Indeed, $H_{n+1} \cap \kappa_{\alpha+1} \in \mathscr{F}_{\alpha+1}$ whenever $\gamma < \kappa_{\alpha+1}$ since $x_\alpha \in \mathscr{F}_{\alpha+1}$ and every $H_{\beta\ell}$ belongs to $\mathscr{F}_{\alpha+1}$. Consequently, $H_{n+1} \in D$ as required in the first part of $(\star)_{n+1}$.

Second, assume that T is any type of size $n+1$ and let α be the maximal ordinal for which $(\alpha, \ell) \in T$. If $\mathrm{type}(s) = T$ then $s = t \cup u$ for t, u such that $|\mathrm{type}(t)| = n$ and $\mathrm{type}(u) = \{(\alpha, \ell)\}$. Let T' be $\mathrm{type}(t)$. By the induction hypothesis applied to T' we have $|W_{H_{\alpha\ell}}^{g_{\alpha\ell}}(T')| \le \theta$.

Now $f(s) = f(t \cup u) = f_t(u) \in f_{\alpha\ell}(t)$. Recall that $h(f_{\alpha\ell}(t)) = g_{\alpha\ell}(t) \in W_{H_{\alpha\ell}}^{g_{\alpha\ell}}(T')$. Since $|W_{H_{\alpha\ell}}^{g_{\alpha\ell}}(T')| \le \theta$ and $|f_{\alpha\ell}(t)| \le \theta$ we infer that $f(s)$ assumes at most $\theta \cdot \theta$ possibilities and hence $|W_{H_{n+1}}^f(T)| \le \theta$ as desired.

$$\square_{1.1}$$

By [Try81], if $\kappa = \mathrm{cf}(\kappa) > 2^\omega$ is a Rowbottom cardinal then κ is strongly inaccessible, and hence item ($\daleth$) in the above theorem is satisfied for these cardinals. We may conclude:

Corollary 1.2. *Assume $\kappa > 2^\omega$ is a singular cardinal of countable cofinality, a limit of regular cardinals each of which carries a Rowbottom filter. Then there is a Rowbottom filter over κ as well.*

$$\square_{1.2}$$

Remark 1.3. If $\mathscr{U}$ is a normal ultrafilter over κ then, in $L[\mathscr{U}]$, every Jónsson cardinal is regular, as proved in [Kun70]. It follows from this fact and the above theorem that the number of cardinals which carry a Rowbottom filter in $L[\mathscr{U}]$ is finite. However, this is subsumed by the known fact that κ (being measurable) is the only cardinal which carries a Rowbottom or Jónsson filter in $L[\mathscr{U}]$.

$$\square_{1.3}$$

We turn to Jónsson filters. We would like to prove a similar theorem about Jónsson filters at singular cardinals. We need a parallel to the parameter θ of the Rowbottomness degree of the filters. This parameter is given by the following version of an argument by Kleinberg (see [Kan03]):

Lemma 1.4. *Assume F is a Jónsson filter over κ.*
Then there exists a cardinal $\alpha < \kappa$ such that for every $f : [\kappa]^{<\omega} \to \alpha$ there is an element $x \in F$ for which $|f''[x]^{<\omega}| < \alpha$.

Proof.
We begin by showing that there exists an ordinal $\alpha < \kappa$ such that for every $f : [\kappa]^{<\omega} \to \alpha$ there is an element $x \in F$ for which $f''[x]^{<\omega} \neq \alpha$. Towards contradiction assume that for every $\alpha < \kappa$ there is a function $f_\alpha : [\kappa]^{<\omega} \to \alpha$ such that $f_\alpha''[x]^{<\omega} = \alpha$ for every $x \in F$. Define a function $f : [\kappa]^{<\omega} \to \kappa$ by $f(\emptyset) = 0$ and if $t \neq \emptyset, \alpha = \min(t)$ then $f(t) = f_\alpha(t - \{\alpha\})$.

Choose any element $x \in F$ and any color $\gamma < \kappa$. Pick an ordinal $\gamma < \alpha \in x$ and let $y = x - (\alpha + 1)$. By the assumptions on F we know that $y \in F$ and hence there is $s \in [y]^{<\omega}$ such that $f_\alpha(s) = \gamma$. Let $t = s \cup \{\alpha\}$, so $\alpha = \min(t)$. It follows that $f(t) = f_\alpha(t - \{\alpha\}) = f_\alpha(s) = \gamma$. However, $t = s \cup \{\alpha\} \subseteq x$ and hence $f''[x]^{<\omega} = \kappa$ for every $x \in F$, a contradiction.

Now let $\alpha < \kappa$ be the first ordinal with the above property. It follows that α must be a cardinal (if not, use a bijection from some $\beta < \alpha$ in order to show that α is not the first with the above property) and we shall prove that α satisfies the lemma. By the minimaity assumption of α, we can choose for every $\beta < \alpha$ a function $f_\beta : [\kappa]^{<\omega} \to \beta$ so that $f_\beta''[x]^{<\omega} = \beta$ whenever $x \in F$.

Towards contradiction assume that there exists $g : [\kappa]^{<\omega} \to \alpha$ such that $|g''[x]^{<\omega}| = \alpha$ for every $x \in F$. We wish to define a new function $f : [\kappa]^{<\omega} \to \alpha$, which omits no color on every $y \in F$. Given $t = \{t_1, \ldots, t_n\} \in [\kappa]^{<\omega}$, if n is not of the form $2^i \cdot 3^j$ then $f(t) = 0$. If $n = 2^i \cdot 3^j$ then $f(t) = f_{g(t_1,\ldots,t_i)}(t_{i+1}, \ldots, t_{i+j})$.

Let y be any element of F, and $\gamma < \alpha$ a color. Choose $\{t_1, \ldots, t_i\} \in [y]^{<\omega}$ and $\beta \in (\gamma, \alpha)$ for which $g(\{t_1, \ldots, t_i\}) = \beta$. By the above notation, $\gamma \in f_\beta''[y]^{<\omega}$. Hence, there is $s = \{s_1, \ldots, s_j\} \subseteq y$ such that $f_\beta(s) = \gamma$. Without loss of generality, $t_i < s_1$. Define $r = t \cup s$ and observe that $f(r) = f_\beta(s) = \gamma$, a contradiction.

$\square_{1.4}$

For a Jónsson filter $\mathscr{F}$ let $\alpha_J(\mathscr{F})$ be the first ordinal α whose existence is proved by the above lemma. The parallel to Theorem 1.1 follows, by assuming that all the filters share the same $\alpha_J(\mathscr{F})$. We phrase it as a corollary:

Corollary 1.5. *Jónsson filters over singular cardinals.*
Assume that:

- (א) $\nu = \mathrm{cf}(\kappa) < \kappa$.
- (ב) $\langle \kappa_\alpha : \alpha < \nu \rangle$ *is a continuous increasing sequence of cardinals which tends to κ.*
- (ג) $\mathscr{F}_{\alpha+1}$ *is a $\kappa_{\alpha+1}$-complete Jónsson filter over $\kappa_{\alpha+1}$ which contains the end-segments, for every $\alpha < \nu$.*
- (ד) $\alpha_J(\mathscr{F}_{\alpha+1}) = \theta$ *for some fixed $\theta \in [\nu, \kappa)$, for every $\alpha < \nu$.*

Then there exists a Jónsson filter over κ.

$\square_{1.5}$

The above corollary demonstrates the similarity between Rowbottomness and Jónssonicity. According to the formal definition, the colorings in

Jónsson cardinals are from $[\lambda]^{<\omega}$ into λ. such colorings may omit colors but one cannot expect a set which assumes *less than* λ colors. To see this, consider the coloring defined by $c(\bar{\alpha}) = \min(\bar{\alpha})$. Rowbottomness, therefore, applies to colorings from $[\lambda]^{<\omega}$ into γ where $\gamma < \lambda$.

However, if the range of the coloring is less than λ then the phenomenon of reducing the number of colors to a set of size strictly less than the range of the coloring is manifested at Jónsson cardinals (or filters) as well. We know that for some $\alpha \in \lambda$ one has $\lambda \rightarrow [\lambda]^{<\omega}_{\alpha,<\alpha}$. The only point is that if λ is Rowbottom then this will happen with a fixed size of colors obtained at every coloring into every $\gamma < \lambda$, while Jónssonicity is weaker from this point of view.

Remark 1.6. Corollary 1.5 does not extend to a regular limit of cardinals carrying Jónsson filters or even to a regular limit of measurable cardinals. Indeed, from a suitable large cardinal assumption, it is possible to force a regular cardinal κ being a limit of measurable cardinals $\kappa = \bigcup_{\alpha \in \kappa} \kappa_\alpha$ and carrying a nonreflecting stationary set $S \subseteq \kappa$. Thus, by a theorem of Tryba and Woodin, κ is not Jónsson.

2. Magidority and singular cardinals

In this section we extend the Kleinberg and Henle analysis to study the Magidor partition property. Firstly we show that if κ is a strong partition cardinal and $(\kappa_n : n \in \omega)$ is the corresponding Kleinberg sequence then κ_n is not Magidor for every $n > 2$. We then show that the limit κ_ω of the Kleinberg sequence is a Magidor cardinal if $\kappa > \aleph_1$, and is ω-Magidor if $\kappa = \aleph_1$ (i.e., $\kappa_\omega = \aleph_\omega$). We commence with relevant background on the Kleinberg sequence, the information is based on [Kle77].

For a subset $D \subseteq \kappa$ of size κ, let $[D]^\kappa$ denote the set of all increasing functions $f : \kappa \to D$. For a function $h \in [\kappa]^\kappa$ let ${}_\omega h$ be the function which enumerates the ω-club of all limits ordinal $\alpha \in \kappa$ which are limits of countable strictly increasing sequences $\alpha = \cup_{n \in \omega} h(\alpha_n)$, of ordinals $\alpha_n \in \kappa$. Clearly, ${}_\omega h \in \mathscr{U}$.

A cardinal $\kappa > \aleph_0$ is a *strong partition cardinal* iff $\kappa \to (\kappa)^\kappa$. Such cardinals are measurable, as witnessed by the restriction of the club filter to the set of countable cofinality ordinals $\alpha < \kappa$, which we denote by $\mathscr{U}$. The filter $\mathscr{U}$ is a σ-complete normal measure on κ. Moreover, it is a strong partition (ultra)filter over κ in the sense that for every coloring function $c : [\kappa]^\kappa \to \nu$ for some $\nu < \kappa$, there exists a some $h \in [\kappa]^\kappa$ so that the restriction of c to $[{}_\omega h]$ is monochromatic.

One assigns to a strong partition cardinal κ an increasing sequence of cardinals $(\kappa_n : n \in \omega)$. The informal idea of the definition is to start with $\kappa_1 = \kappa$ and for each $n \in \omega$, use ultrapowers of κ_n by $\mathscr{U}$ to define κ_{n+1} to be the ordertype of the well-ordering of $\mathscr{U}$-equivalence classes of ${}^\kappa\kappa_n$.

The second cardinal in the sequence κ_2 can be shown to be regular and even measurable. Moreover, it satisfies the weak partition property denoted by $\kappa_2 \to (\kappa_2)^{<\kappa_2}$ (making κ_2 a *weak partition cardinal*). The other cardinals κ_n, $n \geq 3$ are shown to be singular of cofinality $\mathrm{cf}(\kappa_n) = \kappa_2$.[1]

Kleinberg showed that many combinatorial properties of the κ_ns can be translated to properties of the strong partition cardinal κ. Doing so requires working with representing functions for ordinals below κ_n, and may face difficulties involving choosing representing functions from the associated $\mathscr{U}$-equivalences classes. To circumvent choice related difficulties, Kleinberg defines κ_n directly from functions $F \in {}^{\kappa^{n-2}}\kappa$ and the natural $\mathscr{U}$-induced equivalence relation.

In this section we try to apply a similar method with respect to Magidority. It turns out that the situation is not the same. An interesting important difference is that the κ_ns in the Kleinberg sequence are not Magidor. Ahead of proving this fact we shall need the following:

Definition 2.1.

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[1]Moreover, by the work of Kleinberg and Henle these cardinals satisfy relatively strong square bracket partition relations. In particular, under AD each $\aleph_n$ is Jónsson and in fact each $\aleph_n$ carries a Jónsson filter as shown in [Kle77]. In this specific case, the Kleinberg sequence begins at $\aleph_1$.

(1) For $n \geq 3$ and $\gamma \in \kappa_n$, represented as $[F]_{\mathscr{U}}$ for some $F \in [\kappa]^{\kappa^{n-2}}$, we define for every $1 \leq k \leq n-3$ the induced $(-k)$-representation of γ, to be the function $\beta^F_{-k} \in [\kappa]^k \to \kappa_{n-k}$, defined byW

$$\beta^F_{-k}(\alpha_0, \ldots, \alpha_{k-1}) = [F_{\alpha_0, \ldots, \alpha_{k-1}}]_{\mathscr{U}}$$

where $F_{\alpha_0, \ldots, \alpha_{k-1}} \in [\kappa]^{\kappa^{n-k}}$ is given by

$$F_{\alpha_0, \ldots, \alpha_{k-1}}(\alpha_k, \ldots, \alpha_{n-1}) = F(\alpha_0, \ldots, \alpha_{k-1}, \alpha_k, \ldots, \alpha_{n-1})$$

(2) Let $\gamma_0, \gamma_1 < \kappa_n$ and $1 \leq k < n-2$, where F_0, F_1 represent γ_0, γ_1 respectively. We say that γ_0, γ_1 are k-interlaced if

$$\min_{A \in \mathscr{U}^k} (\sup_{(\alpha_0, \ldots, \alpha_{k-1}) \in A} \beta^{F_0}_{-k}(\alpha_0, \ldots, \alpha_{k-1})) = \min_{A \in \mathscr{U}^k} (\sup_{(\alpha_0, \ldots, \alpha_{k-1}) \in A} \beta^{F_1}_{-k}(\alpha_0, \ldots, \alpha_{k-1}))$$

Where $\mathscr{U}^k$ is the usual product measure of $\mathscr{U}$. The **interlacing number** of γ_0, γ_1 is the maximal $0 \leq k < n-2$ for which γ_0, γ_1 are k-interlaced. It is straightforward to see that the definition does not depend on the choice of the representing functions.

(3) Let $\gamma_0, \gamma_1 < \kappa_\omega$ be a pair of ordinals, with $n_0 \leq n_1 < \omega$ minimal so that $\gamma_i \in \kappa_{n_i}$, $i < 2$, and F_0, F_1 representing functions $F_i \in [\kappa]^{\kappa^{n_i}}$, $i < 2$, respectively. If $n_0 < n_1$, let

$$A_{F_0, F_1} = \{(\alpha_0, \ldots, \alpha_{n_1-n_0-1}) \in [\kappa]^{n_1-n_0} : \gamma_0 < \beta^{F_1}_{n_1-n_0}(\alpha_0, \ldots, \alpha_{n_1-n_0-1})\}$$

and for each $k < n_0 - 2$, $B_{F_0, F_1, k} \subseteq [\kappa]^{n_1-n_0}$ be the set of all $(\alpha_0, \ldots, \alpha_{n_1-n_0-1}) \in [\kappa]^{n_1-n_0}$ for which the interlacing number of γ_0 and $\beta^{F_1}_{n_1-n_0}(\alpha_0, \ldots, \alpha_{n_1-n_0-1})$ is k.

The **configuration** of γ_0, γ_1 is the pair $(i', k') \in 2 \times \omega$ given by

$$i' = \begin{cases} 0 & \text{if } A \in \mathscr{U}^{n_1-n_0} \\ 1 & \text{otherwise.} \end{cases}$$

and k' is the unique $k < n-2$ for which $B_{F_0, F_1, k} \in \mathscr{U}^{n_1-n_0}$.

Using the above definition one can prove the following:

Theorem 2.2. *Assume AD. Let κ be a strong partition cardinal below Θ, let $(\kappa_n : n \in \omega)$ be the corresponding Kleinberg sequence and let $\kappa_\omega = \bigcup_{n \in \omega} \kappa_n$.*

($\aleph$) *κ_n is not Magidor for every $n > 2$.*

($\beth$) *There is a coloring $c : [\kappa_\omega]^{\omega\text{-bd}} \to \kappa_\omega$ such that if $H \subseteq \kappa_\omega$ and $|H \cap \kappa_n| = \kappa_n$ for some $n > 2$ then $c''[H]^{\omega\text{-bd}} = \kappa_\omega$.*

Proof.
Fix $n > 2$, and fix a normal ultrafilter over $\mathscr{U}$ over κ. We shall define a coloring $c_n : [\kappa_n - \kappa_{n-1}]^{\omega\text{-bd}} \to {}^\omega 2$. Given an element $\bar{\nu} = (\nu_\ell : \ell \in \omega)$ in $[\kappa_n]^{\omega\text{-bd}}$ we will define $c_n(\bar{\nu})(m)$ for every $m \in \omega$ thus obtaining an element of ${}^\omega 2$.

Suppose, therefore, that $\nu \in \kappa_n$. Recall that $\nu = [F]_{\mathscr{U}}$ for some $F : \kappa^{n-2} \to \kappa$. Let $\beta^F_{-n+1} : \kappa^{n-2} \to \kappa_2$ be the induced $(-(n-1))$-relevant

function, given in Definition 2.1, and $d_n(\nu)$ be the induced $\mathscr{U}$-stable limit value,

$$d_n(\nu) = \inf_{A \in \mathscr{U}^{n-2}} (\beta^F_{-n+1} "A).$$

The function $d_n(\nu)$ allows us to record whether the interlacing number of two ordinals $\nu_1, \nu_2 \in \kappa_n$ is the maximal possible value $n - 2$.

Now for each $m \in \omega$ and each $\bar{\nu} \in [\kappa_n - \kappa_{n-1}]^{\omega\text{-bd}}$ let:

$$c_n(\bar{\nu})(m) = \begin{cases} 0 & \text{if } d_n(\nu_{2m}) < d_n(\nu_{2m+1}) \\ 1 & \text{if } d_n(\nu_{2m}) = d_n(\nu_{2m+1}) \end{cases}$$

Suppose that $H \subseteq \kappa_n - \kappa_{n-1}$ and $|H| = \kappa_n$. If $\beta \in \kappa_2$ then $|\{\nu \in H : d_n(\nu) \leq \beta\}| < \kappa_n$. We conclude that for every $\beta \in \kappa_2$ there are cobounded sets of $\nu \in H$ with $d_n(\nu) > \beta$. Also, since the map $\nu \mapsto d_n(\nu)$ on any set of size κ_n cannot be injective, we can find $\nu_0^\beta, \nu_1^\beta \in H$ for which $d_n(\nu_0^\beta) = d_n(\nu_1^\beta) > \beta$. It follows that $c_n''[H]^{\omega\text{-bd}} = {}^\omega 2$, as for every $(r_m : m \in \omega) \in {}^\omega 2$ we can pick inductively ν_m in pairs, so that for every $m \in \omega$ either $d_n(\nu_{2m}) < d_n(\nu_{2m+1})$ or $d_n(\nu_{2m}) = d_n(\nu_{2m+1})$ according to the desired target value of r_m.

Let $\lambda < \Theta$ be a cardinal and let $h : 2^\omega \to \lambda$ be surjective. The coloring $h \circ c_n : [\kappa_n - \kappa_{n-1}]^{\omega\text{-bd}} \to \lambda$ will satisfy $h \circ c_n''[H]^{\omega\text{-bd}} = \lambda$ whenever $|H| = \kappa_n$. In particular, if $\lambda = \kappa_n$ then this coloring exemplifies the non-Magidority of κ_n for every $n > 2$.

For the second part of the claim we define a coloring $c : [\kappa_\omega]^{\omega\text{-bd}} \to {}^\omega 2$. Given $\bar{\nu} \in [\kappa_\omega]^{\omega\text{-bd}}$ fix $n \in \omega$ such that $\{\nu_n : n \in \omega\} \subseteq^* \kappa_n - \kappa_{n-1}$. Let $\bar{\eta} = (\eta_n : n \in \omega)$ be the part of $\bar{\nu}$ contained in $\kappa_n - \kappa_{n-1}$, re-enumerated as designated. Define $c(\bar{\nu}) = c_n(\bar{\eta})$.

Suppose that $A \subseteq \kappa_\omega$ and $|A| = \kappa_\omega$. Fix $n \in \omega$ such that $|H| = \kappa_n$ where $H = A \cap (\kappa_n - \kappa_{n-1})$. By the above definition $c''[H]^{\omega\text{-bd}} = c_n''[H]^{\omega\text{-bd}} = 2^\omega$. Let $h : 2^\omega \to \kappa_\omega$ be surjective. Now $h \circ c : [\kappa_\omega]^{\omega\text{-bd}} \to \kappa_\omega$ exemplifies the required statement.

$$\square_{2.2}$$

Our next goal is to prove that κ_ω is Magidor (in the case of $\aleph_\omega$ we shall prove just ω-Magidority). We shall need a lemma which generalizes a statement from the proof of [Kle77, Theorem 4.9].

Lemma 2.3. *Let $\kappa > \aleph_0$ be a strong partition cardinal and let $\kappa_2 = {}^\kappa\kappa/\mathscr{U}$ where $\mathscr{U}$ is a normal ultrafilter over κ. Suppose that $(c_\alpha : \alpha \in \kappa)$ is a sequence of colorings where $c_\alpha : [\kappa_2]^{\kappa^{n_\alpha}} \to \kappa$ and $n_\alpha \in \omega$ for each $\alpha \in \kappa$. Then one can find $H \in [\kappa_2]^{\kappa_2}$ so that H is c_α-monochromatic for every $\alpha \in \kappa$ simultaneously.*

Proof.

We may assume, without loss of generality, that the domain of each c_α is $[\kappa_2]^{\kappa^\omega}$ since $n_\alpha \in \omega$ and by monotonicity. If $a \in [\kappa_2]^{\kappa^\omega}$ then $(c_\alpha(a) : \alpha \in \kappa)$ is an element of ${}^\kappa\kappa$ (or a subset of κ). Therefore, one can think of the sequence $(c_\alpha : \alpha \in \kappa)$ as one single coloring $c : [\kappa_2]^{\kappa^\omega} \to {}^\kappa\kappa$ defined simply

by $c(a) = (c_\alpha(a) : \alpha \in \kappa)$. The proof of the lemma will be accomplished once we show that there exists a c-monochromatic $H \in [\kappa_2]^{\kappa_2}$.

Define another coloring $d : [\kappa_2]^{\kappa^\omega + \kappa^\omega} \to \kappa + 1$ as follows. Given $y \in [\kappa_2]^{\kappa^\omega + \kappa^\omega}$ let $y = a {^\frown} b$ where $a, b \in [\kappa_2]^{\kappa^\omega}$ and $a < b$. Let $d(y) = \kappa$ iff $c(a) = c(b)$, and let $d(y) = \nu$ iff $c(a) \neq c(b)$ and $\nu \in \kappa$ is the first ordinal so that $c_\nu(a) \neq c_\nu(b)$.

Since $\kappa_2 \to (\kappa_2)^\gamma_\kappa$ for each $\gamma \in \kappa_2$, one can find a d-monochromatic $H \in [\kappa_2]^{\kappa_2}$. Since κ_2 is regular, the constant value of d cannot be some $\nu \in \kappa$ (otherwise H will produce κ_2 distinct values $c_\nu(a)$ for κ_2-many $a \in [H]^{\kappa^\omega}$). Hence $d''[H]^{\kappa^\omega} = \{\kappa\}$. It follows that $c(a) = c(b)$ whenever $a, b \in [H]^{\kappa^\omega}$ and hence H is c_α-monochromatic for every $\alpha \in \kappa$ as required.

$$\square_{2.3}$$

In the following proposition we deal with colorings from $[\kappa_\omega]^{\aleph_0\text{-bd}}$ into κ_1 and plan to establish the existence of sets of maximal cardinality which omit colors by reducing the given coloring to colorings of κ_2, and applying the conclusion of the previous lemma. Once this is shown, it is straightforward to establish the Magidority of κ_ω (namely, dealing with colorings into κ_ω).

To move forward, we do have to tackle a choice related issue concerning the usage of functions $f \in {}^{\kappa^n}\kappa_2$ to represent ordinals in κ_{n+2}. Consider for example the case of κ_3. According to its definition, this cardinal is the order-type of the $\mathscr{U}^2$-induced well-ordering on its associated equivalence classes of functions $F \in {}^{\kappa^2}\kappa$ (i.e., on κ^2-sequences of ordinals of κ).

An alternative way to present ordinals of κ_3 is by thinking on κ-sequences of ordinals of κ_2. It is natural to consider arbitrary function $f \in {}^\kappa\kappa_2$ as representatives of ordinals in κ_3. This is advantageous as it enables us to invoke directly the partition properties of κ_2. However, there is an issue with this approach. as κ-sequence of ordinals of κ_2 are merely κ-sequence of $\mathscr{U}$ equivalence classes, and replacing such a sequence with a function $F \in {}^{\kappa^2}\kappa$ seems to require making more choice than one is allowed in the context of AD. Therefore, while it is possible to define a well-ordering of $\mathscr{U}$-induced equivalence classes of functions (sequences) $f \in {}^\kappa\kappa_2$, it is possible that some of these classes do not coincide the ordinals in κ_3, which are given by classes of functions $F \in {}^{\kappa^2}\kappa$.

This issue is not essential, and is solvable once realizing that both types of classes produces well-orders of the same ordertype, which is κ_3. This is because each of the two embeds by an order-preserving and unbounded function in the other. To see the fact that they are mutually cofinal we need two directions. One direction is given by the map $F \mapsto \beta^F_{-1}$ from ${}^{\kappa^2}\kappa$ to ${}^\kappa\kappa_2$. For the opposite direction consider a function κ-sequence $(\delta_\alpha : \alpha \in \kappa)$ of ordinals of κ_2. Recall that κ_2 is regular and hence $(\delta_\alpha : \alpha \in \kappa)$ is bounded in κ_2. Choose an upper bound, say $\tau \in \kappa_2$, and let $f \in {}^\kappa\kappa$ be an element from the equivalence class of τ as an ordinal in κ_2. Consider now the κ^2-sequence formed by κ consecutive copies of f. This gives rise to the desired element of ${}^{\kappa^2}\kappa$, and all we need is to choose one representative of one equivalence class.

This consideration generalizes to all cardinals κ_n, $n \geq 3$ and is summarized by the following lemma.

Lemma 2.4. *For every $n \in \omega$, the well-orderings $^{\kappa^{n+1}}\kappa_2$ and $^{\kappa^{n+2}}\kappa$ are mutually cofinal and hence can be employed interchangeably.*

$\square_{2.4}$

We turn to make several definitions and notations which adjust previous definitions to work with functions $f \in {}^{\kappa^{n+1}}\kappa_2$, replacing the role of functions $F \in {}^{\kappa^{n+2}}\kappa$ in the previous section. Fix $n \in \omega$ and $\tau \in \kappa$. Consider a sequence $\bar{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in [\kappa]^n$. Suppose that $f \in {}^{\kappa^n}\kappa_2$ (so f represents an ordinal of κ_{n+2}). For every $i \in \tau$ let $f_i(\bar{\alpha}) = f(\alpha_0 \cdot \tau, \alpha_1, \ldots, \alpha_{n-1})$. Notice that f_i is the same object like f, that is a function from κ^n into κ_2. In particular, one may consider the equivalence class $[f_i]_{\mathscr{U}}$.

We define a *breaking function* $\mathrm{bk}^n_\tau : {}^{\kappa^n}\kappa_2 \to {}^\tau({}^{\kappa^n}\kappa_2)$ as follows. Given $f \in {}^{\kappa^n}\kappa_2$ let $\mathrm{bk}^n_\tau(f) = ([f_i]_{\mathscr{U}} : i \in \tau)$. In order to obtain Magidority we shall need only $\tau \in \omega_1$ and this will take care of every possible order type of an $\aleph_0$-bounded subset of κ_ω. We fix, therefore, an enumeration $(\bar{\tau}_\nu : \nu \in \omega_1)$ of $^{<\omega}\omega_1$. Stipulate $\bar{\tau}_\nu = (\tau_\nu(n) : n < n_\nu)$, so $n_\nu \in \omega$.

Proposition 2.5. *Suppose that:*
   (a) *$\kappa > \aleph_0$ is a strong partition cardinal.*
   (b) *$(\kappa_n : n \in \omega)$ is the corresponding Kleinberg sequence generated from the normal ultrafilter $\mathscr{U}$ over κ.*
   (c) *$\kappa_\omega = \bigcup_{n \in \omega} \kappa_n$.*
   (d) *$c : [\kappa_\omega]^{\aleph_0\text{-}bd} \to \kappa$.*
Then by letting $c_\nu = c \upharpoonright \prod_{n < n_\nu} [\kappa_{n+2}]^{\tau_\nu(n)}$ for $\nu \in \omega_1$, there exists a set $A \in [\kappa_\omega]^{\kappa_\omega}$ which is c_ν-monochromatic for every $\nu \in \omega_1$ simultaneously.

Proof.
For every $\nu \in \omega_1$ let $\zeta_\nu = \bigcup_{n < n_\nu} \kappa^n$, so ζ_ν is an ordinal in κ_2. We define for each $\nu \in \omega_1$ an auxiliary function $d_\nu : {}^{\zeta_\nu}\kappa_2 \to \kappa$ by:

$$d_\nu(f_0, \ldots, f_{n_\nu - 1}) = c_\nu(\star(\mathrm{bk}^n_{\tau_\nu(n)}(f_n) : n < n_\nu)).$$

Let us try to explain what lies behind this definition. One can think of ζ_ν as an ordinal, but it is more convenient to identify the members of ζ_ν with finite sequences of length n_ν where each element f_n in such a sequence is a function from κ^n to κ_2. Such a function is the input of $\mathrm{bk}^n_{\tau_\nu(n)}$, and given a sequence $(f_0, \ldots, f_{n_\nu - 1})$ we apply $\mathrm{bk}^n_{\tau_\nu(n)}$ to f_n for each $n < n_\nu$ thus creating long sequences (of length $\tau_\nu(n)$). Finally, we take their concatenation. Notice that we make a tacit use of Lemma 2.4 in this definition.

Bearing in mind that κ^n-sequences of ordinals from κ_2 represent ordinals in κ_{n+2} we see that $\star(\mathrm{bk}^n_{\tau_\nu(n)}(f_n) : n < n_\nu)$ is an element in $\prod_{n < n_\nu} [\kappa_{n+2}]^{\tau_\nu(n)}$ so we can apply c_ν to this element. From Lemma 2.3 we infer that there exists $H \in [\kappa_2]^{\kappa_2}$ which is d_ν-monochromatic for every $\nu \in \omega_1$. Let β_ν be the constant value of d_ν. Our goal is to manufacture from H a set $A \in [\kappa_\omega]^{\kappa_\omega}$

which is c_ν-monochromatic for every $\nu \in \omega_1$. This will be done piecewise, as we shall create $A_n \subseteq \kappa_{n+2} - \kappa_{n+1}$ and define $A = \bigcup_{n \in \omega} A_n$.

The recurrent idea is that ordinals in κ_{n+2} are κ^n-sequences from κ_2 (in our case, ordinals from $H \subseteq \kappa_2$). The information in the functions c_ν is coded by the functions d_ν for which we already have a monochromatic set at our disposal. It remains, therefore, to translate the objects in the domain of each c_ν into the objects which lie in the domain of the d_νs. This we shall do now.

Firstly we pick ω-many sequences $(\bar{s}_n : 1 \leq n < \omega)$ of elements of H. The first sequence $\bar{s}_1$ consists of the first κ elements of H, and if $\bar{s}_n$ has been defined then $\bar{s}_{n+1}$ consists of the first κ^{n+1} elements of H above $\bigcup \bar{s}_n$. Rather than enumerating the ordinals of each $\bar{s}_n$ by the usual order we would like to ascribe a different enumeration to the elements of each $\bar{s}_n$.

Since the length of $\bar{s}_n$ is the ordinal κ^n we shall write $\bar{s}_n$ as $\{s_{\bar{\alpha}}^n : \bar{\alpha} \in [\kappa]^n\}$ where these elements are ordered by the descending lexicographical ordering. Explicitly, if $\bar{\alpha}, \bar{\beta} \in [\kappa]^n, \bar{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})$ and $\bar{\beta} = (\beta_0, \ldots, \beta_{n-1})$ then $\bar{\alpha} <_S \bar{\beta}$ iff $\ell \leq n - 1$ is the maximal index for which $\alpha_\ell \neq \beta_\ell$ and then $\alpha_\ell < \beta_\ell$. We arrange the enumeration of $\bar{s}_n$ in such a way that $s_{\bar{\alpha}}^n < s_{\bar{\beta}}^n$ iff $\bar{\alpha} <_S \bar{\beta}$.

Secondly, for every $n \in \omega$ we define an operator $f \mapsto_n f^+$ which maps each $f \in {}^{\kappa^n}\kappa$ to $f^+ \in {}^{\kappa^n}\kappa_2$. More specifically, it sends f to $f^+ \in {}^{\kappa^n}H$ (recall that $H \subseteq \kappa_2$). This is done by the following procedure. Given $\bar{\alpha} \in \kappa^n$ and $f \in {}^{\kappa^n}\kappa$ we define $\bar{\alpha}(f) = (\alpha_0, \ldots, \alpha_{n-1}, f(\bar{\alpha}))$ and then we let $f^+(\bar{\alpha}) = s_{\bar{\alpha}(f)}^{n+1}$. Philosophically, if $f \in {}^{\kappa^n}\kappa$ then $[f]_{\mathscr{U}}$ is an ordinal in κ_{n+1} and therefore $[f^+]_{\mathscr{U}}$ represents an ordinal in κ_{n+2}, so this is a sort of a *push-up operator*.

Notice that if $f <_{\mathscr{U}} g$ then $f^+ <_{\mathscr{U}} g^+$. Fix $n \in \omega$ and $\ell < n$. Let $\bar{\alpha} = (\alpha_0, \ldots, \alpha_{\ell-1}) \in [\kappa]^\ell$ and fix $f \in {}^{\kappa^n}\kappa$. By the definition of f^+ we see that:

$$\sup\{f^+(\bar{\alpha}^\frown\bar{\beta}) : \bar{\beta} \in [\kappa-(\alpha_{\ell-1}+1)]^{n-\ell}\} = \sup\{s_{\bar{\alpha}^\frown\bar{\beta}}^{n+1} : \bar{\beta} \in [\kappa-(\alpha_{\ell-1}+1)]^{n-\ell}\}.$$

But the righthand side does not depend on f, hence this supremum depends only on $\bar{\alpha}$ and not on f. From this fact it follows that if $(\gamma_i : i \in \tau)$ is an increasing sequence of ordinals in H then one can find a $<_{\mathscr{U}}$-increasing sequence of functions $(f_i^+ : i \in \tau)$ so that for some $g \in {}^{\kappa^n}\kappa_2$ we have $\mathrm{bk}_\tau^n(g) = (f_i^+ : i \in \tau)$. Explicitly, if $(f_i^+ : i \in \tau)$ is given then we define:

$$g(\alpha \cdot \tau + i, \alpha_1, \ldots, \alpha_{n-1}) = s_{\alpha,\alpha_1,\ldots,\alpha_{n-2},f_i(\alpha,\alpha_1,\ldots,\alpha_{n-1})}^n.$$

One can verify that $\mathrm{bk}_\tau^n(g) = ([f_i^+]_{\mathscr{U}} : i \in \tau)$.

For every $n \in \omega$ let $A_n = \{[f^+]_{\mathscr{U}} : f \in {}^{\kappa^n}\kappa, \kappa = \bigcup(\kappa_n : n \in \omega)\}$. It follows that $A_n \subseteq \kappa_{n+2} - \kappa_{n+1}$ (but notice that $|A_n| = \kappa_{n+1} < \kappa_{n+2}$). Finally, let $A = \bigcup_{n \in \omega} A_n$. By the definition of d_ν and the construction of A we see that A is c_ν-monochromatic for every $\nu \in \omega_1$, so the proof is accomplished.

$$\square_{2.5}$$

Now we can phrase and prove the following:

Theorem 2.6. *Suppose that $\kappa > \aleph_0$ is a strong partition cardinal.*
- ($\aleph$) *If $\kappa > \aleph_1$ then κ_ω is Magidor.*
- ($\beth$) *If $\kappa = \aleph_1$ then $\kappa_\omega = \aleph_\omega$ is ω-Magidor.*

Proof.
Suppose that $\kappa > \aleph_1$ and let $c : [\kappa_\omega]^{\aleph_0\text{-bd}} \to \kappa_\omega$ be a coloring. Define $d : [\kappa_\omega]^{\aleph_0\text{-bd}} \to \kappa$ as follows. Given $y \in [\kappa_\omega]^{\aleph_0\text{-bd}}$ let $d(y) = c(y)$ if $c(y) \in \kappa$ and let $d(y) = 0$ otherwise. Notice that d is expressible as $\bigcup_{\nu \in \omega_1} c_\nu$, so by Proposition 2.5 there is a set $A \in [\kappa_\omega]^{\kappa_\omega}$ for which c_ν is monochromatic whenever $\nu \in \omega_1$. This implies that $|d''[A]^{\aleph_0\text{-bd}}| \leq \aleph_1 < \kappa_1$. Choose $\delta \in \kappa - d''[A]^{\aleph_0\text{-bd}}$ and verify that $\delta \notin c''[A]^{\aleph_0\text{-bd}}$, so κ_ω is Magidor.

This argument will not work if $\kappa = \aleph_1$, but a similar (and simpler) reasoning shows that one can get ω-Magidority in this case. Indeed, we do not have to consider ω_1-many possible order types, so we can get rid of the $\bar{\tau}_\nu$s and only consider ω-sequences for which we have but countably many partitions of bounded sequences scattered along the $\aleph_n$s. We conclude, therefore, that $\aleph_\omega$ is ω-Magidor.

$\square_{2.6}$

We conclude this section with a couple of open problems. If a Kleinberg sequence is inaugurated from $\kappa = \aleph_2$ under AD, then the next step is at least $\aleph_{\omega+1}$ since $\aleph_n$ is singular for every $n > 2$. The possibility of $\aleph_\omega$ being Magidor is not necessarily connected with Kleinberg sequences. However, if one wishes to pursue this direction then it is reasonable to start with a universe in which infinitely many $\aleph_n$s are measurable.

Question 2.7. Is it consistent with ZF that $\aleph_\omega$ is a Magidor cardinal?

Recall that λ is β-Magidor (for some $\beta \in \lambda$) iff $\lambda \to [\lambda]^{<\beta\text{-bd}}_\lambda$. A cardinal λ is strongly Magidor iff λ is β-Magidor for every $\beta \in \lambda$. If λ is I1 then λ is strongly Magidor as proved in [GH18]. It is easy to see from the above statements that if one begins with a larger κ as a strong partition cardinal then the Magidority degree of κ_ω increases. However, it is not clear whether strong Magidority obtains in this way.

Question 2.8. Assume AD. Does there exist a singular strongly Magidor cardinal below Θ?

Under AD, infinitary combinatorics extends to non-wellorderable sets. For example, Holshouser and Jackson, Chan [Chan19], and Chan Jackson Trang [ChJaTr23] show that the non-wellordered cardinalities $\mathbb{R}$, $\mathbb{R} \times \kappa$, and $[\omega_1]^\omega$ are Jónsson. It is not clear whether the notion of $\aleph_0$-bd partition properties and Magidoricity translates well to this setting.

3. Forcing Magidor filters over singular cardinals

The method of the previous section do not seem to natural extend to show that Kleinberg limit cardinals κ_ω carry a Magidor filter. Indeed, the fact that each of the κ_n, $n \geq 3$ is not Magidor implies that κ_ω cannot carry a normal Magidor filter.

Considering the existence of Magidor filters in models closely related to AD, we show that one can force over a model of $\mathsf{AD} + V = L(\mathbb{R})$ to obtain a Magidor filter over a singular cardinal λ. If $\lambda < \Theta$ then our forcing will probably destroy AD, but if one assumes the existence of a strong partition cardinal above Θ then one can force a Magidor filter over a singular cardinal under AD. To the understanding of the authors, it is not known whether the existence of strong partition cardinal above Θ in a model of AD is consistent from a known large cardinal assumption.[2]

We shall see below how to force such filters over models of AD. Remark that the cardinals upon which we force Magidor filters are not of the form κ_ω. Rather, we choose some strong partition cardinal and singularize it.

We begin with some background. Prikry proved in [Pri70] that if one forces with Prikry forcing into a measurable cardinal κ then one preserves the fact that κ is Jónsson. Moreover, the normal ultrafilter which serves for the pure part of the forcing conditions generates a Jónsson filter in the generic extension. Hence despite the fact that κ is not measurable after Prikry forcing, it still secures some combinatorial properties which are quite close to measurability.

We wish to do something similar with Magidor filters. Namely, we would like to begin with a Magidor ultrafilter over a measurable cardinal, to force Prikry (or Magidor) into it and to show that it generates a Magidor filter in the generic extension. Our first step, therefore, is to show that there are many Magidor filters over regular (in fact, measurable) cardinals under $\mathsf{AD} + V = L(\mathbb{R})$.

We shall use, again, the order-type version of Magidority, that is $\lambda \to [\lambda]^{\omega\text{-bd}}_\lambda$. This means that the colorings are defined on bounded subsets of λ *of order type ω.* Recall that κ a *strong partition cardinal* iff $\kappa \to (\kappa)^\kappa$ and a *weak partition cardinal* iff $\kappa \to (\kappa)^\gamma$ for every $\gamma \in \kappa$.

Lemma 3.1. *Assume* $\mathsf{AD} + V = L(\mathbb{R})$.
There are unboundedly many $\kappa < \Theta$ which carry a κ-complete (and normal) Magidor ultrafilter.

Proof.
By [KKMW81] there are unboundedly many strong partition cardinals below Θ. Let us show that if $\kappa < \Theta$ is such a cardinal then κ is measurable, and that if $\mathscr{U}$ is a normal ultrafilter over κ then $\mathscr{U}$ is Magidor. Fix $\kappa < \Theta$ such that $\kappa \to (\kappa)^\kappa$ and hence $\kappa \to (\kappa)^2$. From the latter relation it follows

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[2]The assumption that there exists a strong partition cardinal above Θ appears in the literature, see e.g. [Cun17].

that κ is regular, and under $\mathsf{AD} + V = L(\mathbb{R})$ we know that κ is measurable and the filter generated by the unbounded ω-closed subsets of κ is a normal ultrafilter, see [Ste10, Theorem 8.27]. Denote this ultrafilter by $\mathscr{U}_\omega^\kappa$, or just $\mathscr{U}$ if κ is clear from the context.

Since κ is also a weak partition cardinal we see, in particular, that $\kappa \to (\kappa)^{\omega \cdot \gamma}$ for every countable ordinal γ. From [Jac10, Fact 2.31] we infer that $\kappa \to_{\mathrm{club}} (\kappa)^{\omega\text{-bd}}$. Since $\mathscr{U}$ is normal it extends the club filter over κ and hence $\kappa \to (\mathscr{U})^{\omega\text{-bd}}$, so $\mathscr{U}$ is a Magidor filter as desired.

$$\square_{3.1}$$

Notice that we get a somewhat stronger property than just Magidority, as we elicit a monochromatic set from the ultrafilter.

Theorem 3.2. *Magidor filters over singular cardinals.*
Assume AD and $V = L(\mathbb{R})$. Then one can force the existence of a singular cardinal which carries a Magidor filter.

Proof.
Let κ be a strong (and weak) partition cardinal below Θ. Let $\mathscr{U}$ be a κ-complete ultrafilter over κ, so $\mathscr{U}$ is Magidor as we proved in Lemma 3.1. Let $\mathbb{P}$ be Prikry forcing with $\mathscr{U}$ and let $G \subseteq \mathbb{P}$ be generic over the ground model. Let $\mathscr{F}$ be the filter generated from $\mathscr{U}$ in $V[G]$. We claim that $\mathscr{F}$ is a Magidor filter.

So assume that $\underset{\sim}{c} : [\kappa]^{\omega\text{-bd}} \to \gamma$ for some $\gamma \in \kappa$, and let (s, A) be an arbitrary condition which forces this fact. We define a function $f : [\kappa]^{\omega\text{-bd}} \to \gamma^\omega$ in the ground model as follows. Given $t \in [\kappa]^{\omega\text{-bd}}$ let:

$$f(t) = \{\eta \in \gamma : \exists u, v \exists E, t = u \cup v, (s^\frown u, E) \Vdash \underset{\sim}{c}(v) = \eta\}.$$

One can think of $f(t)$ as an ω-sequence of ordinals of γ. For every $n \in \omega$ let $f_n : [\kappa]^{\omega\text{-bd}} \to \gamma$ be the nth place in the sequence $f(t)$ and choose $B_n \in \mathscr{U}$ so that $f_n \restriction [B_n]^{\omega\text{-bd}}$ is constant. We employ AC_ω at this stage, a principle which holds under AD. Let $B = \bigcap_{n\in\omega} B_n$, so $B \in \mathscr{U}$ and $f \restriction [B]^{\omega\text{-bd}}$ is constant. Hence $\mathbb{P}$ forces that $\underset{\sim}{c} \restriction [B]^{\omega\text{-bd}}$ is constant as well.

$$\square_{3.2}$$

The above theorem shows that a singular cardinal with countable cofinality may carry a Magidor filter. In a seminal work, Magidor showed in [Mag78] that from relevant assumptions of large cardinals one can change the cofinality of a regular cardinal making it a singular cardinal of uncountable cofinality. Using Magidor forcing in the version of Henle from [Hen83] one can obtain a Magidor filter over a singular cardinal with uncountable cofinality, essentially by the same proof.

Results by Chan and Jackson [ChJa21] and Ikegami and Trang [IkTr] show that AD is highly sensitive to nontrivial forcing below Θ, and in particular show that forcing over a model of AD a Prikry/Magidor sequence to a measurable cardinal $\kappa < \Theta$ will cause AD to fail in the generic extension (see Theorem 4.1 in [IkTr]). Of course, under the assumption that there is

a strong partition cardinal above Θ, Prikry forcing into this cardinal will preserve AD, thus we can phrase the following:

Corollary 3.3. *Assume AD. Suppose that $\kappa > \Theta$ and κ is a strong partition cardinal. Then one can force a Magidor filter over κ while preserving AD.*

$$\square_{3.3}$$

We do not know if one can force a Magidor filter with the usual version of bounded subsets of any countable order. The problem seems to be Lemma 3.1, in which we need more than AC_ω in order to take care of all the countable order-types simultaneously. We conclude with the following:

Question 3.4. Is it consistent that some singular cardinal $\lambda < \Theta$ carries a Magidor filter under AD?

This question is particularly interesting at $\aleph_\omega$. Tryba proved in [**?**] that if there is a Jónsson filter over λ then there are unboundedly many Jónsson cardinals below λ, from which he inferred that there is no Jónsson filter over $\aleph_\omega$ in ZFC. Under AD there is a Jónsson filter over $\aleph_\omega$ and this is very natural since each $\aleph_n$ is Jónsson. We saw, however, that $\aleph_n$ is not Magidor whenever $n > 2$. Hence even if the answer to the above question is positive, it is plausible that there are no Magidor filters over $\aleph_\omega$ under AD. Of course, one can force strong properties over infinitely many $\aleph_n$s (e.g., measurability as done in [Abr74]) and in such models of ZF perhaps there is a Magidor filter over $\aleph_\omega$.

We conclude with another interesting question about strong Magidority, in the context of the present section. The idea of preserving square bracket relations using Prikry type forcing notions invites for the following:

Question 3.5. Is it consistent for a singular cardinal λ to carry a strong Magidor filter? Is it consistent under AD?

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