Aronszajn trees and maximality—Part 2*

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Abstract

Assuming the consistency of a weakly compact cardinal above a regular uncountable cardinal μ, we prove the consistency of the existence of a wide μ^+-Aronszajn tree, i.e. a tree of height and cardinality μ^+ with no branches of length μ^+, into which every wide μ^+-Aronszajn tree can be embedded.

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MSC classification: 03E35, 03E40, 03E05, 03E55.

1 Introduction

The topic of this paper is maximality among wide κ-Aronszajn trees, i.e. trees of cardinality and height κ without branches of length κ. Such a tree is called maximal if every such tree can be embedded into it. We show the consistency of maximal trees relative to the consistency of a weakly compact cardinal. This has been an open problem for 30 years.

Trees in this paper are partial orders in which the set of predecessors of every element are well-ordered by the partial order, and there is a unique smallest element. The order-type of the set of predecessors of an element of a tree is called the *height* of the element, and the supremum of all heights in a tree is

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called the *height* of the tree. The set of elements of a fixed height is called a *level* of the tree. There is a natural quasi-ordering of the class of all trees: a tree T is below a tree T' if T can be monomorphically embedded into T'. For any class $\mathcal{C}$ of trees it is natural to ask if $\mathcal{C}$ has a maximal element T under embeddability. Then, up to isomorphism, the class $\mathcal{C}$ consists just of subtrees of T in $\mathcal{C}$.

Our focus is on trees of cardinality and height $\kappa \geq \omega_1$ with no branches, i.e. linearly ordered subsets, of size κ. Following [5], we call such trees *wide κ-Aronszajn trees.* Our main result (Theorem 1 below) is that it is consistent, relative to the consistency of weakly compact cardinals, to have a maximal wide κ-Aronszajn tree. Our proof works for any successor of a regular cardinal $> \aleph_0$. This result complements the fact that it is a consequence of the Generalized Continuum Hypothesis that there are no maximal wide κ^+-Aronszajn trees for any infinite regular κ [12]. Under the stronger assumption $V = L$, no wide κ^+-Aronszajn tree is maximal even just for κ^+-Souslin trees, for we prove in [4], improving a result in [24], that, assuming $V = L$, for every κ^+-Aronszajn tree T, κ regular, there is a κ^+-Souslin tree which is not embeddable into T.

If the levels of a wide κ-Aronszajn tree are of cardinality $< \kappa$, we drop "wide" and call such trees just κ-Aronszajn, or, to be more specific, *narrow* κ-Aronszajn. Furthermore, if $\kappa = \omega_1$, we call the trees Aronszajn, or wide Aronszajn, respectively. For some κ there may be no κ-Aronszajn trees, and then κ is said to have the *tree property.* By Kőnig's Lemma, ω has the tree property. No singular cardinal has the tree property for trivial reasons. An inaccessible cardinal has the tree property if and only if it is weakly compact.

Examples of Aronszajn trees are so-called Souslin trees, which are instrumental in understanding, and proving the independence of, the so-called Souslin Hypothesis i.e. the hypothesis that the order-type of the real numbers is the unique, up to isomorphism, dense complete linear order without end-points in which all families of disjoint non-empty open sets are countable.

Wide κ-Aronszajn trees are important in the study of model theoretic properties of uncountable structures, namely, trees can be used as a weak substitute for ordinals when uncountable models are investigated by means of transfinite games [12] and, more generally, in the study of generalized Baire spaces [16]. For example, the existence of a particular kind of maximal tree, a so-called Canary Tree (see below), is equivalent, assuming CH, to the isomorphism class of the free Abelian group of cardinality $\aleph_1$ being Δ^1_1 in the generalized Baire Space $\omega_1^{\omega_1}$. This emphasises the importance of understanding better the global ordering of trees, especially the existence of maximal trees.

There is no maximal countable ordinal, but if we identify ordinals with trees without infinite branches, and consider generalized ordinals i.e. the class $\mathcal{T}_\alpha$ of trees of cardinality $\aleph_\alpha$ without branches of length $\aleph_\alpha$, the situation is more opaque. When $\alpha > 0$, the structure of the class of such trees is much more complicated than the structure of ordinals. For example, the structure of $\mathcal{T}_1$ is highly non-linear as it is easy to construct pairs of wide (or narrow) Aronszajn trees so that neither can be mapped even by a strict order preserving homomorphism to the other. Furthermore, the structure is highly non-absolute.

Several partial results are known about $\mathcal{T}_\alpha$, $\alpha > 0$, [16, 17, 24, 23, 5].

The main result of this paper is:

Theorem 1. *Suppose that κ is a weakly compact cardinal and $\mu < \kappa$ is regular uncountable. There is a (set) forcing extension of the universe in which $\kappa = \mu^+$ and there is a maximal wide κ-Aronszajn tree i.e. a wide κ-Aronszajn tree T such that any other wide κ-Aronszajn tree can be monomorphically embedded into T.*

To simplify our presentation, we will prove the theorem for the case $\mu = \omega_1$ (i.e., $\kappa = \omega_2$). It will be apparent throughout this work that modification to an arbitrary regular uncountable cardinal μ is straightforward (For a short discussion regarding the case $\mu = \omega$, see Section 7).

This theorem further emphasises the difference between the order of trees with no infinite branches and the class of trees with no branches of length $\kappa > \omega$.

We shall now define in detail the central concepts of this paper. We have already defined the concept of a wide κ-Aronszajn tree as well as its special case, the (narrow) κ-Aronszajn tree, agreeing to drop κ if $\kappa = \omega_1$. While Aronszajn trees always exist, the existence of an $\aleph_2$-Aronszajn tree is independent of ZFC, assuming the consistency of weakly compact cardinals: Specker proved the existence of an $\aleph_2$-Aronszajn tree from CH [22]. Mitchell and Silver proved the consistency of the non-existence of $\aleph_2$-Aronszajn trees, relative to the consistency of a weakly compact cardinal [18, Theorem 5.8]. They also showed that if there are no $\aleph_2$-Aronszajn trees, then $\aleph_2$ is weakly compact in L. As opposed to the case of (narrow) κ-Aronszajn trees, it is easy to construct a *wide* κ-Aronszajn tree in ZFC by bundling together isolated branches of all lengths $< \kappa$.

As discussed already, our main topic in this paper is the existence of trees that are maximal in some specific sense. There are several ways in which two trees T and T' can be compared to each other in order for the concept of maximality to make sense. Originally the question of maximality was raised [16] in connection with comparing trees by asking whether there is a *homomorphism* from one to the other i.e. a mapping from one tree T to another T', that preserves strict ordering:

$$t <_T t' \Rightarrow f(t) <_{T'} f(t').$$

Note that such a mapping need not be one-one because incomparable elements can be mapped to the same element. We follow [5] in calling such a mapping a *weak embedding*[1]. The reason for the emergence of weak embeddings as a way to order classes of trees is its close connection to certain games, introduced below. While proving Theorem 1 the authors realized that they can actually prove the consistency of the existence of a maximal tree under a stronger order,

[1] There was a claim in [16] that a maximal tree exists in the sense of weak embeddings in $\mathcal{T}_1$ if Martin's Axiom (MA) and $2^\omega > \omega_1$ were assumed. This claim was proved wrong in [5].

namely the order according to (monomorphic) embeddability. In the end, ordering trees by the existence of an embedding is very natural. From a general mathematical perspective it can be considered even more natural than ordering by weak embeddings.

Let us write $T \leq^* T'$ if there is an embedding (i.e. a monomorphism) $T \to T'$. If $T \leq^* T'$ and $T' \leq^* T$ we write $T \equiv^* T'$. Respectively, if $T \leq^* T'$ but $T' \not\leq^* T$, we write $T <^* T'$. If there is a weak embedding from T to T', we write $T \leq T'$. If $T \leq T'$ and $T' \leq T$ we write $T \equiv T'$. Finally, if $T \leq T'$ but $T' \not\leq T$, we write $T < T'$. Of course, $T \leq^* T'$ implies $T \leq T'$.

If B_α is the tree of descending chains of elements of α, ordered by end-extension, then $\alpha \leq \beta$ if and only if $B_\alpha \leq B_\beta$. Thus in the class of trees without infinite branches the weak embedding order reflects the received ordering of the class of all ordinal numbers. Again, we may ask, whether there is a maximal tree under the weaker ordering $\leq$ in the class of all Aronszajn trees. If we assume $\mathrm{MA}_{\aleph_1}$, then no wide Aronszajn tree is $\leq$-above all Aronszajn trees [5, 23]. Similarly, if $V = L$, then for every wide Aronszajn tree T there is a Souslin tree S such that $S \not\leq T$ [24].

We may now ask in two different senses whether there is a maximal tree in a given class of trees:

> **The Maximality Question:** Given a class $\mathcal{C}$ of trees, is there a tree T in $\mathcal{C}$ such that $S \leq^* T$ for every $S \in \mathcal{C}$?

> **The Weak Maximality Question:** Given a class $\mathcal{C}$ of trees, is there a tree T in $\mathcal{C}$ such that $S \leq T$ for every $S \in \mathcal{C}$?

Trivially, a positive solution to the Maximality Question gives a positive solution to the Weak Maximality Question.

Both the full and the Weak Maximality Questions are meaningful even if the maximal tree T is not in $\mathcal{C}$ but satisfies some weaker constraints. For example, it is consistent, relative to the consistency of ZF, that CH holds and every Aronszajn-tree is special (Jensen, [20, Theorem 8.5]). Thus in this model there is a wide Aronszajn-tree that is $\leq$-above all Aronszajn-trees. However, this tree T is (a priori) not Aronszajn, so we do not obtain a solution to the Weak Maximality Question for the class of Aronszajn trees. Let us call a wide $\aleph_2$-Aronszajn-tree T *special* if there is $f : T \to \omega_1$ such that $t < t'$ always implies $f(t) \neq f(t')$. Consistency of a weakly compact cardinal implies the consistency of $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} > \aleph_2 +$ every wide $\aleph_2$-Aronszajn-tree is special [15]. In this model there is a tree T $\leq$-above all $\aleph_2$-Aronszajn-trees such that T has no $\aleph_2$-branches. Here $|T| > \aleph_2$, so again T is not an answer to the Weak Maximality Question for wide $\aleph_2$-Aronszajn trees. Our Theorem 1 gives a positive solution to the (full) Maximality Question for wide κ-Aronszajn trees, κ a successor of a regular cardinal $> \aleph_0$. As we see below, it is impossible to combine this with GCH.

Both $T \leq^* T'$ and $T \leq T'$ measure in their own ways how big the trees T and T' are with respect to each other. If B_{ω^*} denotes the tree consisting of the single branch of length ω, then $B_\alpha \leq B_{\omega^*}$ holds for all α but of course $B_\alpha \not\leq^* B_{\omega^*}$ when $\alpha > 1$. Thus B_{ω^*} is $\leq$-above a proper class of non-$\equiv$-equivalent trees. There can be only $2^{|T|}$ trees $\leq^*$-below a given tree T, up to $\equiv^*$. This illustrates the different senses in which $\leq$ and $\leq^*$ measure the bigness of trees.

A still further ordering of Aronszajn trees is the following: If T is an Aronszajn tree and $C \subseteq \omega_1$, then we use $T \restriction C$ to denote the suborder of T consisting of nodes in T the height of which is in C. Suppose T and T' are Aronszajn trees. We say that a partial map $\pi : T \to T'$ is an embedding (or an isomorphism) *on a club* if there is a club $C \subseteq \omega_1$ such that π is an embedding (or respectively an isomorphism) $T \restriction C \to T' \restriction C$. It follows from the Proper Forcing Axiom that any two Aronszajn trees are isomorphic on a club [1, 13].

The following useful operation on trees is due to Kurepa [14]: If T is a tree, let $\sigma(T)$ be the tree of ascending chains in T, ordered by end-extension. For well-founded trees this is like the successor function on ordinals in the sense that $\sigma(B_\alpha) \equiv B_{\alpha+1}$. It is easy to see that if T is any tree, then $T < \sigma(T)$. Moreover, if T has no branches of length κ, neither has $\sigma(T)$. So from the point of view of lengths of branches $\sigma(T)$ is similar to T. However, it is perfectly possible that $|T[< |\sigma(T)|$. For example, if every node in T splits, then $|\sigma(T)| \geq 2^\omega$.

The σ-operation shows that if $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$, the class $\mathcal{T}_\alpha$ does not have a $\leq$-maximal element. So in that case even the Weak Maximality Question has a negative answer for the class $\mathcal{T}_\alpha$. In consequence, $\mu^{<\mu} > \mu$ holds in the final model of our Theorem 1.

If $A \subseteq \omega_1$ is co-stationary, let $T(A)$ be the tree of closed increasing sequences of elements of A. The class of such trees $T(A)$ is an interesting subclass of trees without uncountable branches. A tree without uncountable branches which is of cardinality $\leq 2^\omega$ and $\leq$-above all such $T(A)$ is known as a *Canary Tree*. The existence of Canary Trees is independent of $ZFC + GCH$ [17, 7]. Assuming CH, a Canary Tree is perhaps not maximal in the entire class of trees in $\mathcal{T}_1$ but it still $\leq$-majorises the large class of trees of the form $T(A)$.

Trees as game clocks

We already alluded to the fact that, assuming CH, Canary Trees can be used to show that the isomorphism class of a particular structure, in this case the free Abelian group of cardinality $\aleph_1$, is Δ^1_1 in the generalized Baire space $\omega_1^{\omega_1}$. This is an example of the use of trees as clocks in games in the way we now describe. A maximal tree would represent a kind of universal clock. To see what this means, suppose δ is an ordinal and G is a game of length δ between I and II in which I and II produce a δ-sequence of elements of a fixed set M, alternating moves, I starting each round. We fix a set $W \subseteq M^\delta$ and say that II wins if the sequence played is in W. Otherwise I wins. We assume the game is closed in the sense that if $s \notin W$ then there is an initial segment s' of s such that no extension of s' is in W. If T is any tree (of height δ), we can define a new game G_T, a kind of approximation of G, as follows. Every time I moves in G he also picks a node t in T in such a way that t is above all nodes he has picked during previous rounds of G. If he cannot pick such a t then he loses. Otherwise the game is played as G. Clearly, if I has a winning strategy in G_T, he has also in G. The role of T in G_T is to make it harder for I to win. If T is well-founded, player I can only win G_T if he can win G in finitely many moves but he does not have to tell in advance how many moves he needs in order to win. If T has no branches of length δ, player I can only win G_T if he can win G in $< \delta$ moves but, again, he does not have to tell in advance how long δ'-sequence, $\delta' < \delta$, of moves he needs in order to win. He can change his mind about this during the game.

The following implications are immediate:

1. If II has a winning strategy in $G_{T'}$ and $T \leq T'$, then II has a winning strategy in G_T.
2. If I has a winning strategy in G_T and $T \leq T'$, then I has a winning strategy in $G_{T'}$.
3. If II has a winning strategy in G_T and I has a winning strategy in $G_{T'}$, then $T \leq T'$.

These implications emphasise the role of maximal trees for the games G_T. Let us see how this works, first on a general level and then more specifically. Let $\mathcal{C}_G$ be the class of trees T such that II has a winning strategy in G_T. If II has a winning strategy even in the non-approximated game G, the class $\mathcal{C}_G$ is simply the class of all trees. The other extreme is that $\mathcal{C}_G = \emptyset$, which happens if $W = \emptyset$. Suppose S is $\leq$-above all trees in $\mathcal{C}_G$. Then $\sigma(S) \notin \mathcal{C}_G$. So, maximality of the tree gives us negative information about winning strategies of II. Let $\mathcal{C}'_G$ be the possibly bigger class of trees T such that I does not have a winning strategy in G_T. Again, $\mathcal{C}'_G$ may be the class of all trees and it is also possible that $\mathcal{C}'_G = \emptyset$. Suppose S' is $\leq$-above all trees in $\mathcal{C}'_G$. Then $\sigma(S') \notin \mathcal{C}'_G$ i.e. I has a winning strategy in $G_{\sigma(S')}$. So, maximality of the tree gives us positive information about winning strategies of I.

A particular closed game of interest in this connection is the transfinite EF-game. Suppose M and N are models of the same vocabulary, $|M| = |N| = \delta$ and $M \not\cong N$. Let τ be the canonical enumeration strategy (i.e. I enumerates $M \cup N$) of I in the EF-game $\mathrm{EF}^\delta(M,N)$ of length δ on M and N such that both players are allowed to play a sequence of $< |\delta|$ elements at a time. Because we assume $M \not\cong N$, τ is a winning strategy of I. The pairs (T,T'), $T \leq T'$, of trees such that Player II has a winning strategy in $\mathrm{EF}^\delta(M,N)_T$ but Player I has a winning strategy in $\mathrm{EF}^\delta(M,N)_{T'}$ give information about how far or close M and N are from being isomorphic. Such pairs outline a kind of boundary where advantage in the game $\mathrm{EF}^\delta(M,N)$ moves from Player II to Player I. Every tree with a branch of length δ is above the boundary. If $\delta = \omega$, the boundary is (up to $\equiv$) exactly one tree, namely B_α for some (unique) countable ordinal α. If $\delta > \omega$, the boundary may be quite wide. Let us assume $\delta = \omega_1$. If the first order theory of M is classifiable in the sense of stability theory, the boundary lies between well-founded trees and non-well-founded trees [19]. In the opposite case the boundary may be quite high in the class of trees without uncountable branches. For models of size $\aleph_1$ of unstable theories it is above any tree in $\mathcal{T}_1$, if CH is assumed [11].

> **Open Question:** Are there, for every tree $T \in \mathcal{T}_1$ non-isomorphic models M and N of cardinality $\aleph_1$ such that Player II has a winning strategy in $\mathrm{EF}^{\omega_1}(M,N)_T$?

A positive answer is known only for the extremely simple trees which consist of countable branches bunched together at the root [21]. A positive answer follows also from CH [11]. If there is a weakly maximal tree T in $\mathcal{T}_1$, solving the above question for T gives automatically a positive answer for all trees in $\mathcal{T}_1$.

The analogue of the Scott height of a countable model in this context is the following, introduced in [11]: A tree T without branches of length ω_α is called a *universal non-equivalence tree* for a model M of cardinality $\aleph_\alpha$ if for all models N of cardinality $\aleph_\alpha$ in the same vocabulary, if $M \not\cong N$, then Player I has a winning strategy in $\mathrm{EF}^{\omega_\alpha}(M,N)_T$. For example, a Canary Tree is (if it exists) a universal non-equivalence tree for the free Abelian group of cardinality $\aleph_1$. A tree T without branches of length ω_α is called a *universal equivalence tree* for a model M of cardinality $\aleph_\alpha$ if for all models N of cardinality $\aleph_\alpha$ in the same vocabulary, if Player II has a winning strategy in $\mathrm{EF}^{\omega_\alpha}(M,N)_T$, then $M \cong N$. If $\alpha = 0$, every countable model has a universal non-equivalence tree $B_{\alpha+1}$ and a universal equivalence tree B_α, where α is the Scott height of the model. For uncountable models the existence of such universal trees depends on stability theoretic properties of the first order theory of the model [11, 8, 9, 10]. By and large, models whose first order theory is unstable have no universal equivalence tree [11]. Models whose first order theory is superstable, NOTOP and OTOP, have a universal equivalence tree [11].

This paper continues in spirit [4] but is self-contained and can be read without knowledge of [4]. Our main result improves the main result of [4] by extending the maximality from narrow κ-Aronszajn trees to wide κ-Aronszajn trees. This leaves still open the possibility of having a narrow κ-Aronszajn tree which is maximal with respect to wide κ-Aronszajn trees under strict order preserving homomorphisms.

An outline of the paper

After some preliminaries in Section 2, we use in Section 3 a weakly compact cardinal κ to force a wide $\aleph_2$-Aronszajn tree T by Levy-collapsing κ to $\aleph_2$. The tree T is the tree that will be the desired maximal tree in the final model. The levels of T are sufficiently collapse-generic to permit the wide $\aleph_2$-Aronszajn trees arising in the construction to be embedded into T. We then define in Section 4 a σ-closed countable support iteration of length $\aleph_3$ of forcing with side conditions designed by means of an appropriate book-keeping to force for every wide $\aleph_2$-Aronszajn tree S an embedding $S \to T$. Naturally, we have to make sure $\aleph_2$ is not collapsed during this forcing. Section 5 is devoted to showing that our iterated forcing has the right kind of strong properness to guarantee the κ^+-chain condition and thereby the preservation of $\aleph_2$. We have to also make sure that our tree T will not acquire a long branch during the iteration. This is shown in Section 6. Theorem 1 is then proved. We conclude this work with a short open problems section 7.

Our methodology: Pairwise strong properness

When we want to get the consistency of the existence of a maximal wide κ-Aronszajn tree, we face the challenges of preserving κ, and of showing that the intended universal tree does not obtain a cofinal branch by the iteration. We deal with the former challenge by maintaining that the forcing is strongly proper with respect to sufficiently many structures of cardinality ω_1. Proper forcing methods involving specializing Aronszajn trees has been used in [3]. The transition to wide trees and tree embeddings requires the development of a new type of argument for maintaining strong properness, which is developed in Section 5 of this paper. In addition, having no cofinal branch is a typical example of a second order property of an object that is supposed to be preserved under the iteration.

The concept that turned out to be useful for this is the following strengthening of the notion of strongly generic conditions.

Definition 2. A forcing notion $\mathbb{P}$ is said to be **pairwise strongly proper** with respect to the structure M if there is an M residue function $p \mapsto [p]_M$ such that if $[p]_M = [q]_M$ and $w \in M \cap \mathbb{P}, w \leq [p]_M$, then there are $p' \leq p, w$ and $q' \leq q, w$ such that $[p']_M = [q']_M$.

In Section 6 we further develop this notion, prove it is satisfied by our iteration, and use it to prove that our universal tree remains Aronszajn in the final generic extension.

To secure these properties we introduce a forcing with certain special features:

- The original object we intend to be universal is highly generic. Specifically, a key requirement of each individual poset is that it whenever it maps a node s at level $\alpha < \kappa$ of a given tree S to a node t in the intended universal tree then the (collapse induced) local branch b_t below t is mutually generic from the generic information the local branch b_s below s.

- We make use of substructures M as "side-conditions" to guide the generically constructed embeddings. It is crucial that the chosen structures reflect second order assertions about the objects involved in the forcing. The existence of such structures requires the involvement of large cardinals.

We adopt the following general schema for proving the consistency of the existence of a universal object of cardinality κ in a class of structures satisfying some second order sentence Φ. The scheme consists of:

1. Force an object A intended to be the universal object for the property Φ.

2. By dovetailing, iterate forcings which embed each individual structure satisfying Φ into A.

3. Show that the iteration is proper (or strongly proper) so that we do not collapse the relevant cardinals.

4. Show that A satisfies Φ after the iteration by proving the iteration satisfies a certain variation of the pairwise strong properness property.

2 Preliminaries

Trees: Preliminaries

A tree $(T, <_T)$ is a partial ordered set with a minimal element and with the property that for every $t \in T$, the set of its $<_T$-predecessors $b_t = \{\bar{t} \in T \mid \bar{t} <_T t\}$ is well-ordered by $<_T$. We refer to b_t as the *branch below* t. For an ordinal α, the α-th level of T, denoted $\mathrm{Lev}_\alpha(T)$ is the set of all $t \in T$ so that b_t has ordertype α in $<_T$. The union $\bigcup_{\alpha'<\alpha} Lev_{\alpha'}(T)$ is denoted by $T_{<\alpha}$. The height of the tree T is the minimal κ such that $Lev_\kappa(T) = \emptyset$. Let T be a tree of height κ. We say that T is *narrow* if $|Lev_\alpha(T)| < \kappa$ for every $\alpha < \kappa$. Otherwise, we say T is *wide*. A subset $b \subseteq T$ is a cofinal branch if it is well ordered by $<_T$ and has order-type κ. We say that T is κ-Aronszajn if it has no cofinal branches. If M is a transitive set that is closed under the tree order $<_T$, and $t \in T \setminus M$, we define the *exit node* $e_T(t, M)$ of t from M to be the $<_T$-minimal node $e \in b_t \cup \{t\}$ that does not belong to M.

Weakly compact cardinals: Preliminaries

A cardinal κ is weakly compact if for every $B \subseteq V_\kappa$ and every Π^1_1 statement ψ of $(V_\kappa, \in, B)$ the set

$$A_\psi = \{\alpha < \kappa \mid (V_\alpha, \in, B \cap V_\alpha) \models \psi\}$$

is nonempty. It follows from the definition that the collection of sets

$$\{A_\psi \mid \psi(V_\kappa, \in, B) \models \psi,\ \psi \text{ is } \Pi^1_1 \text{ and } B \subseteq V_\kappa\}$$

generates a κ-complete normal filter on κ, denoted by $\mathcal{F}_{WC}$.

Definition 3. (Reflecting sequence of structures and associated function)
Let $\theta \geq \kappa^{++}$ be a regular cardinal and $<_\theta$ a well-ordering of H_θ. For every $P \in H_\theta$ we define the reflecting sequence of P,

$$\vec{M}^P = \langle M^P_\alpha \mid \alpha \in \mathrm{dom}(\vec{M}^P)\rangle$$

to consist of all Skolem-hull substructures M^P_α of the form $M^P_\alpha = Hull^{(H_\theta, \in, <_\theta, P)}(\alpha)$ with the following properties:

- $M^P_\alpha \cap V_\kappa = V_\alpha$,

- for every $Q \subseteq V_\kappa$ in M^P_α and a Π^1_1 statement of $(V_\kappa, \in, Q)$, if $(V_\kappa, \in, Q) \models \psi$ then $(V_\alpha, \in, Q \cap V_\alpha) \models \psi$.

It follows from a standard argument that $\mathrm{dom}(\vec{M}^P)$ belongs $\mathcal{F}_{WC}$ for every $P \in H_\theta$.

Levy Collapse: Preliminaries

Let $\mathbb{P} = \mathrm{Coll}(\omega_1, <\kappa)$ be the Levy-collapse poset. Conditions $p \in \mathbb{P}$ are countable partial functions $p : \omega_1 \times \kappa \to \kappa$ with the property that $p(\nu, \alpha) < \alpha$ for every $(\nu, \alpha) \in \mathrm{dom}(p)$. Let $G \subseteq \mathbb{P}$ be a generic filter. For each $\alpha < \kappa$ let $f^G_\alpha : \omega_1 \to \alpha$ be given by $f^G_\alpha(\nu) = \alpha'$ iff there is $p \in G$ and $p(\nu, \alpha) = \alpha'$. We refer to f^G_α as the collapse generic surjection from ω_1 onto α that is derived from G. By a well-known argument, $\mathbb{P}$ is isomorphic to any number $\tau \leq \kappa$ of copies of itself. Fix an isomorphism between $\mathbb{P}$ and $\kappa \times \kappa \times \kappa$ copies of itself,

$$\mathbb{P} \cong \prod_{(\eta,\beta,\alpha)\in\kappa^3} \mathbb{P}(\eta, \beta, \alpha)$$

i.e., $\mathbb{P}(\eta, \beta, \alpha) = \mathrm{Coll}(\omega_1, <\kappa)$ for all $\eta, \beta, \alpha \in \kappa$. The isomorphism breaks a generic filter $G \subseteq \mathrm{Coll}(\omega_1, <\kappa)$ to mutually generic filters

$$\langle G(\eta, \beta, \alpha) \mid (\eta, \beta, \alpha) \in \kappa^3 \rangle,$$

$G(\eta, \beta, \alpha) \subseteq \mathbb{P}(\eta, \beta, \alpha)$. For each $\tau < \kappa$, let

$$f^{G(\eta,\beta,\alpha)}_\tau : \omega_1 \to \tau$$

denote the collapse generic surjection from ω_1 to τ, derived from $G(\eta, \beta, \alpha)$.

Let κ be a cardinal and X be a set of cardinality κ (e.g., $\kappa^3 = \kappa \times \kappa \times \kappa$). We say that two functions $h, h' : X \to \kappa$ *disagree almost everywhere* if

$$|\{x \in X \mid h(x) = h'(x)\}| < \kappa.$$

3 Building the Wide Tree T

In this section, we construct a wide tree T in a collapse generic extension $V[G]$, $G \subseteq \mathrm{Coll}(\omega_1, <\kappa)$. This will be the maximal tree in the final model.

We want to define a wide tree T that will provide very generic branches over any small part of it. Our forcing iteration for embedding trees on κ into T will have to deal with κ^+ many trees $\langle S_\eta | \eta < \kappa^+ \rangle$. Ideally, each S_η will have its own part of T into which it will be embedded. This cannot be accommodated because T is supposed to be of size κ. So the next best thing is to associate with each of the $\eta < \kappa^+$ a function h_η which for level $\alpha < \kappa$ picks a part of the tree T into which the α'th level of S_η is to be embedded, such that for $\eta \neq \eta'$ the parts associated with S_η and $S_{\eta'}$ are eventually disjoint. So the α'th level of T will be made up of κ parts according to the value of h_η outside α.

For this reason, we construct the wide tree T to be a union of wide trees T^h, for functions $h : \kappa \to \kappa$ in V, where each T^h is continuously constructed from h in the sense that for every $\alpha < \kappa$, $T^h_{<\alpha}$ will be determined from $h \restriction \alpha$ and therefore also denoted $T^{h\restriction\alpha}$.

Fix an enumeration $\langle g_\gamma \mid \gamma < \kappa\rangle$ of ${}^{<\kappa}\kappa$.
A key ingredient of the construction of the tree T^h is that its local branches are generically independent. More precisely, considering all $\alpha < \kappa$ of uncountable cofinality, and $t \in \mathrm{Lev}_\alpha(T^h)$, their associated branches $b_t = \{t' \in T^h_{<\alpha} \mid t' <_{T^h} t\}$ are generically independent of each other in the sense that the parts of the collapse generic filter G that is required to determine their identity are independent. To do this, we associate to each node $t \in \mathrm{Lev}_\alpha(T^h)$ four parameters $(\alpha, \beta, \gamma, \delta) \in \kappa^4$ called the *collapse index of* t and use them to determine a segment of G that will define b_t.

We can now give the construction of the wide tree $T^h \in V[G]$ for a function $h : \kappa \to \kappa$ in V. The domain of T^h is the set of triples $\{(\alpha, \nu, h \restriction \alpha) \mid \alpha, \nu < \kappa\}$, where α is the identifier of the level

$$\mathrm{Lev}_\alpha(T^h) = \{(\alpha, \nu, h \restriction \alpha \mid \nu < \kappa\},$$

and $h \restriction \alpha$ is an indicator of the function part that has been used to construct the tree up to and including level α.

Convention 4. *When there is no risk of confusing the identity of the function h, we shall omit the third index of nodes in T^h and identify the domain of T^h with $\kappa \times \kappa$, and $T^h_{<\alpha}$ with $\alpha \times \kappa$.*

The ordering $<_{T^h}$ is constructed level-by-level and makes use of a fixed isomorphisms between $\mathbb{P} = \mathrm{Coll}(\omega_1, < \kappa)$ and κ^3 many copies of itself,

$$\prod_{(\alpha,\beta,\gamma)\in\kappa^3} \mathbb{P}(\alpha, \beta, \gamma)$$

with the conventions given in Section 2 above.

We maintain an inductive assumption that the restriction of $<_{T^h}$ to $T^h_{<\alpha}$ depends only on the part $h \restriction \alpha$ (namely, if $h_1 \restriction \alpha = h_2 \restriction \alpha$ then $T^{h_1}_{<\alpha} = T^{h_2}_{<\alpha}$) and that it belongs to the intermediate extension $V[G \restriction \alpha \times \kappa \times \kappa]$, generic over V for the partial product poset

$$\mathbb{P} \restriction (\alpha \times \kappa \times \kappa) = \prod_{(\alpha',\beta,\gamma)\in\alpha\times\kappa^2} \mathbb{P}(\alpha', \beta, \gamma).$$

Suppose that $<_{T^h} \restriction \alpha \times \kappa$ has been defined for some $\alpha < \kappa$.

If $\alpha = \alpha' + 1$ is a successor ordinal then we define the new level α of T^h by adding κ many successive nodes above each $t' \in \{\alpha'\} \times \kappa = \mathrm{Lev}_{\alpha'}(T^h)$. We use a pairing bijection $\langle , \rangle : \kappa \times \kappa \to \kappa$ to enumerate the $\kappa \times \kappa$ nodes at level α in a

κ enumeration identified with $\{\alpha\} \times \kappa$.

Suppose that $\alpha < \kappa$ is a limit ordinal. To define the extension of $<_{T^h}$ to $(\alpha+1) \times \kappa$ it suffices to assign each $t \in \{\alpha\} \times \kappa$ a cofinal branch $b_t \subseteq T^h_{<\alpha}$, as we can then define $t' <_{T^h} t$ if and only if $t' \in b_t$.

If $\mathrm{Cof}(\alpha) = \omega$ and $\langle \alpha_n \mid n < \omega \rangle$ is an increasing cofinal sequence in α, then each cofinal branch $b \subseteq T^h_{<\alpha}$ is determined by its ω-subseqence $\langle b(\alpha_n) \mid n < \omega \rangle \in (\alpha \times \kappa)^\omega$. As $|(\alpha \times \kappa)^\omega| = \kappa$, we enumerate all cofinal branches in $T^h_{<\alpha}$, $\langle b'_\nu \mid \nu < \kappa \rangle$, and for $t = (\alpha, \nu) \in \{\alpha\} \times \kappa$, define $b_t = b'_\nu$.

Suppose that $\alpha < \kappa$ is a limit ordinal of uncountable cofinality. The following definition will be used to define the cofinal branches $b \subseteq T^h_{<\alpha}$ that will be extended to nodes at level α.

Definition 5. Suppose that T' is a σ-closed normal tree on $\alpha \times \beta$, $f : \omega_1 \to \delta$ with $\delta \geq \alpha \cdot \beta$ and for every $\mu < \delta$, $f^{-1}(\mu) \subseteq \omega_1$ is unbounded.

1. Define the branch $b^f \subseteq T'$ determined by f to be the sequence $\langle t_i \mid i < \omega_1 \rangle$ defined as follows.

   To define t_0 we look at the minimal $j < \omega_1$ for which $f(j) < \delta$ is of the form $f(j) = \alpha \cdot \beta_0 + \alpha_0 < \alpha \cdot \beta$. We then set $t_0 = (\alpha_0, \beta_0)$.

   Suppose that $\langle t_i \mid i < i^* \rangle$ has been defined. If i^* is a limit ordinal then t_{i^*} is the limit of the countable sequence $\langle t_i \mid i < i^* \rangle$. Otherwise $i^* = i + 1$. Define $t_{i^*} = (\alpha^*, \beta^*)$ by looking at $j^* < \omega_1$ minimal such that $f(j^*) = \alpha \cdot \beta^* + \alpha^* < \alpha \cdot \beta$, and $t_i <_{T'} (\alpha^*, \beta^*)$ for all $i < i^*$. We then set $t_{i^*} = (\alpha^*, \beta^*)$.

2. If $q = f \restriction \nu : \nu \to \delta$ is an initial segment of f, then q naturally determines an initial sequence $\langle t_i \mid i \leq i_q \rangle$ of b^f which has a maximal element t_{i_q}. We denote the last node t_{i_q} by $\pi_q(b^f)$ and call it the *projection* of b_f determined by q.

The next two lemmas follows from a standard density argument and the definition of cofinal branches b_f (5).

Lemma 6. *If $f : \omega_1 \to \delta$ is a $\mathrm{Coll}(\omega_1, \delta)$-generic over a model V' that has T' then $b^f \subseteq T'$ is a cofinal branch i§n T'.*

Lemma 7. *Suppose that $T' \in V'$. Let $\mathbb{Q} = \prod_{n<\omega} \mathrm{Coll}(\omega_1, \delta_n)$ be the product of collapse posets with all $\delta_n \geq \alpha \cdot \beta$. Let $\langle \dot{f}_n \mid n < \omega \rangle$ be the $\mathbb{Q}$-name for the sequence of generic collapse functions $f_n : \omega_1 \to \delta_n$, and $\langle \dot{b}^{f_n} \mid n < \omega \rangle$ be the corresponding names of generic cofinal branches b^{f_n} in T'.*

For every condition $\vec{q} = \langle q_n \mid n < \omega \rangle \in \mathbb{Q}$ and a sequence of nodes $\langle t_n \mid n < \omega \rangle$ in T' such that $\pi_{q_n}(\dot{b}^{f_n}) <_{T'} t_n$, there is some $\vec{q^*} = \langle q^*_n \mid n < \omega \rangle$ extending $\vec{q}$ so that $\pi_{q^*_n}(\dot{b}^{f_n}) = t_n$ for each n.

We shall use Definition 5 to determine the branches b_t for $t \in Lev_\alpha(T^h)$ by choosing subtrees T' of $T^h_{<\alpha}$ and using collapse generic functions f to form branches b^f through T'. In this setup, T' belongs to the intermediate extension V' of V by $\mathbb{P}' = \mathbb{P} \restriction \alpha \times \kappa \times \kappa$. This means that the tree projection maps $\dot\pi_p(t)$ from definition 5 will be $\mathbb{P}'$-names of nodes in $T^h_{<\alpha}$. We make the following observation about the nature of the construction that follows from this setup.

Lemma 8. *Suppose that T' is $\mathbb{P}'$-name of a tree on $\alpha \times \beta$ for some poset $\mathbb{P}'$, $\delta \geq \alpha \cdot \beta$ is an ordinal, and $\dot f$ is a $\mathrm{Coll}(\omega, \delta)$-name for the generic collapse. Let b^f be the $\mathbb{P}' \times \mathrm{Coll}(\omega_1, \delta)$-name for the associated generic branch through T', and $q \mapsto \dot\pi_q(b^f)$ denote the $\mathbb{P}'$-name for the projection assignment to conditions $q \in \mathrm{Coll}(\omega, \delta)$. Suppose that $(p', q) \in \mathbb{P}' \times \mathrm{Coll}(\omega, \delta)$ is such that for some $t^* \in T'$,*

$$p' \Vdash_{\mathbb{P}'} \dot\pi_q(b^f) = \check t^*.$$

For every $t' \in T'$ there is an extension $q' \leq q$ such that

$$p' \Vdash_{\mathbb{P}'} \text{ if } t' >_{T'} \check t^* \text{ then } \pi_{q'}(b^f) = t'.$$

We use definition 5 above to define the branches b_t for $t \in \mathrm{Lev}_\alpha(T^h)$.

Definition 9. Call a quadruple $(\alpha, \beta, \gamma, \delta) \in \kappa^4$ *valid for* $\mathrm{Lev}_\alpha(T^h)$ if the following conditions hold:

- The subset $\alpha \times \beta$ of $T^h_{<\alpha}$ is closed under $<_{T^h}$,[2]
- $\delta \geq \alpha \cdot \beta$, and
- $g_\gamma = h \restriction (\alpha + 1)$.[3]

To each valid quadruple $(\alpha, \beta, \gamma, \delta)$, the function $f = f_\delta^{G(\alpha,\beta,\gamma)} : \omega_1 \to \delta$ satisfies the assumption of Lemma 6 with respect to the tree $T' = \alpha \times \beta \subseteq T^h_{<\alpha}$. Since the tree ordering on $T^h_{<\alpha}$ is assumed to be defined in $V' = V[G \restriction \alpha \times \kappa \times \kappa]$ and $f_\delta^{G(\alpha,\beta,\gamma)}$ is $\mathrm{Coll}(\omega_1, \delta)$ generic over V', it follows from Lemma 6 that the branch $b^{f_\delta^{G(\alpha,\beta,\gamma)}}$ is cofinal in $T^h_{<\alpha}$.

It is clear that the set of valid quadruples for $\mathrm{Lev}_\alpha(T^h)$ has size κ. Let $\langle(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu) \mid \nu < \kappa\rangle$ be a definable enumeration of all valid quadruples for $\mathrm{Lev}_\alpha(T^h)$.

Definition 10. (Collapse index and Collapse height)
Define for each node $t = (\alpha, \nu, h \restriction \alpha) \in \mathrm{Lev}_\alpha(T^h)$ its *collapse-index* to be $(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu)$, its associated function $f_t = f_{\delta_\nu}^{G(\alpha_\nu,\beta_\nu,\gamma_\nu)}$, and set $b_t = b^{f_t}$. The *collapse height* of t is the triple $(\alpha_\nu, \beta_\nu, \delta_\nu)$.

Therefore given $t = (\alpha, \nu, h \restriction \alpha) \in T^h$ of collapse height (α, β, δ), one can derive its collapse-index $(\alpha, \beta, \gamma, \delta)$ as $g_\gamma = h \restriction (\alpha + 1)$.

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[2] I.e., $T^h \restriction (\alpha \times \beta)$ is a subtree of $T^h_{<\alpha}$.
[3] g_γ is the γ-th function in the enumeration $\langle g_\gamma \mid \gamma < \kappa\rangle$ of ${}^{<\kappa}\kappa$.

Remark 11. *The assignment of $t \in T$ to its collapse index $(\alpha, \beta, \gamma, \delta) \in \kappa^4$ is injective. Indeed, if $t = (\alpha, \nu, h \restriction \alpha)$ then γ recovers $h \restriction \alpha$ as $g_\gamma = h \restriction (\alpha + 1)$, and the valid sequence $(\alpha, \beta, \gamma, \delta)$ recovers ν as the index $(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu) = (\alpha, \beta, \gamma, \delta)$ in the enumeration of all sequences which are valid for $T^{h \restriction \alpha}_{<\alpha}$.*

This concludes the construction of the tree T^h. We state a number of basic properties of T^h that follow from its construction. The first is an immediate consequence of our level-by-level definition of T^h.

Lemma 12. *Suppose that $M_\alpha \prec (H_{\kappa^{++}}, h)$ is an elementary substructure with ${}^\omega M_\alpha \subseteq M_\alpha$ and $M_\alpha \cap \kappa = \alpha$ is a regular cardinal, then*

$$\mathbb{P} \cap M_\alpha = \prod_{(\alpha', \beta', \gamma') \in \alpha^3} \mathbb{P}(\alpha', \beta', \gamma') \cap V_\alpha$$

is a regular subforcing of $\mathbb{P}$, and $T^h \cap M_\alpha$ is forced to be equivalent to a $(\mathbb{P} \cap M_\alpha)$-name of a subtree of T^h.

Let $t \in \mathrm{Lev}_\alpha(T^h)$ be of collapse index $(\alpha, \beta, \gamma, \delta)$. Then b_t is a cofinal branch in the subtree $T^h \restriction (\alpha \times \beta)$ whose order is defined in the intermediate generic extension $V' = V[G \restriction \alpha \times \kappa \times \kappa]$. Working in V', definition 5 allows us to assign each $q \in \mathbb{P}(\alpha, \beta, \gamma)$ for which $\mathrm{dom}(q \restriction \{\delta\} = \{i \in \omega_1 \mid (\delta, i) \in \mathrm{dom}(q)\} < \omega_1$ a node $\pi_q(t) \in T^h \restriction (\alpha \times \beta)$ which is the maximal node forced by q to be in b_t. Back in V, $\pi_q(t)$ is a $\mathbb{P} \restriction (\alpha \times \kappa \times \kappa)$-name for a node in $T^h_{<\alpha}$ that is decided on a dense open set.

Definition 13. For each node $t \in \mathrm{Lev}_\alpha(T^h)$ with collapse index $(\alpha, \beta, \gamma, \delta)$, let $D(t) \subseteq \mathbb{P}$ be the set of conditions p so that $p \restriction \alpha \times \kappa \times \kappa$ decides the name of the node $\dot{\pi}_q(t) \in T^h_{<\alpha}$ where $q = p \restriction \{(\alpha, \beta, \gamma)\} \times \kappa$. Being decided by p, we denote it by $\pi_p(t)$.

Recall that our construction of T^h as limit stages of countable cofinality add a node about every cofinal increasing sequence of nodes. It follows from our construction of a tree T^h satisfies that that It is straightforward to verify that for each $t \in T^h$, the projection $\pi_p(t)$ is defined for almost every $p \in \mathbb{P}$.

Lemma 14. *$D(t)$ is dense and σ-closed for each $t \in T^h$.*

The following is straightforward application of Lemma 8 to countably many mutually generic branches.

Lemma 15. *Suppose that $\langle t_n \mid n < \omega \rangle$ is a sequence of nodes in T^h so that for each $n < \omega$, t_n is of collapse index $(\alpha_n, \beta_n, \gamma_n, \delta_n)$. For every condition $p \in \bigcap_n D(t_n)$ and a sequence of nodes $\langle t'_n \mid n < \omega \rangle$ such that for each $n < \omega$, $t'_n \in T^h \restriction (\alpha_n \times \beta_n)$. There is an extension $p' \leq p$, with the following properties:*

1. *p' and p are equal, except maybe at the collapse indices $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ of the points t_n.*
2. *for every $q \leq_{T^{h_n}} p'$, if $q \Vdash \dot{\pi}_p(t_n) \leq_{T^{h_n}} t'_n$ then $q \Vdash t'_n \leq_{T^{h_n}} \dot{\pi}_q(t_n)$.*

Taking another step, we form a stronger version of Lemma 15 that allows the nodes t_n to come from possibly different trees T^{h_i}, under the assumption that distinct functions $h_i \neq h_j$ disagree almost everywhere (i.e., agree only on a bounded set in κ). We formulate the statement in a specific way that best fits our later proofs (Specifically, in Lemma 49).

Lemma 16. *Let $H \subseteq \kappa^\kappa$ be a countable set of functions that pairwise disagree almost everywhere. Suppose that $\langle t_n \mid n < \omega\rangle$ is a sequence of nodes so that each t_n is an exit node from a structure $M_n \prec (H_{\kappa^{++}}, H)$ with ${}^\omega M_n \subseteq M_n$, and that the collapse-index $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ of t_n is outside M_n.*
For every sequence of nodes $\langle t'_n \mid n < \omega\rangle$ such that $t'_n \in T^{h_n} \cap M_n$ for some $h_n \in H$, and for each $n < \omega$, there is an extension $p' \leq p$ with the property that

1. *p' and p are equal, except maybe at the collapse indices $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ of the points t_n.*
2. *for every $q \leq_{T^{h_n}} p'$, if $q \Vdash \pi_p(t_n) \leq_{T^{h_n}} t'_n$ then $q \Vdash t'_n \leq_{T^{h_n}} \pi_q(t_n)$.*

4 The embedding poset

We fix in V a sequence of bijections $\vec{\psi} = \langle \psi_\tau \mid \tau < \kappa^+\rangle$, such that for each τ, $\psi_\tau : \kappa \to \tau$ is a bijection. We also fix a sequence of functions $\langle h_\tau \mid \tau < \kappa^+\rangle$, $h_\tau : \kappa \to \kappa$, which are pairwise almost everywhere disagree.

Definition 17. We say that a set $a \subseteq \kappa^+$ is *α-closed with respect to $\vec{\psi}$* for some $\alpha < \kappa$ if $\psi_\delta``\alpha \subseteq a$ for every $\delta \in a$.

We are going to define by induction a sequence of forcing notions $\mathbb{P}_\tau$ for $\tau < \kappa^+$, together with $\mathbb{P}_\tau$-names of wide trees $S_\tau \subseteq \kappa \times \kappa$ and sequences of structures $\vec{M}^\tau = \langle M^\tau_\alpha : \alpha \in \mathrm{dom}(\vec{M}^\tau)\rangle$. This is our iteration for the proof of Theorem 1.

Remark 18. *Before giving the exact definitions, we give a brief informal description of $\mathbb{P}_\tau$. $\mathbb{P}_0$ will be equivalent to the Levy collapse forcing $\mathrm{Coll}(\omega_1, <\kappa)$. For $\delta \geq 1$, $\mathbb{P}_\delta$ will consists of pairs $p = \langle f^p, N^p\rangle$ of countable sets, so that $f^p = \langle f^p_\gamma : \gamma \in \mathrm{supp}(p)\rangle$ is a sequence of functions, such that $\mathrm{supp}(p) \subseteq \delta$ is a countable set, and for each $\gamma \in \mathrm{supp}(p)$, f^p_γ is a countable partial function, which will be forced to be order preserving from the tree S_γ to T^{h_γ}. The tree (names) S_γ, $\gamma < \kappa^+$ will be chosen by a book-keeping function, which is planned to exhaust all wide trees S of a certain kind. The set N^p corresponds to the side condition part, which specifies a collection of structures M for which we would like to secure the existence of strong generic conditions. This is realized by having N^p consists of pairs (α, a_α) so that $\alpha < \kappa$ and $a_\alpha \subseteq \delta$ is a nonempty α-closed set with respect to $\vec{\psi}$ of size $|a_\alpha| \leq \alpha$. The role of (α, a_α) is to specify a set of $\gamma < \delta$ for which we would like $M^\gamma_\alpha \in \vec{M}^\gamma$ to have a strong generic condition (a.k.a. a master condition). For this, we add several natural requirements to the working parts f^p_γ. For example, we require that nodes $s \in \mathrm{dom}(f^p_\gamma) \cap M^\gamma_\alpha$ are mapped to $f^p_\gamma(s) \in T^{h_\gamma} \cap M^\gamma_\alpha$.*

We proceed with the complete recursive definition of $\mathbb{P}_\tau$, $\vec{M}^\tau$, $\tau < \kappa^+$. The definition will be given in steps, and require introducing a number of auxiliary definitions and notations. The auxiliary definitions will be extensively used to prove that $\mathbb{P}_\tau$ has the desired properties.

To start, we fix in advance a well-order $<_{H_{\kappa^{++}}}$ of $H_{\kappa^{++}}$, as well as a book-keeping function Ψ whose domain is the set of all posets $\mathbb{P} \in H_{\kappa^{++}}$ which preserve κ and satisfy κ^+.c.c, and $\Psi(\mathbb{P})$ is a $\mathbb{P}$-name of (wide) tree S on κ whose domain is $\kappa \times \kappa$.

Definition 19. ($\mathbb{P}_0$)
$\mathbb{P}_0$ is equivalent to the Levy collapse forcing. Formally, it consists of pairs $p = \langle f^p, N^p\rangle$ where $f^p = \langle f^p_0\rangle$ is a sequence with an element $f^p_0 \in \mathrm{Coll}(\omega_1, <\kappa)$, and $N^p = \emptyset$.

Suppose that $\mathbb{P}_\delta$ has been defined for every $\delta < \tau$ for some $1 \leq \tau < \kappa^+$. Before defining $\mathbb{P}_\tau$ we list seven inductive assumptions for $\mathbb{P}_\delta$, $\delta < \tau$.

Inductive Assumption I: For every $\delta < \tau$, $\mathbb{P}_\delta$ is a σ-closed poset of size $|\mathbb{P}_\delta| = \kappa$, and $\mathbb{P}_\delta \in H_{\kappa^{++}}$.

For each $\gamma < \delta$ let $\dot{S}_\gamma = \Psi(\mathbb{P}_\gamma)$ denote the $\mathbb{P}_\gamma$-name for a wide tree with domain $\kappa \times \kappa$ chosen by a fixed book-keeping function Ψ.

Let $\mathcal{A}^\gamma = \langle H_{\kappa^{++}}, \in, <_{H_{\kappa^{++}}}, \Psi, \mathbb{P}_\gamma, h_\gamma, \vec{\psi}\rangle$. Note that $S_\gamma = \Psi(\mathbb{P}_\gamma)$ is definable in this structure. Let $\vec{M}^\gamma = \langle M^\gamma_\alpha : \alpha \in \mathrm{dom}(\vec{M}^\gamma)\rangle$ is the associated sequence of Π^1_1-elementary substructures of $\mathcal{A}^\tau$ from Definition 3. As mentioned after the definition, $\mathrm{dom}(\vec{M}^\gamma)$ belongs to the weakly compact filter on κ, $\mathcal{F}_{WC}$.

Inductive Assumption II: For every $\delta < \tau$ and $p \in \mathbb{P}_\delta$, p is of the form $\langle f^p, N^p\rangle$ where

1. $f^p = \langle f^p_\gamma : \gamma \in \mathrm{supp}(p)\rangle$, a sequence whose domain $\mathrm{supp}(p) \subseteq \delta$ (called the *support of* p) is a countable set so that
   - $0 \in \mathrm{supp}(p)$ and $f^p_0 \in \mathrm{Coll}(\omega_1, <\kappa)$,
   - for each positive ordinal $\gamma \in \mathrm{supp}(p)$, $f^p_\gamma : \kappa \times \kappa \to T^{h_\gamma}$ is a partial countable function.
2. N^p is a countable set of pairs of the form (α, a_α), where $a_\alpha \subseteq \delta$ is a nonempty α-closed set (w.r.t $\vec{\psi}$) of size $|a_\alpha| \leq \alpha$. For each α there is at most one pair (α, a_α) in N^p.

Definition 20. 1. Let $p \in \mathbb{P}_\delta$, $\delta' < \delta$ and $M^{\delta'}_\alpha \in \vec{M}^{\delta'}$. We say that $M^{\delta'}_\alpha$ *appears* in p if $\delta' \in a_\alpha$ where $(\alpha, a_\alpha) \in N^p$.

2. We say that a condition $p \in \mathbb{P}_\delta$ is *amenable* to a structure $M^\delta_\alpha \in \vec{M}^\delta$ if $M^{\delta'}_\alpha$ appears in p for every $\delta' \in M^\delta_\alpha \cap \delta$ (i.e., $M^\delta_\alpha \cap \delta \subseteq a_\alpha$ where $(\alpha, a_\alpha) \in N^p$)

3. We say that M^δ_α is $\vec{M}^\delta$*-reflective* if $M^\delta_\alpha = N^\delta_\alpha \cap \mathcal{A}^\delta$ for some $N^\delta_\alpha \prec (H_{\kappa^{++}}, \mathcal{A}^\delta, \vec{M}^\delta)$.

Definition 21. For every $\delta < \delta^*$ and $p \in \mathbb{P}_{\delta^*}$, define $p \restriction \delta = \langle f^{p\restriction\delta}, N^{p\restriction\delta}\rangle$ by

$$f^{p\restriction\delta} = \langle f^p_\gamma : \gamma \in \operatorname{supp}(p) \cap \delta\rangle,$$

and

$$N^{p\restriction\delta} = \{(\alpha, a_\alpha \cap \delta) : (\alpha, a_\alpha) \in N^p \text{ and } a_\alpha \cap \delta \neq \emptyset\}.$$

Inductive Assumption III: For every $\delta < \delta^* < \tau$ and $p \in \mathbb{P}_{\delta^*}$, $p \restriction \delta \in \mathbb{P}_\delta$, and for every structure M^δ_α that appears in p, M^δ_α is $\vec{M}^\delta$-reflective and $p \restriction \delta$ is amenable to M^δ_α.

Inductive Assumption IV: For every $\delta < \tau$, $M^\delta_\alpha \in \vec{M}^\delta$ and $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, there is an extension $p' \leq p$ which is amenable to M^δ_α.

Definition 22. ($\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$)
For each $\delta < \tau$ let $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ be the $\mathbb{P}_\delta$-name of the sub-sequence of $\vec{M}^\delta$ consisting of all $M^\delta_\alpha \in \vec{M}^\delta$ which are $\vec{M}^\delta$-reflective and there is an M^δ_α-amenable condition p' in the (canonical $\mathbb{P}_\delta$-name) generic filter $\dot{G}(\mathbb{P}_\delta)$.

Lemma 23.

1. *$\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ is (forced to be) to have cardinality κ.*

2. *If p forces some M^δ_α to be in $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ then it forces $M^{\delta'}_\alpha$ to be in $\vec{M}^{\delta',\dot{G}(\mathbb{P}'_\delta)}$ for every $\delta' \in \delta \cap M^\delta_\alpha$.*

Proof. The first part is an immediate consequence of Inductive assumption IV. For the second item, note that for every $M^\delta_\alpha \in \vec{M}^\delta$, $\psi_\delta \in M^\delta_\alpha$ and therefore $M^\delta_\alpha \cap \delta = \psi_\delta``\alpha$. It follows that for every a which is α-closed (w.r.t $\vec{\psi}$) and $\delta \in a$, one must have $\delta \cap M^\delta_\alpha \subseteq a$. The statement of this part now follows from Inductive Assumption III. $\square$

Inductive Assumption V: The following requirement holds for every $\delta' < \delta < \tau$ and $p \in \mathbb{P}_\delta$ with $\delta' \in \operatorname{supp}(p)$:
For every $s_1 \neq s_2 \in \operatorname{dom}(f^p_{\delta'})$ there is some $s \in \operatorname{dom}(f^p_{\delta'})$ which is forced by $p \restriction \delta'$ to be the meet of s_1, s_2 in the tree order of $S_{\delta'}$, and $f^p_0 \in \operatorname{Coll}(\omega_1, <\kappa)$ forces that $f^p_{\delta'}(s)$ is the meet in $T^{h_{\delta'}}$ of $f^p_{\delta'}(s_1)$ and $f^p_{\delta'}(s_2)$.

Remark 24. *Inductive Assumption V implies $p \restriction \delta'$ forces that $f^p_{\delta'}$ is order preserving. This follows from the simple observation that a partial function $f : S \to T$ between two trees S, T, whose domain is closed under meets in S, mapping those meets to the meets of the images, is order preserving.*

The next inductive assumptions describe the connection/restrictions between the "working" parts f^p_δ in conditions p and the "side condition" parts M^δ_α that appear in p. We start with a brief discussion before giving the precise details.

Our main goal is to secure a strong properness property for a structure M^δ_α that appears in a condition p. As usual, strong properness will imply that the restriction of the generic embedding f_δ to M^δ_α will be generic over M^δ_α. It is therefore natural to include a restriction saying that for every $s \in \operatorname{dom}(f^p_\delta)$, $s \in M^\delta_\alpha$ if and only if $f^p_\delta(s) \in M^\delta_\alpha$. Now, to secure this property while allowing every node $s' \in S_\delta$ to be added to the domain of an extension, we impose a similar requirement for branches b_s of nodes $s \in \operatorname{dom}(f^p_\delta)$. Namely, we require that for every $(s,t) \in f^p_\delta$, $b_s \subseteq M^\delta_\alpha$ if and only if $b_t \subseteq M^\delta_\alpha$. We recall that by the construction of our trees T^{h_δ}, for every $t \in T^{h_\delta}$, the identity of $\beta < \kappa$ for which $b_t \subseteq M^\delta_\beta$ is determined from the collapse height $(\alpha', \beta', \gamma')$ of t (specifically, we need $\alpha', \beta' \leq \beta$).

We point out that the last restriction introduces the following additional complication: Say, $(s,t) \in f^p_\delta$ are outside of a structure M^δ_α which appears in p. By our strong properness aspirations for M^δ_α we would like our ability to extend the conditions $f^p_\delta \cap M^\delta_\alpha$ inside M^δ_α to be "independent" from considerations outside of M^δ_α. Because of this, we are at a risk of adding a new structure M^δ_β to the side condition part, when working inside M^δ_α that will violate the branch requirement coming from (s,t), which are outside of the structure. One approach to avoid such a problem will be to add a notion of excluded intervals to the side condition part (such as when adding a club to ω_1 with finite side conditions, as in Baumgartner, [2]) of the poset to exclude "problematic" structures M^δ_β below M^δ_α. We take here a similar and slightly more implicit approach to avoid this problem: We can use the function f^p_δ to produce excluded intervals, essentially by mapping a node $s' \in M^\delta_\beta$ to $f^p_\delta(s')$ outside of M^δ_β, but inside M^δ_α so that this restriction appears when we move to $f^p_\delta \cap M^\delta_\alpha$ and excludes adding M^δ_β to the side condition part. The last description can be seen as a motivation for the next definition, of a refined sub-sequence $\vec{M}^{\delta, G(\mathbb{P}_\delta), f_\delta} \subseteq \vec{M}^{\delta, G(\mathbb{P}_\delta)}$ given by a partial countable function f_δ.

Definition 25. $(\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta), f_\delta})$
Let $f_\delta : S_\delta \to T_\delta$ be a countable partial function. Define a $\mathbb{P}_\delta$-name for a sub-sequence $\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta), f_\delta}$ of $\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta)}$ by having $\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta), f_\delta}$ consists of all $M^\delta_\beta \in \vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta)}$ such that for every $(s,t) \in f_\delta$

1. $s \in M^\delta_\beta$ if and only if $t \in M^\delta_\beta$, and
2. s is an exit node from M^δ_β if and only if t is an exit node from M^δ_β.

Definition 26. $(\beta_{f_\delta}(s))$
For every condition $p \in \mathbb{P}_\delta$, node $s \in \operatorname{Lev}_\alpha(S_\delta)$ where $\alpha < \kappa$, and partial countable function $f_\delta : S_\delta \to T_\delta$ there is an extension $q \leq p$ which either forces some value, which we denote $\beta_{f_\delta}(s)$, to be the minimal $\beta < \kappa$ such that

$b_s \subseteq \alpha \times \beta$, $s \notin M^\delta_\beta$, and $M^\delta_\beta \in \vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta), f_\delta}$, or else forces that such β does not exist.

Remark 27. *Given a condition $p \in \mathbb{P}_\delta$, $\delta' < \delta$, and a node $s \in S_{\delta'}$, we can extend $p \restriction \delta'$ to determine the minimal $\beta < \kappa$ such that $b_s \subseteq \beta \times \beta$ and $s \notin \beta \times \beta$. This makes β a natural candidate for $\beta_{f_{\delta'}}(s)$ but not necessarily the correct one. We might have to increase it to some $\beta' \geq \beta$ since either $M^{\delta'}_\beta$ does not exist, or it does, but is not forced to be inside $\vec{M}^{\delta, G(\mathbb{P}_\delta), f^p_{\delta'}}$ Further extending $p \restriction \delta'$ there is no problem determining whether there is $\beta' < \kappa$ so that $b_s \subseteq \beta' \times \beta'$, $s \notin \beta' \times \beta'$, and (if exists) determine the minimal such β'.*

Definition 28. For every $\delta < \tau$ and $s \in S_\delta$, we define a $\mathbb{P}_\delta$-name of a structure M^s_1 to be M^δ_β which is the first in $\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta)}$ so that $s \in M^\delta_\beta$.

Definition 29. (s-knowledgeable conditions)
Let $\delta' < \delta$. We say that a condition $q \in \mathbb{P}_{\delta'}$ is *s-knowledgeable* (or knowledgeable about s) for some $s \in S_{\delta'}$ with respect to a partial countable function $f_{\delta'} : S_{\delta'} \to T_{\delta'}$, if

- q decides the identity of M^s_1,
- q decides if $\beta_{f_{\delta'}}(s)$ exists, and if so, determines its value.

Let $p \in P_{\delta+1}$, $s \in \mathrm{dom}(f^p_\delta)$, and $p_\delta(s) = t$. Suppose $(\eta, \delta) \in N^p$. Let $M = M^{\delta+1}_\eta$. Suppose that s is an exit node of M. Hence t is an exit node from M. Let β be minimal such that $b(s) \subseteq \alpha \times \beta$. Note that $\beta \leq \eta$. We require that the "slice" of t is β. (We assume that $p \restriction \delta$ determined β.)

Claim: If $M^* \in M$ such that s is an exit node of M^* then t is an exit node from M^*.

$\beta \subseteq M^*$, otherwise s can not be an exit node from it because $b(s)$ is cofinal in $\alpha \times \beta$. $t \notin M^*$ because $M^* \subseteq M$ and $t \notin M$. $b(t) \subseteq \alpha \times \beta$, hence $b(t) \subseteq M^*$.

Inductive Assumption VI: For every $p \in \mathbb{P}_\delta$ and $s \in \mathrm{dom}(f^p_{\delta'})$ for some $\delta' < \delta$ then

1. $p \restriction \delta'$ to be s-knowledgeable with respect to $f^p_{\delta'}$, and if $p \restriction \delta'$ forces $\beta = \beta_{f^p_{\delta'}}(s)$ exists then $M^{\delta'}_\beta$ appears in p.
2. Suppose that $M^{\delta'}_\alpha$ appears in p, then
   - $s \in M^{\delta'}_\alpha$ if and only if $f^p_{\delta'}(s) \in M^{\delta'}_\alpha$
   - s is an exit node from $M^{\delta'}_\alpha$ if and only if $f^p_{\delta'}(s)$ is.

and $\beta < \kappa$ such that $(\beta, a_\beta) \in N^p$ and $\delta' \in a_\beta$ we have $p \restriction \delta' \Vdash M^{\delta'}_\beta \in \vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta), f^p_{\delta'}}$.

Inductive Assumption VII: For every $p \in \mathbb{P}_\delta$ and $s \in \mathrm{dom}(f^p_{\delta'})$ for some $\delta' < \delta$, the collapse-index $(\alpha, \beta, \gamma, \delta)$ of $f^p_{\delta'}(s)$ is forced by $p \restriction \delta'$ to satisfy the following requirements:

1. α is the level of s,

2. $\beta = \beta_{f^p_{\delta'}}(s)$ if exists, and

3. $\gamma > (M_1^s \cap \kappa)$.

We are ready to define $\mathbb{P}_\tau$. We split the definition between the case τ is a limit ordinal, and $\tau = \delta + 1$ is a successor ordinal.

Definition 30. ($\mathbb{P}_\tau$ for a limit ordinal τ)
Conditions $p \in \mathbb{P}_\tau$ are all pairs $\langle f^p, N^p \rangle$ which satisfy the following requirements:

- $f^p = \langle f^p_\delta : \delta \in \mathrm{supp}(p) \rangle$ is a sequence with $\mathrm{supp}(p) \subseteq \tau$ is countable and $0 \in \mathrm{supp}(p)$,
- N^p is a countable set of pairs (α, a) satisfying $\alpha < \kappa$ and $a \in [\tau]^{\leq \alpha}$ nonempty and $(\vec{\psi}, \alpha)$-closed,
- For every $\delta < \tau$, $p \restriction \delta$ belongs to $\mathbb{P}_\delta$.

A condition $p' \in \mathbb{P}_\tau$ extends p, denoted $p' \leq p$, iff $p' \restriction \delta \leq_{\mathbb{P}_\delta} p \restriction \delta$ for all $\delta < \tau$.

Definition 31. ($\mathbb{P}_\tau$ for a successor ordinal τ)
Suppose that $\tau = \delta + 1$ is a successor ordinal. $\mathbb{P}_\tau = \mathbb{P}_{\delta+1}$ consists of all pairs $p = \langle f^p, N^p \rangle$ which satisfy the following conditions:

- $f^p = \langle f^p_\delta : \delta \in \mathrm{supp}(p) \rangle$ is a sequence with $\mathrm{supp}(p) \subseteq \delta + 1$ is countable and $0 \in \mathrm{supp}(p)$,
- N^p is a countable set of pairs (α, a) satisfying $\alpha < \kappa$ and $a \in [\tau]^{\leq \alpha}$ nonempty and $(\vec{\psi}, \alpha)$-closed, and whenever $\delta \in a$, $M^\delta_\alpha \in \vec{M}^\delta$ exists,
- $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$,
- If $\delta \in \mathrm{supp}(p)$ then $p \restriction \delta$ forces the statements in V,VI, and VII from inductive assumptions, having δ' is replaced with δ.

A condition $p' \in \mathbb{P}_{\delta+1}$ extends p if it satisfies the following requirements:

1. $p' \restriction \delta \leq_{\mathbb{P}_\delta} p \restriction \delta$,
2. $f^p_\delta \subseteq f^{p'}_\delta$,
3. for every $(a, \alpha) \in N^p$ there is some $a' \supseteq a$ such that $(a', \alpha) \in N^{p'}$.[4]

Lemma 32. *$\mathbb{P}_\tau$ satisfies inductive assumptions I-VII.*

[4] Equivalently, every $M^{\delta'}_\alpha$ that appears in p appears in p'.

Proof. **Assumption I**: It is clear that $|\mathbb{P}_\tau|$ has size κ and is σ-closed.

Assumption II: The structural assumptions of conditions $p \in \mathbb{P}_\tau$ are immediate consequences of the definition of $\mathbb{P}_\tau$.

Assumption III: The fact that $p \restriction \delta \in \mathbb{P}_\delta$ for every $p \in \mathbb{P}_\tau$ and $\delta < \tau$ is immediate from the definition of $\mathbb{P}_\tau$.

Assumption IV: Let $p \in \mathbb{P}_\tau \cap M^\tau_\alpha$ for some $M^\tau_\alpha \in \vec{M}^\tau$. Take $a = \psi_\tau``\alpha$. Then a is α closed since $\vec{\psi} \in M^\tau_\alpha$. Let $p' = \langle f^{p'}, N^{p'}\rangle$ where $f^{p'} = f^p$ and $N^{p'} = N^p \cup \{(\alpha, a)\}$. The fact that p' is a condition follows from the fact that $p \in M^\tau_\alpha$. This means that items 3,4 from IA V are not challenged by adding (a, α) to N^p. It is clear that p' is amenable for M^τ_α.

Assumptions V,VI,VII: If τ is limit then IA V,VI,VII are immediate. Suppose $\tau = \delta + 1$ is a successor ordinal. The fact that V,VI, and VII hold at $\delta' < \delta$ follows from the fact $p \restriction \delta \in \mathbb{P}_\delta$. The fact that V,VI,VII hold at δ follows from the definition of $\mathbb{P}_{\delta+1}$. □

Remark 33. *Let $p \in \mathbb{P}_\tau$ be a condition. Suppose that $s' \in \mathrm{dom}(f^p_\delta)$ for some $\delta \in \mathrm{supp}(p)$, and $s \in S_\delta$ are such that*

- *$p \restriction \delta$ decides the meet of s, s' in S_δ, denoted by $m(s, s') \in S_\delta$*
- *$m(s, s') \in \mathrm{Lev}_{\bar{\alpha}}(S_\delta)$ for some $\bar{\alpha} < \kappa$*
- *The collapse part f^p_0 of p determines the identity of a unique node below $f^p_\delta(s')$ at level $\bar{\alpha}$ to be some $\bar{t} \in \mathrm{Lev}_{\bar{\alpha}}(T)$.*

Although the assumptions do not imply $m(s, s')$ belongs to $\mathrm{dom}(f^{p'}_\delta)$ as we do not assume s does. However, the image $f^q_\delta(m(s, s'))$ of $m(s, s')$ in all possible extensions q of p with $m(s, s') \in \mathrm{dom}(f^q_\delta)$ is already decided by p to be $\bar{t}$. We call $\bar{t}$ the implicit image *of the meet $m(s, s')$ of s, s' as determined by p, and denote it by $t(s, s')$ or $t_p(s, s')$.*

Traces and simple amalgamations

We end this section with definitions of operations that attempt to capture small pieces of conditions $p \in \mathbb{P}_\tau$, and join those pieces together (possibly pieces from different conditions). These are given by *traces* to structures, and *simple amalgamations*, respectively. The two operations need not produce conditions in general, but as we will show, they do under quite natural additional assumptions.

For a condition $p \in \mathbb{P}_\tau$ and a structure $M = M^\tau_\alpha \in \vec{M}^\tau$ the trace of p to M, denoted $[p]_M$, is meant to capture all the information p has which is relevant to M.

Definition 34. Let $p \in \mathbb{P}_\tau$ for some $\tau < \kappa^+$ and $M \prec (H_{\kappa^{++}}, \in \mathbb{P}_\tau, \vec{\psi})$. The *trace* of $p = \langle f, N\rangle$ to M, is the pair $[p]_M = \langle \bar{f}, \bar{N}\rangle$ where

- $\mathrm{dom}(\bar{f}) = M \cap \mathrm{dom}(f)$,
- for every $\gamma \in \mathrm{dom}(\bar{f})$, $\bar{f}_\gamma = f_\gamma \cap M$,
- $\bar{N}$ consists of pairs $(\alpha, \bar{a})$ for which there is some a such that $(\alpha, a) \in N$, $\alpha \in M$, and $\bar{a} = a \cap M$ is nonempty.[5]

It is not true in general that $[p]_M$ is a condition in $\mathbb{P}_\tau$. For example, it is possible that the trace part $[p \upharpoonright \delta]_M = [p]_M \upharpoonright \delta$ does not decide the meet of nodes $s_1 \neq s_2$ for $s_1, s_2 \in \mathrm{dom}(\bar{f}_\delta)$. Our main argument below shows that when p is amenable to M, then $[p]_M$ is a condition in $\mathbb{P}_\tau \cap M$. Moreover, in the next section we introduce a property of a condition p begin super-nice with respect to M and prove it implies that $[p]_M$ is a residue of p (see Definitions 40 and 42).

Next, we define the simple amalgamation operation of two conditions p, p'. Given two conditions $p, p' \in \mathbb{P}_\tau$, the natural attempt to find a common extension q to p, p' involves taking coordinate-wise unions of the "working parts" and the "side condition" parts. The result, which need not be a condition, is called the *simple amalgamation* of p, p'.

Definition 35. (Simple Amalgamations)
The *simple amalgamation* of conditions $p = \langle f, N\rangle$ and $p' = \langle f', N'\rangle$ is the pair $\langle f^*, N^*\rangle$ defined by

- $\mathrm{dom}(f^*) = \mathrm{dom}(f) \cup \mathrm{dom}(f')$
- for each $\delta \in \mathrm{dom}(f^*)$, $f^*_\delta = f_\delta \cup f'_\delta$ (with f_δ or f'_δ taken to be empty in the case they do not exist)
- $(\alpha^*, a^*) \in N^*$ if and only if there is either a so that $(\alpha^*, a) \in N$ or a' so that $(\alpha^*, a') \in N'$, and then $a^* = a \cup a'$ (with a or a' taken to be empty in the case they do not exist)

It is clear that if q is an extension of p, p' then, with the above notation, $f^*_\delta \subseteq f^q_\delta$ for every $\delta \in \mathrm{dom}(f^*)$, and for every $(\alpha^*, a^*) \in N^*$ there is $b \supseteq a^*$ with $(\alpha^*, b) \in N^q$. However, the simple amalgamation (f^*, N^*) need not be a condition even when p, p' are compatible. This is because given that the simple amalgamation of p, p' up to some coordinate $\delta < \tau$ forms a condition in $\mathbb{P}_\delta$ (this is clearly the case for $\delta = 0$) it need not decide the meets in S_δ of nodes $s \in \mathrm{dom}(f^p_\delta)$ with nodes $s' \in \mathrm{dom}(f^{p'}_\delta)$, or the exit node of some $s' \in \mathrm{dom}(f^{p'}_\delta)$ with all side condition structures M^δ_α that appear in p, and vice versa.

In a very simple case where $p' \in \mathbb{P}_\delta$ extends an initial segment $p \upharpoonright \delta$ of p, it is an immediate consequence of the definition that the simple amalgamation is a condition.

Lemma 36. *Suppose that $p \in \mathbb{P}_\tau$, and $p' \in \mathbb{P}_\delta$ extends $p \upharpoonright \delta$ for some $\delta < \tau$. Then the simple amalgamation of p, p' belongs to $\mathbb{P}_\tau$*

[5]Since $\vec{\psi} \in M$ and a is α-closed, then so is $(\alpha, \bar{a})$.

To establish the main preservation properties of $\mathbb{P}_\tau$ (e.g., preservation of κ) will make use of another criterion for when the simple amalgamation of p, p' is a condition.

Definition 37. (Local extensions with projections)
Let $p, p' \in \mathbb{P}_\tau$. We say that p' is a *local extension of p with projections* if for every $\delta \in \mathrm{supp}(p) \cap \mathrm{supp}(p')$ we have

1. $(\delta = 0)$ f_0^p and $f_0^{p'}$ are compatible in $\mathrm{Coll}(\omega_1, <\kappa)$.

2. $(\delta > 0)$ for every $s' \in \mathrm{dom}(f_\delta^{p'}) \setminus \mathrm{dom}(f_\delta^p)$ and $s \in \mathrm{dom}(f_\delta^p) \setminus \mathrm{dom}(f_\delta^{p'})$ there is a connecting node $\tilde{s} \in \mathrm{dom}(f_\delta^p) \cap \mathrm{dom}(f_\delta^{p'}) \cap M_\alpha^\delta$ for some M_α^δ which appears in both p and p', such that

   (a) (p' extends p in M_α^δ) The model M_α^δ appears in both p and p' and
   $$[p' \restriction \delta]_{M_\alpha^\delta} \leq [p \restriction \delta]_{M_\alpha^\delta}$$

   (b) ($\tilde{s}$ is a local projection of s in M_α^δ)
   $$p \restriction \delta \Vdash \tilde{s} \leq_{S_\delta} s$$
   and
   $$\mathrm{Lev}_{S_\delta}(\tilde{s}) \geq \mathrm{Lev}_{S_\delta}(s').$$

   Therefore, any extension of p, p' will force the meet of s', s to be the meet of $s', \tilde{s}$ as determined by p'.

3. $(\delta > 0)$ for every model $M_{\alpha'}^\delta$ that appears in p' but not in p. and $s \in \mathrm{dom}(f_\delta^p) \setminus \mathrm{dom}(f_\delta^{p'})$ there are a model M_α^δ, $\alpha > \alpha'$, which appears in both p and p', and $\tilde{s} \in \mathrm{dom}(f_\delta^p) \cap \mathrm{dom}(f_\delta^{p'}) \cap M_\alpha^\delta$ such that

   (a) $[p' \restriction \delta]_{M_\alpha^\delta} \leq [p \restriction \delta]_{M_\alpha^\delta}$

   (b)
   $$p \restriction \delta \Vdash \tilde{s} <_{S_\delta} s$$
   and
   $$p' \restriction \delta \Vdash \tilde{s} \text{ is an exit node from } M_{\alpha'}^\delta$$

   Therefore, any extension of p, p' will force the exit node of s from $M_{\alpha'}^\delta$ to be $\tilde{s}$, as it is determined by p'.

4. $(\delta > 0)$ if a model $M_{\alpha'}^\delta$ appears in p but not in p' then $f_\delta^{p'} \subseteq M_{\alpha'}^\delta$.

The word "projections" refers to the nodes $\tilde{s}$ in the definition, which faithfully connect the information from p down to the relevant parts of p' inside M_α^δ. As an immediate consequence of the definitions of simple amalgamations and local extensions with connections we conclude

Lemma 38. *If $p, p' \in \mathbb{P}_\tau$ are such that p' is a local extension of p with projections then the simple amalgamation of p, p' is a condition in $\mathbb{P}_\tau$ which extends both p, p'.*

Proof. One proves by induction on $\delta \leq \tau$ that the simple amalgamation of $p \restriction \delta$ and $p' \restriction \delta$ in $\mathbb{P}_\delta$ is a condition in $\mathbb{P}_\delta$. v For $\delta = 0$ this is an immediate consequence of item 1 in Definition 37. Limit stages $\delta \leq \tau$ immediately follow from the definition of the poset $\mathbb{P}_\delta$. Suppose $\delta + 1 \leq \tau$ is a successor ordinal. Denote the simple amalgamation of $p \restriction \delta$ and $p' \restriction \delta$ by q. We assume $q \in \mathbb{P}_\delta$ by induction, and want to show that the simple amalgamation of $p \restriction \delta + 1$ and $p' \restriction \delta{+}1$ is a condition as well. Let $q' = (f', N')$ denote the simple amalgamation of $p \restriction \delta{+}1$ and $p' \restriction \delta{+}1$. Obviously $q' \restriction \delta = q$. In order to verify q is a condition it suffices to check assumptions I-VII as part of the definition of conditions in $\mathbb{P}_{\delta+1}$. Assumption I-IV are immediate. Assumptions VI.1 and VII is also clear as all nodes $s \in \mathrm{dom}(f'_\delta)$ belong to $\mathrm{dom}(f^p_\delta)$ or $\mathrm{dom}(f^{p'}_\delta)$ and so, the relevant information to be determined is decided by either $p \restriction \delta$ or $p' \restriction \delta$, and hence by q. We are left with verifying Assumptions V and VI.2. Assumption V requires that $\mathrm{dom}(f'_\delta)$ is closed under meets in S_δ and that f'_δ maps these meets to the meets in T_δ of the images. Let $s, s' \in \mathrm{dom}(f'_\delta)$, if $s, s' \in \mathrm{dom}(f^p_\delta)$ or $s, s' \in \mathrm{dom}(f^{p_\delta}$ then the result is immediate. Suppose that $s \in \mathrm{dom}(f^p_\delta) \setminus \mathrm{dom}(f^{p'}_\delta)$ and $s' \in \mathrm{dom}(f^{p'}_\delta) \setminus \mathrm{dom}(f^p_\delta)$. By the assumption of the Lemma and part 2 of Definition 37 there is a connecting node $\tilde{s} \in M^\delta_\alpha \cap \mathrm{dom}(f^p_\delta) \cap \mathrm{dom}(f^{p'}_\delta)$ for some M^δ_α which appears in both p and p'. Let $\bar{s} \in \mathrm{dom}(f^{p'}_\delta)$ be the meet of s' and $\tilde{s}$ (as decided by $p' \restriction \delta$). It follows from the properties of $\tilde{s}$ listed in part 2 of Definition 37 that q forces $\bar{s}$ to be the meet of s' and s, and similarly that that $f'_\delta(\bar{s})$ is the meet of $f^p_\delta(s) >_T f^p_\delta(\tilde{s}) = f^{p'}_\delta(\tilde{s})$, and $f^{p'}_\delta(s')$.
Moving to Assumption VI.2, it suffices to check that for every structure $M^\delta_{\alpha'}$ that appears in exactly one of the conditions p, p' and a node $s \notin M^\delta_{\alpha'}$ which appears in the domain of the other condition, then the exit node e of s from $M^\delta_{\alpha'}$ is decided by q, it belongs to $\mathrm{dom}(f'_\delta)$ and its image $f'_\delta(e)$ is forced to be the exit node of $f'_\delta(s)$ from $M^\delta_{\alpha'}$. First, we note that this cannot happen if $M^\delta_{\alpha'}$ appears in p but not in p', since property 4 in Definition 37, says that in such a case $f^{p'}_\delta \subseteq M^\delta_{\alpha'}$. Hence no $s \in \mathrm{dom}(f^{p'}_\delta)$ can be outside $M^\delta_{\alpha'}$. Suppose now that $M^\delta_{\alpha'}$ appears in p' but not in p and $s \in \mathrm{dom}(f^p_\delta) \setminus M^\delta_{\alpha'}$. By property 3 of 37 there is a "connecting" node $\tilde{s} \in \mathrm{dom}(f^p_\delta) \cap \mathrm{dom}(f^{p'}_\delta) \cap M^\delta_\alpha$ for some M^δ_α, $\alpha > \alpha'$, that appears in both p and p', is forced by p to be below s, and is forced by $p' \restriction \delta$ to be an exit node from $M^\delta_{\alpha'}$. It follows that p forces $f'_\delta(\tilde{s}) <_T f'_\delta(s)$, and that $p' \restriction \delta$ forces that $f'_\delta(\tilde{s})$ is an exit node from M^δ_α. Hence, the common extension q of $p \restriction \delta$ and $p' \restriction \delta$ forces that $\tilde{s}$ is the exit node of s from $M^\delta_{\alpha'}$, and that its image $f'_\delta(\tilde{s})$ is the exit node of $f'_\delta(s)$ from $M^\delta_{\alpha'}$. $\square$

The last lemma will play a key role in the proof that for a condition $p \in \mathbb{P}_\tau$ which is super-nice with respect to some M, if $w \in M \cap \mathbb{P}_\tau$ extends $[p]_M$ then p, w are compatible.

5 Strong Properness

We commence by recalling the usual definition of strong properness:

Definition 39. A condition $p \in \mathbb{P}$ is *strongly proper* with respect to a model $M \prec (H_\theta, \in, \mathbb{P})$ (for some sufficiently large θ) if p forces that $\dot{G} \cap M$ is a V-generic filter for $\mathbb{P} \cap M$. We say that $\mathbb{P}$ is *strongly proper* with respect to a stationary class $\mathcal{T}$ of models M, if for every $M \in \mathcal{T}$ and $q \in M \cap \mathbb{P}$, q has an extension $p \in \mathbb{P}$ which is strongly proper with respect to M.

This definition is equivalent to the definition of a *strong master condition* in Neeman and Gilton [6]. We will use the following notion of residue function to prove strong properness results for our poset.

Definition 40. Let $p \in \mathbb{P}$ and $M \prec (H_\theta, \in, \mathbb{P})$. A *residue function* for M over p is a function $r : D \to M \cap \mathbb{P}$, where $D \subseteq \mathbb{P}/p$ is dense below p and for every $q \in D$ and $w \in M \cap \mathbb{P}$ such that $w \leq r(q)$, w and q are compatible in $\mathbb{P}$.

Lemma 41. *If p,M are as above, and $r : D \to \mathbb{P} \cap M$ is a residue function for M over p, then p is strongly proper with respect to M.*

See Proposition 1.7 in [6] for details.

For appropriate conditions $p \in \mathbb{P}_\tau$ and M, our residue function to M will be given by a natural trace $[p]_M$ operator. We introduce the notion of a super-nice conditions with respect to a structure M, and prove that when p is super-nice with respect to M then $[p]_M$ is a residue of p in M.

Definition 42.

1. We say that $p \in \mathbb{P}_\tau$ *nicely projects* to a structure M^τ_β if it satisfies the following requirements:
   - For every $\delta \in M^\delta_\beta \cap \mathrm{supp}(p)$ and $s' \in \mathrm{dom}(f^p_\delta)$ which is outside of M^δ_β there exists some $s \in \mathrm{dom}(f^p_\delta)$ such that $p \restriction \delta$ forces s to be an exit node from M^δ_β and $s \leq_{S_\delta} s'$.
   - $f^p_0 \in D(t)$ for every $t \in \bigcup_{\delta \in \mathrm{supp}(p)} \mathrm{rng}(f^p_\delta)$ (see Definition 13 for the σ-closed dense set $D(t)$)
   - For every $\delta \in M^\tau_\beta \cap \mathrm{supp}(p)$ and $s \in \mathrm{dom}(f^p_\delta)$ which is forced by $p \restriction \delta$ to be an exit node from M^δ_β, then $\bar{t} = \pi_{f^p_0}(f^p_\delta(s))$ is of sufficiently high level so that it does not belong to M^δ_β for any $\beta < \beta_p(s)$ (see Definition 26 for $\beta_p(s)$), and either
     (a) ($\bar{t}$ has a preimage)
     there is some $\bar{s} \in \mathrm{dom}(f^p_\delta) \cap M^\tau_\beta$ such that $p \restriction \delta \Vdash \bar{s} <_{S_\delta} s$ and $f^p_\delta(\bar{s}) = \bar{t}$, or
     (b) ($b_{\bar{t}}$ has cofinal preimages)[6]
     there is a sequence of nodes $\langle \bar{s}_n \rangle_n \subseteq \mathrm{dom}(f^p_\delta) \cap M^\tau_\beta$ such that $p \restriction \delta$ forces it is increasing in S_δ, bounded by s, and the images by f^p_δ are cofinal in $\bar{t}$.

[6] This option was added to make sure that the set of nicely projects conditions is σ-closed.

2. Let $p \in \mathbb{P}_\tau$ be a condition which satisfies that for every $\delta \in \operatorname{supp}(p) \setminus 1$ and $s \in \operatorname{dom}(f^p_\delta)$, the condition $p \restriction \delta$ decides the identity of M^s_1 In particular, $p \restriction \delta$ is M^s_1-amenable. We define $E(p, M^\tau_\alpha)$ to be the countable set of pairs (δ, M^δ_β) obtained by starting with the singleton $\{(\tau, M^\tau_\alpha)\}$ and closing under the following operation: Given (γ, M^γ_β) in our set, and δ, s such that $\delta \in M^\gamma_\beta \cap \gamma$ and $s \in \operatorname{dom}(f^p_\delta)$ is forced by $p \restriction \delta$ to be an exit node from M^δ_β, we add (δ, M^s_1).

3. p is *super-nice* with respect to M^τ_α if it satisfies the assumption needed to define $E(p, M^\tau_\alpha)$, and for every $(\delta, M^\delta_\beta) \in E(p, M^\tau_\alpha)$, $p \restriction \delta$ nicely projects to M^δ_β. The set of super-nice conditions $p \in \mathbb{P}_\tau$ with respect to M^τ_α is denoted by $D_\tau(M^\tau_\alpha)$.

Remark 43. *It follows from the definition of $E(p, M^\tau_\alpha)$ that for every pair $(\delta, M^\delta_\beta) \in E(p, M^\tau_\alpha)$ that is not the initial one (τ, M^τ_β), then M^τ_α, $\delta < \tau$ and M^δ_β is (forced by $p \restriction \delta$) to be a successor structure in the sequence $\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta)}$. This observation together with the fact that for every $\delta' \in M^\delta_\beta \cap \delta$, $M^{\delta'}_\beta$ is a limit structure in the sequence $\vec{M}^{\delta', \dot{G}(\mathbb{P}_{\delta'})}$, imply that for all $(\delta', M^{\delta'}_{\beta'}) \in E(p, M^\tau_\alpha)$ we must have that $\beta \neq \beta'$.*

Lemma 44. *Let $\tau < \kappa^+$ be such that for every $\delta < \tau$ and M^δ_α which is $\vec{M}^\delta$-reflective. Suppose that*

- *the set $D_\delta(M^\delta_\alpha)$ is σ-closed and dense below every condition amenable to M^δ_α,*
- *for every $p \in D_\delta(M^\delta_\alpha)$, $[p]_{M^\delta_\alpha} \in \mathbb{P}_\delta \cap M^\delta_\alpha$ and every $w \in M^\delta_\alpha$ with $w \leq [p]_{M^\delta_\alpha}$ is compatible with p.*

Then

1. *(Node Density) for every $p \in \mathbb{P}_\tau$, $\delta < \tau$, and $s \in S_\delta$, there is an extension $p' \leq p$ with $s \in \operatorname{dom}(f^p_\delta)$.*
2. *(Super-nice density for $\vec{M}^\tau$) For every $M^\tau_\alpha \in \vec{M}^\tau$, the set $D_\tau(M^\tau_\alpha)$ is σ-closed and dense below every condition amenable to M^τ_α.*
3. *For every condition $p \in D_\tau(M^\tau_\alpha)$, $[p]_{M^\tau_\alpha} \in \mathbb{P}_\tau \cap M^\tau_\alpha$ is a condition.*

Proof. Starting with the node density assertion, let $p \in \mathbb{P}_\tau$ and $s \in S_\delta$, $\delta < \tau$. We would like to extend p and add s to $\operatorname{dom}(f^p_\delta)$. The idea is to extend $p \restriction \delta$ to make it s-knowledgeable, and choose the image $f_\delta(s)$ accordingly. We have to make sure that the decision is compatible with Assumption VI in the sense that for every M^δ_α which appears in p, if $s \in M^\delta_\alpha$ then so is our choice for $f_\delta(s)$. Let $\alpha < \kappa$ be minimal such that M^δ_α appears in p and $s \in M^\delta_\alpha$. Recall that this implies M^δ_α is $\vec{M}^\delta$-reflective. Namely, that there is some $N^\delta_\alpha \prec (H_{\kappa^{++}}, \vec{M}^\delta)$ so that $N^\delta_\alpha \cap \mathcal{A}^\delta = M^\delta_\alpha$.

Claim 45. There is an extension $w \in \mathbb{P}_\delta \cap N^\delta_\alpha$ of $[p \restriction \delta]_{M^\delta_\alpha}$ which decides the identity of $M^s_1 \in \vec{M}^\delta$.

Proof. Working inside N^δ_α, we can apply Lemma 23 to find an extension w^0 of $[p \restriction \delta]_{M^\delta_\alpha}$ which determines the minimal β such that $s \in M^\delta_\beta$ and $M^\delta_\beta \in \vec{M}^{\delta,G(\mathbb{P}_\delta)}$. We then extend repeatedly and find conditions $w^n \in N^\delta_\alpha$ which determines the first ω-many successors of $M^\delta_\beta = M^s_1$ in $\vec{M}^{\delta,G(\mathbb{P}_\delta)}$. Since $\mathbb{P}_\delta$ is σ-closed, the sequence has a limit $w \in N^\delta_\alpha$ with the desired property. $\square$

Taking $w \leq [p \restriction \delta]_{M^\delta_\alpha}$ as in the claim, the assumptions of the lemma guarantee there is a common extension r of w and $p \restriction \delta$. Let q be the simple amalgamation of r and p. By Lemma 36 it is a common extension of both. Next, by extending $q \restriction \delta$ we may assume it determines if there exists some $\beta < \kappa$ with the following properties:

- M^δ_β belongs to $\vec{M}^{\delta,G(\mathbb{P}_\delta)}$,
- s is an exit node from M^δ_β (i.e., $b_s \subseteq M^\delta_\beta$, and $s \notin M^\delta_\beta$),
- for every $s' \in \mathrm{dom}(f^p_\delta) \cap M^\beta_\delta$, $f(s') \in M^\delta_\beta$,
- for every $s' \in \mathrm{dom}(f^p_\delta) \setminus M^\delta_\beta$, $f(e_{S_\delta}(s', M^\delta_\beta)) = e_{T^{h_\delta}}(f^p_\delta(s'), M^\delta_\beta)$.

Moreover, if such β exists then we can assume $q \restriction \delta$ determines the identity of the minimal such β. If exists, then by its first property, $q \restriction \delta$ is amenable to M^δ_β, which means there is $(\beta, b) \in N^q$ so that $\delta \cap M^\delta_\beta \subseteq b$. It further follows from the last two listed assumptions of M^δ_β that M^δ_β can be added to the side condition part of q. Let $q' \leq q$ be obtained by replacing (β, b) in N^q with $(\beta, b \cup \{\delta\})$. Then M^δ_β appears in q'.

As $s \in M^\delta_\alpha$ then such $\beta < \kappa$ (if exists) has to be below α. By further extending q' we may assume that $q' \restriction \delta$ determines the following information regarding b_s:

1. all meets $m(s, s')$ of s and $s' \in \mathrm{dom}(f^q_\delta)$,
2. the determined images $t(s, s')$ of each $m(s, s')$ using the values $f^p_\delta(s')$ (see Remark 33).
3. the exit nodes $e(s, M^\delta_\beta)$ of s from each M^δ_γ that appears in p, and $s \notin M^\delta_\gamma$.

The nodes in $\{t(s, s') \mid s' \in \mathrm{dom}(f^p_\delta)\}$ are (forced to be) pairwise compatible in T^{h_δ}. Moreover, if β exists then all nodes $m(s, s')$ must belong to M^δ_β, and since M^δ_β appears in q', their images $t(s, s')$ must also belong to M^δ_β.

Let α^* be the level of s, and β^* be either β if it exists, or otherwise, some $\beta^* < \alpha$ above all $\beta' < \alpha$ so that $M^\delta_{\beta'}$ appears in p. Clearly, $\alpha^*, \beta^* \in M^\delta_\alpha$. Now pick $\tau < \alpha$ above both $\alpha^* \cdot \beta^*$ and $(\kappa \cap M^s_1)$, and so that q' does not provide any information regarding the collapse generic function $f^{G(\alpha^*,\beta^*,\gamma^*)}_\tau$, where $\gamma^* = h_\delta(\alpha^*, \beta^*, \tau)$. Let $t \in T^{h_\delta}$ be the node of collapse-index $(\alpha^*, \beta^*, \gamma^*, \tau)$ and therefore, of collapse height $(\alpha^*, \beta^*, \tau)$. Since $(\alpha^*, \beta^*, \tau)$ and h_δ are in M^δ_α then so is t. By the construction of T^{h_δ}, the branch b_t is cofinal in $T^{h_\delta} \cap (\alpha^* \times \beta^*)$, and therefore cofinal in $T^{h_\delta} \cap M^\delta_\beta$ if β exists. Given the tree ordering of $T^{h^\delta} \cap M^\delta_\beta$, q' does not provide any information regarding the collapse generic function $f_\tau^{G(\alpha^*,\beta^*,\eta^*)}$, which determines b_t, we can extend the collapse part f^q_0 as in Lemma 15, to a collapse condition f^* so that it forces the following information about b_t:

1. $t(s, s') \in b_t$ for all $s' \in \mathrm{dom}(f^p_\delta)$,
2. For every $e = e(s, M^\delta_\beta)$ as above, if its level is α_e then $b_t(\alpha_e) \in T^{h_\delta}$ is an exit node from M^δ_β.

Let q^* be obtained from q by taking $f_0^{q^*} = f^*$, and $f_\delta^{q^*} = f^q_\delta \cup \{(s,t)\}$. q^* is a condition. Indeed, the last listed properties of b_t guarantee it satisfies requirements V, VI for δ, and our choice of t to be of collapse index $(\alpha^*, \beta^*, \tau)$ guarantees that q^* satisfies requirement VII. Therefore $q^* \leq p$ has $s \in \mathrm{dom}(f_\delta^{q^*})$.

Next, we prove the second assertion of $D_\tau(M^\tau_\alpha)$ being dense below every condition p which is amenable to M^τ_α. Since $\mathbb{P}_\tau$ is σ-closed one can use a standard bookkeeping argument and the node-density assertion to construct an extension p' of p such that

- $f^p_0 \in D(t)$ for all $(s,t) \in f^p_\delta$ for some $\delta < \tau$, and moreover, $\bar{t} = \pi_{f^p_0}(t)$ is sufficiently high so that it does not belong to any M^δ_β for which p is amenable to, with $\beta < \beta_p(s)$.
- for every structure M^δ_β that appears in p, and $s \in \mathrm{dom}(f^p_\delta) \setminus M^\delta_\beta$, $p' \restriction \delta$ determines the exit node $e = e_{S_\delta}(s, M^\delta_\beta)$ and $e \in \mathrm{dom}(f_\delta^{p'})$.

These two items take care of the first two bullets in the definition of super-nice conditions, 42. To construct p' which further satisfies item (b) of the last clause in the definition, one has to further maintain that for every pair $(\delta, M^\delta_\beta) \in E(p', M^\tau_\alpha)$, p' nicely projects from M^δ_β. Namely, if $(s,t) \in f_\delta^{p'}$ and s is an exit node from M^δ_β, then there is a sequence of nodes $\langle \bar{s}_n \mid n < \omega \rangle \in \mathrm{dom}(f_\delta^{p'}) \cap M^\delta_\beta$ below s such that their images $f_\delta^{p'}(\bar{s}_n)$, $n < \omega$ are forced to be cofinal in $\pi_{f_0^{p'}}(t)$. To achieve this, consider a single step of the construction. Given $(s,t) \in \mathrm{dom}(f^p_\delta)$ with s an exit node from some relevant M^δ_β. We define a decreasing sequence of conditions $\langle q^n \mid n < \omega \rangle$ below $q^0 = p \restriction \delta$, whose collapse parts are all in $D(t)$. Let $\bar{t}_0 = \pi_{q^0}(t)$ and denote its level by $\bar{\alpha}_0$. Extend q^0 to q^1 that determines the unique node $\bar{s}_0 \in b_s \cap \mathrm{Lev}_{\bar{\alpha}_0}(S_\delta)$, which is the only possible candidate for the pre-image of $\bar{t}_0$. We need to make sure that the pair $(\bar{s}_0, \bar{t}_0)$ can be added to f^p_δ. The addition can potentially be in conflict with the requirement that for every M^δ_β that appears in p, $\bar{s}_0 \in M^\delta_\beta$ if and only if $\bar{t}_0 \in M^\delta_\beta$. Having $\bar{t}_0 = \bar{t}$, our assumption about $\bar{t}$ being sufficiently high, shows that the only relevant structure M^δ_β is the one with $\beta = \beta_p(s)$ (if exists). In this case $p \restriction \delta \Vdash b_s, b_t \subseteq M^\delta_\beta$. This guarantees the desired requirement. We may assume that the extension q^1 of q^0 increases the projection of t. Namely, $\bar{t}_1 = \pi_{q^1}(t) >_T \bar{t}_0$. We can repeat the construction once more to find an extension $q^2 \in D(t)$ of q^1 which determines a preimage $\bar{s}_1 \in b_s$ for $\bar{t}_1$. By proceeding in this fashion ω-many times, and dovetailing, we produce the desired sequence $\langle q^n \mid n < \omega\rangle$ extending $p \restriction \delta$, and finally take p' to be the simple amalgamation of $\cup_{n<\omega} q^n$, and p. This takes care a single pair (s,t) in f^p_δ for some $\delta < \tau$. We repeat this process for all pairs $(s',t') \in \mathrm{dom}(f^p_\delta)$ by similarly extending $p' \restriction \delta$. These stages could potentially "step over" each other, and harm the result done for other pairs such as (s,t), but we can go back to each harmed pair at a later stage to repair it cofinally many times in the process.

Moving to the third assertion of the lemma. Given $p \in D_\tau(M^\tau_\alpha)$ we would like to show that $[p]_{M^\tau_\alpha}$ is a condition in $\mathbb{P}_\tau$. Having that $[p \restriction \delta]_{M^\delta_\alpha} \in \mathbb{P}_\delta$ for every $\delta \in M^\tau_\alpha \cap \tau$, it suffices to verify the assertion for the case $\tau = \delta + 1$ is a successor ordinal. In this case, one has to verify that $[p]_{M^{\delta+1}_\alpha}$ satisfies the requirements in assumptions V,VI, and VII regarding decisions for pairs $(s,t) \in f^p_\delta \cap M^{\delta+1}_\alpha$ and structures M^δ_β, $\beta < \alpha$ that appear in p. Having $p \in D_{\delta+1}(M^{\delta+1}_\alpha)$ implies $p \restriction \delta \in D_\delta(M^\delta_\alpha)$, which by our assumption, implies that $\mathbb{P}_\delta \cap M^{\delta+1}_\alpha$ is a regular sub-forcing of $\mathbb{P}_\delta/(p \restriction \delta)$. It follows that all decisions about nodes $s \in \mathrm{dom}(f^\delta_p) \cap M^{\delta+1}_\alpha$, including (i) the meets, (ii) their exit nodes from every M^δ_β that appears in p, (iii) the identity of M^s_1, and (iv) $\beta_p(s)$, which are determined by $p \restriction \delta$, must already be determined by the trace $[p \restriction \delta]_{M^\delta_\alpha}$. Otherwise, there will be an extension $w \in \mathbb{P}_\delta \cap M^{\delta+1}_\alpha$ which gives incompatible information. But such a condition w could not be compatible with $p \restriction \delta$, contradicting the inductive assumption of the lemma for δ. $\square$

Proposition 46. *Suppose $M^\tau_\alpha \in \vec{M}^\tau$ and $q \in \mathbb{P}_\tau$ is amenable to M^τ_α. The function which takes $p \in D_\tau(M^\tau_\alpha)$ to $[p]_{M^\tau_\alpha}$ is a residue function to M^τ_α over q.*

Corollary 47.

1. *By proposition 46 every condition $q \in \mathbb{P}_\tau$ which is amenable to M^τ_α is strongly proper with respect to M^τ_α. In particular, $\mathbb{P}_\tau \cap M^\tau_\alpha$ is a regular subforcing of $\mathbb{P}_\tau/q$*

2. *For every condition $q \in D_\tau(M^\tau_\alpha)$, $[q]_{M^\tau_\alpha}$ forces that q belongs to the quotient forcing $\mathbb{P}_\tau/(\mathbb{P}_\tau \cap M^\tau_\alpha)$.*

Before proving Proposition 46 we make some preparations.

Given a condition $p \in D_\tau(M^\tau_\alpha)$ as in the statement of the proposition, and an extension $w \leq [p]_{M^\tau_\alpha}$, the goal of the proof is to find a common extension of w and p. Taking the simple amalgamation of w and p will not work in general. The main part of the argument is based on an inductive construction that results in an extension p' of w which is shown to locally extend p with projections. We then finish by applying Lemma 38.

To carry the inductive construction we formulate a technical generalization of the strong properness statement. We commence with the relevant definitions:

Definition 48.

1. Let $\tau < \kappa^+$. A τ-sequence is a countable sequence $\langle(\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu\rangle$ of pairs where each $\tau_i \leq \tau$ and
   - $\alpha_i < \alpha_j$ for $i < j < \nu$,
   - for $i < j < \nu$, if $\tau_i < \tau_j$ then $\tau_i \in M^{\tau_j}_{\alpha_j}$.
2. Suppose that $\langle(\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu\rangle$ is a τ-sequence, and let $p \in \mathbb{P}_\tau$ so that $p \restriction \tau_i \in D_{\tau_i}(M^{\tau_i}_{\alpha_i})$ for all $i < \nu$.
   Define the *p-closure* of $\langle(\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu\rangle$ to be the τ-sequence obtained by adding all pairs from the sets $E(p \restriction \tau_i, M^{\tau_i}_{\alpha_i})$. The ordering of the extended sequence is the unique one which makes it a τ-sequence.
3. Let $p \in \mathbb{P}_\tau$. A pair of countable sequences $\langle(\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu\rangle$ and $\langle q^i \mid i < \nu\rangle$ is a *p-Multi-Extension*, p-M.E. in short, if it satisfies
   (a) $p \restriction \tau_i \in D_{\tau_i}(M^{\tau_i}_{\alpha_i})$ for every $i < \nu$,
   (b) for each $i < \nu$, $q^i \in \mathbb{P}_{\tau_i} \cap M^{\tau_i}_{\alpha_i}$, $q^i \leq [p \restriction \tau_i]_{M^{\tau_i}_{\alpha_i}}$.
   (c) $\langle(\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu\rangle$ is a τ-sequence and an initial segment of its p-closure,
   (d) for every $i < j < \nu$, letting $\bar\tau = \min(\tau_i, \tau_j)$, then $q^j \restriction \bar\tau \leq q^i \restriction \bar\tau$
   (e) for every $i < \nu$ and $s \in \mathrm{dom}(f^p_{\tau_i}) \cap M^{\tau_i}_{\alpha_i}$, q^i decides the following information about b_s:
      - the meet $m(s, s')$ in S_{τ_i} for every $s' \in \mathrm{dom}(f^{q^j}_{\tau_i})$ for some $j < i$.
      - the implicit image $t(s, s')$ (forced to be below $f^{q^i}_{\tau_j}(s')$) of $m(s, s')$ (as described in Remark 33).
      - the exit node $e(s, M^{\tau_i}_\beta)$ for every structure $M^{\tau_i}_\beta$ which appears in q^j for some $j < i$.
4. The *simple amalgamation* of a sequence of conditions $\langle q^i \mid i < \nu\rangle$ is the pair (f, N) defined by
   - $\mathrm{dom}(f) = \bigcup_{i<\nu} \mathrm{supp}(q^i)$.
   - for each $\delta \in \mathrm{dom}(f)$, $f_\delta = \bigcup\{f^{q^i}_\delta \mid i < \nu \text{ and } \delta \in \mathrm{supp}(q^i)\}$.
   - for each $\alpha < \kappa$, the set N includes a pair (α, a_α) if and only if there is some pair (α, a') in $\bigcup_i N^{q^i}$, in which case a_α is taken to be the union of all such a'.

Lemma 49.
For every $p \in \mathbb{P}_\tau$ and p-M.E. pair of sequences $\langle(\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu\rangle$ and $\langle q^i \mid i < \nu\rangle$ there is a condition $q \in \mathbb{P}_\tau$ such that $q \leq p$ and for each $i < \nu$, $q \restriction \tau_i \leq q^i$.

Proof. The proof is by induction on pairs (τ, α) (with the usual lexicographic ordering) where $\alpha < \kappa$ is minimal so that $\alpha \geq \cup_{i<\nu}\alpha_i$, and $p \in M^{\tau}_{\alpha}$.

Let $\langle (\tau_j, M^{\tau_j}_{\alpha_j}) \mid j < \nu^* \rangle$ be the p-closure of $\langle (\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu \rangle$. By the assumption, the p-closure is an end extension of $\langle (\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu \rangle$. Moreover, it follows from the definition of super-nice conditions and the fact $p \restriction \tau_i \in D_{\tau_i}(M^{\tau_i}_{\alpha_i})$ for all $i < \nu$ that $p \restriction \tau_j \in D_{\tau_j}(M^{\tau_j}_{\alpha_j})$ for all $j < \nu^*$.
Our next step is to extend the corresponding sequence of conditions $\langle q^i \mid i < \nu \rangle$ to a sequence $\langle q^j \mid j < \nu^* \rangle$, so that $\langle (\tau_j, M^{\tau_j}_{\alpha_j}) \mid j < \nu^* \rangle$ and $\langle q^j \mid j < \nu^* \rangle$ are p-M.E. Suppose that $\langle q^j \mid j < \eta \rangle$ has been defined for some $\nu \leq \eta < \nu^*$, such that $\langle (\tau_j, M^{\tau_j}_{\alpha_j}) \mid j < \eta \rangle$ and $\langle q^j \mid j < \eta \rangle$ are p-M.E. We define q^η in four steps that go through auxiliary conditions $q^{\eta,0}, q^{\eta,1}$, and $q^{\eta,2}$. The entire construction happens inside $M^{\tau_\eta}_{\alpha_\eta}$.

($q^{\eta,0}$) Let $q^{\eta,0} = [p \restriction \tau_\eta]_{M^{\tau_\eta}_{\alpha_\eta}}$. Also, denote for each $j < \eta$, $\tau^\eta_j = \min(\tau_j, \tau_\eta)$. It follows from the definition of a p-M.E. pair and the fact $p \restriction \tau_j \in D_{\tau_j}(M^{\tau_j}_{\alpha_j})$ for all $j < \eta$, that the pair of sequences $\langle (\tau^\eta_j, M^{\tau^\eta_j}_{\alpha_j}) \mid j < \eta \rangle$, $\langle q^j \restriction \tau^\eta_j \mid j < \eta \rangle$ is $q^{\eta,0}$-M.E.

($q^{\eta,1}$) To obtain $q^{\eta,1}$, we extend the collapse part of $q^{\eta,0}$ to make decisions about nodes in T that fit relevant decisions about meets and exit nodes in various trees S_δ, which were made by previous conditions q^j, $j < \eta$, according to property 3(e) in the definition of $M.E$ sequences. More precisely, let $\langle s_n \mid n < M \rangle_{n<M}$, $M \leq \omega$ enumerate all nodes s for which there is $j < \eta$ such that

- $\tau_j \leq \tau_\eta$, and $s \in \mathrm{dom}(f^p_{\tau_j})$, such that j is minimal for which $s \in M^{\tau_j}_{\alpha_j}$
- $\eta < \nu^*$ is minimal such that $f^p_{\tau_j}(s) \in M^{\tau_\eta}_{\alpha_\eta}$.

For each $n < M$, denote its relevant $j < \eta$ for s_n by j_n, and the image $f^p_{\tau_j}(s_n)$ by t_n.

By the assumption of $\langle (\tau^\eta_j, M^{\tau^\eta_j}_{\alpha_j}) \mid j < \eta \rangle$, $\langle q^j \restriction \tau^\eta_j \mid j < \eta \rangle$ being $q^{\eta,0}$-M.E. It follows from property 3(e) that for each $n < M$, $q^{j_n} \restriction \tau^\eta_{j_n}$ decides the meets $m(s_n, s')$ of s_n and every $s' \in \mathrm{dom}(f^{q^i}_{\tau_{j_n}})$, $i < j_n$, as well as their implicit images $t(s_n, s')$, and the exit nodes $e(s, M^{\tau_{j_n}}_\beta)$, whenever $M^{\tau_{j_n}}_\beta$ appears in $q^i \restriction \tau^\eta_i$ for some $i < j_n$.

For each $n < N$, let $s^*_n \in M^{\tau_{j_n}}_\beta$, be the $f^p_{\tau_{j_n}}$-preimage of the collapse projection $\pi_{f^p_0}(t_n)$.[7] s^*_n is (forced to be) compatible with all meets $m(s_n, s')$ and exit nodes $e(s_n, M^{\tau_{j_n}}_\beta)$ mentioned above, as all are forced to be below s_n in their tree order. Hence, their implicit images $t(s_n, s')$ are forced to

[7] s^*_n exists by the definition of super-nice conditions and since $p \restriction \gamma_\eta \in D_{\gamma_\eta}(M^{\gamma_\eta}_{\alpha_\eta})$ (specifically that p nicely projects).

be compatible with the tree projection $\pi_{f_0^p}(t_n)$.

For each exit node $e(s_n, M^{\tau_j}_\beta)$ let $\alpha_n(M^{\tau_j}_\beta)$ be its level in S_{τ_j}. Note that $\alpha_n(M^{\tau_j}_\beta) < \alpha_j$.
Also, for each $n < M$, let $(\alpha_n, \beta_n, \delta_n)$ be the collapse index of t_n. Since $(s_n, t_n) \in f^p_{\tau_j}$, then $p \restriction \tau_j$ forces $b_{s_n} \subseteq \alpha_n \times \beta_n$.
In particular, all relevant meets $m(s_n, s')$ and relevant exit nodes $e(s_n, M^{\tau_j}_\beta)$ belong to $\alpha_n \times \beta_n$. If $\beta_p(s_n)$ exists, then $\beta_n = \beta_p(s_n)$ which means that p is $M^{\tau_j}_{\beta_n}$ amenable, and therefore so is $q^{\eta,0}$. This, in turn, implies that the implicit images $t(s_n, s')$ of $m(s_n, s')$, as well as the exist node levels $\alpha_n(M^{\tau_j}_\beta)$ are all in $M^{\tau_j}_{\beta_n}$. Similarly, for each $n < M$, the collapse-based projection $\pi_{f^{q^{\eta,0}}}(t_n)$ belongs to $M^{\tau_j}_{\beta_n}$.

We apply Lemma 16 with $f_0^{q^{\eta,0}}$ in the role of p, countably many times, to guarantee that each b_{t_n} "climbs" correctly through the implicit images $t(s_n, s')$ and through exit nodes from relevant structures $M^{\tau_j}_\beta$ at the correct level $\alpha_n(M^{\tau_j}_\beta)$. By the Lemma there is a collapse extension f' of $f_0^{q^{\eta,0}}$ (f' in the role of p' in the Lemma) so that for each $n < M$, the extension $f = f' \cup (\bigcup_{j<\eta} f_0^{q^j})$ of f' (in the role of q in the Lemma) satisfies that for each $n < \omega$, $\pi_f(t_n)$ extends (in T) all the relevant meet images $t(s_n, s')$, and its exist node from each relevant $M^{\tau_j}_\beta$ is at the correct level $\alpha_n(M^{\tau_j}_\beta)$.

Let $q^{\eta,1}$ be the extension of $q^{\eta,0}$ obtained by replacing the collapse part $f_0^{q^{\eta,0}}$ with f'. The minimality assumption for η, being minimal such that $M^{\gamma_\eta}_{\alpha_\eta}$ is the first structure that sees the node t_n, implies that we can take f' and thus $q^{\eta,1}$ to satisfy

$$[q^{\eta,1} \restriction \tau_j]_{M^{\tau_j}_{\alpha_j}} = [q^{\eta,0} \restriction \tau_j]_{M^{\tau_j}_{\alpha_j}} \text{ for all } j < \eta.$$

Hence $\langle (\tau^\eta_j, \alpha_j) \mid j < \eta \rangle$ and $\langle q^j \restriction \tau^\eta_j \mid j < \eta \rangle$ is $q^{\eta,1}$-M.E. as well.

($q^{\eta,2}$) Still working inside the structure $M^{\tau_\eta}_{\alpha_\eta}$, We want to apply the inductive assumption of the Lemma, with respect to the $q^{\eta,1} \in \mathbb{P}_{\tau_\eta}$ and the $q^{\eta,1}$-M.E. pair of sequences. This is possible since $(\tau_\eta, \alpha_\eta) <_{lex} (\tau, \alpha)$. Let $q^{\eta,2} \in M^{\tau_\eta}_{\alpha_\eta}$ be the resulting condition. In particular we have that

1. $q^{\eta,2}$ extends $[p \restriction \tau_\eta]_{M^{\tau_\eta}_{\alpha_\eta}}$ and $q^j \restriction \tau_\eta$ for every $j < \eta$.
2. The collapse part $f^{q_0^{\eta,2}}$ extends $f' \cup (\bigcup_{j<\eta} f_0^{q^j})$, and therefore forces that for each $n < M$, b_{t_n} contains the implicit images $t(s_n, s')$ of relevant meets, and the exit node of b_{t_n} from relevant structure $M^{\tau_j}_\beta$, are at the correct levels $\alpha_n(M^{\tau_j}_\beta)$.

(q^η) Finally, working inside $M^{\tau_\eta}_{\alpha_\eta}$, we take an extension of $q^{\eta,2}$ which determines the following information:

- The meets of all pairs of nodes $m(s,s')$ where $s \in \mathrm{dom}(f^p_{\tau_\eta})$, $M^{\tau_\eta}_{\alpha_\eta} = M^s_1$, and $s' \in \bigcup_{i<j} \mathrm{dom}(f^{q^i}_{\tau_\eta})$
- The implicit images $t(s,s')$ of $m(s,s')$ (see Remark 33).
- The exit node of s from every structure $M^{\tau_\eta}_\beta$, $\beta < \alpha_\eta$, for which some q^i, $i < \eta$ is amenable to and $s^* \in M^{\tau_\eta}_\beta$ where s^* is the $f^p_{\tau_\eta}$ preimage of the projection $\pi_{f^p_0}(t)$, $t = f^p_\delta(s)$.

It is clear from the construction that the sequences $\langle(\gamma_j, M^{\gamma_j}_{\beta_j}) \mid j < \eta+1\rangle$ and $\langle q^j \mid j < \eta+1\rangle$ constitute a p-M.E..

This concludes the construction of the extended sequence $\langle q^j \mid j < \nu^*\rangle$. The construction guarantees the resulting sequences are p-M.E.. Being a p-M.E., it is easily seen that the simple amalgamation of the memeber of the sequence $\langle q^j \mid j < \nu^*\rangle$ is a condition $r \in \mathbb{P}_\tau$ (i.e., see Lemma 36).

We need to make two last extensions to r in order to get a desired condition p' which will be a local extension of p with projections. We first extend r to a condition r^1, obtained by extending the collapse part of r in the same way $q^{\eta,1}$ is constructed from $q^{\eta,0}$ above, with respect to the entire sequence $\langle q^j \mid j < \nu^*\rangle$. Then, finally, let $p' \leq r^1$ obtained by adding to each function $f^{r^1}_\delta$, $\delta < \tau$ the pairs $(\bar{s}, \bar{t})$ where either

- $\bar{s}$ is forced to be a relevant meet $\bar{s} = m(s,s')$, by some q^j at a stage $j < \nu^*$, and $\bar{t} = t(s,s')$ is the implicit image determined in a later stage $\eta > j$ (or at the very end, by r^1) and forced by $q^{\eta,2}$ (or r^1) to be the meet of $f^p_{\tau_j}(s)$ and $f^{q^i}_{\tau_j}(s')$ for some $i < \nu^*$.
- $\bar{s}$ is forced to be an exit node $e(s, M^{\tau_j}_\beta)$ from a relevant structure $M^{\tau_j}_\beta$ by some q^j at stage j, and its level in S_{τ_j} to be $\alpha(M^{\tau_j}_\beta)$, and $\bar{t}$ is the node forced by a later stage $\eta > j$ (or at the very end by r^1) to be the node on b_t, $t = f^p_{\tau_j}(s)$, at the same level $\alpha(M^{\tau_j}_\beta)$. By construction this node is forced by $q^{\eta,2}$ to be an exit node from the $M^{\tau_j}_\beta$.

We claim that p' is a local extension p with projections (Definition 37). Let $s \in \mathrm{dom}(f^p_\delta) \setminus \mathrm{dom}(f^{p'}_\delta)$ and $s' \in \mathrm{dom}(f^{p'}_\delta) \setminus \mathrm{dom}(f^p_\delta)$ for some $\delta < \tau$. There is some $j < \nu^*$ so that $s' \in M^{\tau_j}_{\alpha_j}$. Having $p \restriction \tau_j \in D_{\tau_j}(M^{\tau_j}_{\alpha_j})$ it follows there is some $\tilde{s} \in \mathrm{dom}(f^p_\delta)$ so that

$$p \restriction \delta \Vdash \tilde{s} <_{S_\delta} s, \text{ and}$$

$$p \restriction \delta \text{ is an exit node from } M^{\tau_j}_{\alpha_j}.$$

By the choice of structures in the enriched sequence, there is some $\eta > j$ so that $M^{\tau_\eta}_{\alpha_\eta} = M^1_{\tilde{s}}$. In particular $\delta = \tau_\eta$. By construction we have that $[p' \restriction \tau_\eta]_{M^{\tau_j}_{\alpha_j}} \leq q^\eta \leq [p \restriction \tau_\eta]_{M^{\tau_\eta}_{\alpha_\eta}}$, and q^η determines the meet of $\tilde{s}$ and s'. In particular,

$\mathrm{Lev}_{S_\delta}(\tilde{s}) > \mathrm{Lev}_{S_\delta}(s')$. This concludes the verification of the first clause of Definition 37. The proof of the second clause follows in a similar fashion.

Having p' being a local extension of p with projections, we conclude (Lemma 38) that the simple amalgamation of p and p' is a condition, which clearly extends p and each q^i, $i < \tau$.

$\square$

We can now prove Proposition 46.

Proof. (Proposition 46)
Let $p \in D_\tau(M^\tau_\alpha)$. To show that $[p]_{M^\tau_\alpha}$ is a residue for p, we need to verify that it is compatible with every condition $w \in \mathbb{P}_\tau \cap M^\tau_\alpha$ which extends $[p]_{M^\tau_\alpha}$. This is an immediate consequence of the previous Lemma with the p-M.E. pair of sequences of length $\nu = 1$ with $(\tau_0, M^{\tau_0}_{\alpha_0}) = (\tau, M^\tau_\alpha)$, and $q^0 = w$. $\square$

Theorem 50.
Let $\mathbb{P}_{\kappa^+} = \bigcup_{\tau<\kappa^+} \mathbb{P}_\tau$.

1. $\mathbb{P}_{\kappa^+}$ *is* κ^+*.c.c*
2. σ*-closed and thus, does not collapse* ω_1*.*
3. *It collapses all cardinals between* ω_1 *and* κ*.*
4. $\mathbb{P}_{\kappa^+}$ *does not collapse* κ
5. $2^{\aleph_1} = \kappa^+$*.*
6. *For each* $\tau < \kappa^+$*, the* τ*-th wide-tree* S^τ *chosen by the book-keeping function* Ψ *embeds into* T*. In particular, if* Ψ *covers all* $\mathbb{P}_\tau$ *names of wide trees on* κ *for all* $\tau < \kappa^+$*, then* T *is a maximal wide tree in the generic extension.*

Proof.

1. This is a standard consequence of the fact that each $\mathbb{P}_\tau$, $\tau < \kappa^+$ has size κ, and that there is a stationary set of $\tau < \kappa^+$ for which $\mathbb{P}_\tau$ is a direct limit of $\mathbb{P}_\delta$, $\delta < \tau$ (the set of all limit $\tau < \kappa^+$ of uncountable cofinality).
2. Immediate from the fact each $\mathbb{P}_\tau$, $\tau < \kappa^+$ is σ-closed (inductive assumption I).
3. Immediate. As $\mathbb{P}_{\kappa^+}$ embeds $\mathbb{P}_0 = \mathrm{Coll}(\omega_1, <\kappa)$.
4. This follows from the fact that $\mathbb{P}_{\kappa^+}$ is κ^+.c.c, and from the strong properness of all $\mathbb{P}_\tau$, $\tau < \kappa^+$, for structures of size $\aleph_1$.
5. For every $\tau < \kappa^+$, the poset $\mathbb{P}_{\tau+1}$ introduces a tree embedding $f_\tau : S_\tau \to T^{h_\tau}$, so that for every structure $M^{\tau+1}_\alpha \in \vec{M}^{\tau+1, G(\mathbb{P}_{\tau+1})}$, $f_\tau \restriction M^{\tau+1}_\alpha$ is generic over $V[G(\mathbb{P}_\tau)]$ and introduces an embedding of $S_\tau \cap M^{\tau+1}_\alpha$ to $T^{h_\tau} \cap M^{\tau+1}_\alpha$, which is a new subset of $M^{\tau+1}_\alpha$ of size $\aleph_1$.
6. Immediate by the construction of the posets $\mathbb{P}_\tau$, $\tau < \kappa^+$.

$\square$

6 No new branches

In this section, we prove that if all chosen wide-trees S^τ are wide Aronszajn trees, then T does not get a cofinal branch, and so can become a universal wide Aronszajn tree on $\kappa = \aleph_2$. We start with a lemma.

Lemma 51. *Let $M^\tau_\alpha, M^\tau_\beta \in \vec{M}^\tau$, $\alpha < \beta$. Suppose that $p \in D_\tau(M^\tau_\beta) \cap D_\tau(M^\tau_\alpha)$. Let $G_\alpha \subseteq \mathbb{P}_\tau \cap M^\tau_\alpha$ be generic over V with $[p]_{M^\tau_\alpha} \in G_\alpha$.*

1. *The quotient $\mathbb{P}_\tau / G_\alpha$ has a σ-closed dense subset.*
2. *For every $w \in (\mathbb{P}_\tau / G_\alpha) \cap M^\tau_\beta[G_\alpha]$ with $w \leq [p]_{M^\tau_\beta}$, there is a common extension $q \leq_{\mathbb{P}_\tau / G_\alpha} w, p$, such that $q \in D_\tau(M^\tau_\beta)$.*

Proof.

1. The quotient is σ-closed with respect to conditions in $D_\tau(M^\tau_\alpha)$. This is an immediate consequence of the fact that $D_\tau(M^\tau_\alpha)$ is σ-closed dense below p, of Proposition 46, and the fact that the trace map $p \mapsto [p]_{M^\tau_\alpha}$ respects countable joins.
2. Let $p \in D_\tau(M^\tau_\alpha) \cap D_\tau(M^\tau_\beta)$, $p \in \mathbb{P}_\tau / G_\alpha$ and $w \leq [p]_{M^\tau_\beta}$, $w \in \mathbb{P}_\tau / G_\alpha$, $w \in M^\tau_\beta[G_\alpha]$. By extending w we may assume that it belongs to $D_\tau(M^\tau_\alpha)$, and therefore that $[w]_{M^\tau_\alpha}$ forces that w belongs to the quotient $\mathbb{P}_\tau / G_\alpha$.

   Suppose towards contradiction that there exists an extension $r \in M^\tau_\beta$ of w which is incompatible with p in the quotient forcing $\mathbb{P}_\tau / G_\alpha$.

   This means that in V, there is $w' \leq [w]_{M^\tau_\alpha}$, $w' \in \mathbb{P}_\tau \cap M^\tau_\alpha$, which forces that r and p are incompatible as conditions of the quotient forcing, over an extension by $\mathbb{P}_\tau \cap M^\tau_\alpha$. By strong properness (Proposition 46) with respect to M^τ_α, r and w' are compatible by some $r' \in M^\tau_\beta$. Since $r' \leq r \leq w \leq [p]_{M^\tau_\beta}$ we can apply strong properness again with respect to M^τ_β and conclude that r' and p are compatible by some $q \in D^\tau(M^\tau_\beta) \cap D^\tau(M^\tau_\alpha)$. But now $[q]_{M^\tau_\alpha}$ extends w' and forces contradictory information.

□

Lemma 52. *Let $M^\tau_\alpha, M^\tau_\beta \in \vec{M}^\tau$, $\alpha < \beta$, and $p \in D_\tau(M^\tau_\beta) \cap D_\tau(M^\tau_\alpha)$. Suppose that $q^L, q^R \in D_\tau(M^\tau_\alpha) \cap M^\tau_\beta$ are two extensions of $[p]_{M^\tau_\beta}$ which satisfy $[q^L]_{M^\tau_\alpha} = [q^R]_{M^\tau_\alpha}$, and $D^L, D^R \in M^\tau_\beta$ are two dense open sets in $\mathbb{P}_\tau$. Then there are $p^L, p^R \in \mathbb{P}_\tau$ that satisfy:*

$$p^L, p^R \leq p,$$
$$p^L \leq q^L \textit{ and } p^L \in D^L \cap D_\tau(M^\tau_\alpha),$$
$$p^R \leq q^R \textit{ and } p^R \in D^R \cap D_\tau(M^\tau_\alpha),$$
$$[p^L]_{M^\tau_\alpha} = [p^R]_{M^\tau_\alpha}.$$

Proof. Let $G_\alpha \subseteq \mathbb{P}_\tau \cap M^\tau_\alpha$ be a generic filter over V with $[q^L]_{M^\tau_\alpha} \in G_\alpha$. Therefore q^L, q^R, p belong to the quotient forcing $\mathbb{P}_\tau / G_\alpha$. We define two sequences $\langle q^{L,n} \rangle_n$ of extensions of q^L and p, and $\langle q^{R,n} \rangle_n$ of extensions of q^R and p. $q^{L,0}$ is obtained by applying Lemma 51 in the quotient forcing $\mathbb{P}_\tau / G_\alpha$ for p and q^L. We may also take $q^{L,0} \in D^L$. Therefore $q^{L,0}$ extends q^L, p and has $[q^{L,0}]_{M^\tau_\alpha} \in G_\alpha$. We similarly take $q^{R,0} \in D^R$ to extend q^R, p and satisfy $[q^{R,0}]_{M^\tau_\alpha} \in G_\alpha$.

Let $r^0 \in G_\alpha$ be a common extension of $[q^{L,0}]_{M^\tau_\alpha}$ and $[q^{R,0}]_{M^\tau_\alpha}$. Since $q^{L,0}, q^{R,0}$ belong to the quotient forcing $\mathbb{P}_\tau / G_\alpha \cap M^\tau_\beta[G_\alpha]$ they have common extensions $q^{L,1}$,$q^{R,1} \in \mathbb{P}_\tau / G_\alpha \cap M^\tau_\beta[G_\alpha]$ so that $[q^{L,0}]_{M^\tau_\alpha}, [q^{R,0}]_{M^\tau_\alpha} \leq r^0$. Let $r^1 \in G_\alpha$ be a common extension of the two.

Repeating this process ω-many times, we can secure that for each n, $q^{L,n+1} \leq q^{L,n}$, $q^{R,n+1} \leq q^{R,n}$, $[q^{L,n+1}]_{M^\tau_\alpha} \leq [q^{R,n}]_{M^\tau_\alpha}$, and $[q^{R,n+1}]_{M^\tau_\alpha} \leq [q^{L,n}]_{M^\tau_\alpha}$.

Going back to V, the sequences $\langle q^{L,n} \rangle_n$, $\langle q^{R,n} \rangle_n$ are in V, since $\mathbb{P}_\tau \cap M^\tau_\alpha$ is σ-closed. It is clear from the constructing that the limit conditions p^L of $\langle q^{L,n} \mid n < \omega \rangle$ and p^R of $\langle q^{R,n} \mid n < \omega \rangle$ satisfy the desired properties. □

Lemma 53.
Let $p \in \mathbb{P}_\tau$ and $\alpha < \kappa$ such that $p \in D_\tau(M^\tau_\alpha)$. Suppose that $\langle (\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu \rangle$ is a τ-sequence with $\alpha_0 > \alpha$, and $\langle q^{i,L} \mid i < \nu \rangle$, $\langle q^{i,R} \mid i < \nu \rangle$ are two sequences of conditions that satisfy

- *$\langle (\tau_i, M^{\tau_i}_{\alpha_i}) \mid i < \nu \rangle$ and $\langle q^{i,Z} \mid i < \nu \rangle$ form a p-M.E. for $Z \in \{L, R\}$,*
- *for each $i < \nu$, $[q^{i,L}]_{M^{\tau_i}_\alpha} = [q^{i,R}]_{M^{\tau_i}_\alpha}$.*

Then there are conditions p^L, p^R that extend p and satisfy

1. *$[p^L]_{M^\tau_\alpha} = [p^R]_{M^\tau_\alpha}$, and*
2. *for $Z \in \{L, R\}$ and $i < \nu$, $p^Z \restriction \tau_i \leq q^{i,Z}$.*

Proof. The proof follows from a straightforward modification of the proof of Lemma 49 by applying it in parallel to the sequences $\langle q^{i,L} \mid i < \nu \rangle$ and $\langle q^{i,R} \mid i < \nu \rangle$, and construct corresponding sequences $\langle q^{j,L} \mid j < \nu^* \rangle$ and $\langle q^{j,R} \mid j < \nu^* \rangle$, while maintaining the property $[q^{j,L}]_{M^{\gamma_j}_\alpha} = q^{j,R}]_{M^{\gamma_j}_\alpha}$ along the way. This modification is achieved by using Lemma 52 at each step of the construction in which a decision about meets of nodes of exit nodes made in the construction of $q^{\eta,1}, q^{\eta,2}$ and q^η in the proof of Lemma 49. □

Theorem 54. *Suppose that the book-keeping function Ψ picks each S_τ, a $\mathbb{P}_\tau$-names for (wide) κ-Aronszajn tree. Then for every $\tau < \kappa^+$, no cofinal branches are added to the wide tree T by $\mathbb{P}_\tau$.*

Proof. Assume that $\mathbb{P}_\tau$ introduces a cofinal branch to T. Fix a $\mathbb{P}_\tau$-name $\dot{b}$ which is minimal in the well ordering of $H_{\kappa^{++}}$ for which there is a condition $q \in \mathbb{P}_\tau$ forcing that it is a cofinal branch in T. It follows that every structure in the sequence $\vec{M}^\tau$ contains $\dot{b}$ and q.

Let α be any ordinal in $\mathrm{dom}(\vec{M}^\tau)$. Fix $p \in D_\tau(M^\tau_\alpha)$ such that p forces $\dot{b}(\alpha) = t$ for some $t \in T$. We consider two case:

Case 1: t is not in the range of any f^p_δ for $\delta \in \mathrm{supp}(p) \cap M^\tau_\alpha$.

We can extend the collapse part f^p_0 of p, in the component relevant to t to get two extensions f^L_0, f^R_0 of f^p_0, such that $\pi_{f^L_0}(t) = t^L$ and $\pi_{f^R_0}(t) = t^R$ such that $t^L \neq t^R$ on the same level $\beta < \alpha$. Let p^L, p^R be the extensions of p obtained by extending f^p_0 to f^L_0 and f^R_0 respectively. Since the collapse index map $t^* \in T \mapsto (\alpha_{t^*}, \beta_{t^*}, \gamma_{t^*}, \delta_{t^*})$ is injective (see Remark 11), it follows from the case assumption that collapse index of t is distinct from those of points that appear in range of $f^p_\delta \setminus M^\tau_\alpha$, $\delta \in M^\tau_\alpha \cap \alpha$. We can therefore assume that f^L_0, f^R_0 satisfy $f^L_0 \cap M^\tau_\alpha = f^R_0 \cap M^\tau_\alpha$, and $f^L_0 \setminus M^\tau_\alpha$ and $f^R_0 \setminus M^\tau_\alpha$ agree everywhere, except at $(\alpha_t, \beta_t, \gamma_t, \delta_t, \epsilon_t)$. This, in turn, implies that the resulting conditions p^L, p^R are in $D_\tau(M^\tau_\alpha)$ and $[p^L]_{M^\tau_\alpha} = [p^R]_{M^\tau_\alpha}$. Denote the common trace by w.

$\mathbb{P}_\tau \cap M^\tau_\alpha$ is a complete subforcing of $\mathbb{P}_\tau/p$ and $\dot{b} \in M^\tau_\alpha$ there is $w' \in \mathbb{P}_\tau \cap M^\tau_\alpha$ extending w such that $w' \Vdash \dot{b}(\beta) = \bar{t}$ for some $\bar{t}$ of level β. By Proposition 46 w' is compatible with both p^L and p^R, but then it follows that $\bar{t}$ is equal to both t^L t^R, which is contradiction.

Case 2: t is in the range of f^p_δ for some $\delta \in \mathrm{supp}(p) \cap M^\tau_\alpha$. Such δ has to be unique since for every $\delta_1 \neq \delta_2$ in $M^\tau_\alpha \cap \tau$, the almost everywhere different function $h^{\delta_1}, h^{\delta_2}$ must disagree outside of M^τ_α. Say $f^p_\delta(s) = t$. S_δ is forced by $\mathbb{P}_\delta$ to be an Aronszajn tree. Since $\delta \in M^\tau_\alpha$ and $p \in D_\tau(M^\tau_\alpha)$, $\mathbb{P}_\delta \cap M^\delta_\alpha$ is a complete subforcing of $\mathbb{P}_\delta/p$. Moreover, $\dot{S}_\delta \in M^\tau_\alpha$ and M^τ_α reflects every Π^1_1 statement which has parameters in M^τ_α. Let $G^\delta_\alpha \subseteq \mathbb{P}_\delta \cap M^\delta_\alpha$ be a generic filter that contains $[p \restriction \delta]_{M^\delta_\alpha}$. We obtain that

(a) $S_\delta \cap V_\alpha \in V[G^\delta_\alpha]$

(b) $V[G^\delta_\alpha] \models$ "$S_\delta \cap V_\alpha$ is a wide Aronszajn tree on α".

It follows that the branch b_s leading to s is a branch in $S_\delta \cap V_\alpha$ that does not belong to $V[G^\delta_\alpha]$. Hence it is introduced by forcing over $V[G^\delta_\alpha]$ with the quotient forcing $\mathbb{P}_\delta/G^\delta_\alpha$. Let $M^\delta_{\alpha'} = M^s_1 \in \vec{M}^\delta$ as determined by p. In $M^\delta_{\alpha'}$, we can find two extensions q^L, q^R of $[p \restriction \delta]_{M^\delta_{\alpha'}}$ in the quotient forcing, such that for some $\bar{\alpha} < \alpha$ and some nodes $s^L \neq s^R$ in $\mathrm{Lev}_{\bar{\alpha}}(S_\delta)$, $q^L \Vdash s^L <_{S_\delta} s$ and $q^R \Vdash s^R <_{S_\delta} s$. Moreover, we may assume that $[q^L]_{M^\delta_\alpha} = [q^R]_{M^\delta_\alpha}$. By Lemma 53 applied to the condition p, the τ-sequence of length one, $\langle(\delta, M^\delta_{\alpha'})\rangle$ and the sequences of conditions $\langle q^L\rangle$, $\langle q^R\rangle$, we can find two extensions p^L, p^R of p so that $p^L \restriction \delta \leq q^L$, $p^R \restriction \delta \leq q^R$, and $[p^L]_{M^\tau_\alpha} = [p^R]_{M^\tau_\alpha}$. Since $\dot{b} \in M^\tau_\alpha$, we can find $w \in \mathbb{P}_\tau \cap M^\tau_\alpha$ extending $[p^L]_{M^\tau_\alpha}$ so that $w \Vdash \dot{b}(\bar{\alpha}) = \bar{t}$ and $s^L, s^R \in \mathrm{dom}(f^w_\delta)$. Denote the images of s^L, s^R according to w by $f^w_\delta(s^L) = t^L$ and $f^w_\delta(s^R) = t^R$. Clearly $t^L \neq t^R$ both belong to $\mathrm{Lev}_{\bar{\alpha}}(T^\delta)$ and cannot both be $\bar{t}$. Assume without loss of generality that $t^L \neq \bar{t}$.

Having $w \leq [p^L]_{M^\tau_\alpha}$ we make one last application of strong properness to find a common extension p'. We conclude that

- $f_\delta^{p'}(s) = t$,
- $p' \Vdash t = b(\alpha)$ since $p' \leq p$,
- $p' \restriction \delta \Vdash s^L <_{S^\delta} s$ as $p' \leq q^L$, and
- $f_\delta^{p'}(s^L) = t^L$ as $p' \leq w$.

This implies that $p' \Vdash t^L <_T t$ on the one hand. On the other hand, $p' \Vdash \bar{t} <_T t$ as $p' \leq w$. But this is an absurd as $t^L \neq \bar{t}$ belong to the same level $\mathrm{Lev}_{T^\delta}(\alpha)$. Contradiction.

□

Theorem 55. *Suppose that the book-keeping function Ψ which picks the trees S_τ, picks only names for wide κ-Aronszajn trees and covers all $\mathbb{P}_\tau$-names for wide κ-Aronszajn trees, for all $\tau < \kappa^+$. Let $G \subseteq \mathbb{P}_{\kappa^+}$ be a generic filter. In $V[G]$, $\kappa = \aleph_2$, T is a wide $\aleph_2$-Aronszajn tree on κ which embeds all wide $\aleph_2$-Aronszajn trees.*

Proof. It follows from the proof 50 that T embeds all wide $\aleph_2$-Aronszajn trees on κ. By the last theorem T does not get a cofinal branch, and hence remains $\aleph_2$-Aronszajn. □

7 Open Problems

The following problems are left open by this work:

1. Is the weakly compact cardinal needed for Theorem 1?
   We conjecture that the answer is yes.

2. Is it consistent to have a universal (narrow) Aronszajn tree?
   For $\aleph_2$-Aronszajn trees, we expect this to be possible by incorporating ideas from Mitchell's construction of a model without Aronszajn trees on ω_2 and work towards verifying the details. The case of $\aleph_1$-Aronszajn trees remains unclear.

3. Can we replace $\aleph_2$ by $\aleph_1$ in our result?
   It seems plausible that an adaptation of the construction to ω_1, in which finite supports and approximations are used can lead to a desired result. However, several new arguments are needed to avoid the prolific use of the σ-closure property of our poset.

4. Can the universality result for wide Aronszajn trees hold at successors of singular cardinals?
   A positive answer would likely require developing new methods for iteration at successors of singular cardinals.

5. Can one consistently have a maximal wide $\aleph_2$-Aronszajn and, at the same time, a maximal wide $\aleph_3$-Aronszajn tree?

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