INFINITE DECREASING CHAINS IN THE MITCHELL ORDER

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Abstract. It is known that the behavior of the Mitchell order substantially changes at the level of rank-to-rank extenders, as it ceases to be well-founded. While the possible partial order structure of the Mitchell order below rank-to-rank extenders is considered to be well understood, little is known about the structure in the ill-founded case. The purpose of the paper is to make a first step in understanding this case, by studying the extent to which the Mitchell order can be ill-founded. Our main results are (i) in the presence of a rank-to-rank extender there is a transitive Mitchell order decreasing sequence of extenders of any countable length, and (ii) there is no such sequence of length ω_1.

1. Introduction

Definition 1.1. Let E, E' be two extenders. We write $E \lhd E'$ if E is represented in the (well-founded) ultrapower of V by E'.

The relation $\lhd$, known as the Mitchell order, was introduced by Mitchell in [Mit74] to construct canonical inner models with many measurable cardinals. The Mitchell order, which was initially introduced as an ordering on normal measures, has been extended to extenders and plays a significant role in inner model theory. As a prominent notion in the theory of large cardinals, the study of the Mitchell order and its structure has expanded in recent decades. The behaviour of the Mitchell order on extenders depends on the type of extenders in consideration and naturally becomes more complicated when restricted to stronger types of extenders. A fundamental dividing line in the behaviour of the Mitchell order is its well-foundedness: Mitchell ([Mit83]) has shown that $\lhd$ is well-founded when restricted to normal measures. The question of the well-foundedness of $\lhd$ was further studied by Steel [St93], and Neeman [Ne04], who showed that it fails exactly at the level of rank-to-rank extenders.

Definition 1.2. Let E be an extender with associated ultrapower embedding $i_E \colon V \to \mathrm{Ult}(V, E)$. We say E is a *rank-to-rank extender* iff whenever $\lambda > \mathrm{crit}(E)$ is least such that $i_E(\lambda) = \lambda$, then $V_\lambda \subseteq \mathrm{Ult}(V, E)$.

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Due to their similarity with embeddings $j : V \to M$ with $V_{\lambda+2} \subseteq M$, which have been shown by Kunen to be inconsistent with ZFC, the large cardinal strength witnessed by rank-to-rank extenders is considered to be located near the top of the large cardinal hierarchy. More precisely, rank-to-rank extenders naturally arise from the large cardinal axiom I2.

The known dividing line of well-foundedness naturally breaks the question of the general possible behaviour into two (i) which well-founded partial orderings can be isomorphic to the Mitchell order on measures/extenders below the rank-to-rank level? and (ii) which ill-founded partial orderings can be isomorphic to the Mitchell order on a set of rank-to-rank extenders?[1]

Concerning question (i), the possible structure of the Mitchell order on normal measures has been extensively studied in [Mit83], [Bal85], [Cum93], [Cum94], [Wit96], [BN16], [BN15]. It has been shown in [BN15] that it is consistent for every well-founded partial ordering to be isomorphic to the restriction of $\triangleleft$ to the set of normal measures on some measurable cardinal κ (the exact consistency strength of this property has not been discovered).

In this work, we make a first step towards expanding the study of the Mitchell order in the ill-founded case, and address question (ii). Specifically, we focus on the extent to which the well-foundedness of the Mitchell order fails on rank-to-rank extenders, by considering possible ordertypes of infinite decreasing chains in $\triangleleft$. The main results of this paper are the following two theorems.

Theorem 1.3. *Assume there exists a rank-to-rank extender E. Then for every countable ordinal γ there is a sequence of rank-to-rank extenders of length γ, $(E_\alpha \mid \alpha < \gamma)$, on which the Mitchell order is transitive and strictly decreasing.*

Theorem 1.4. *There is no ω_1-sequence of extenders which is strictly decreasing and transitive in the Mitchell order.*

Our presentation of the proof of Theorem 1.4 goes through a proof of a weak version of Steel's conjecture, which addresses *transitive* ω-sequences of extenders.[2] This presentation replaces a previous ad-hoc proof. The authors would like to thank Grigor Sargysyan for pointing out the connection with Steel's conjecture, which led to the current concise proof of Theorem 1.4. The (full) conjecture was recently proved by Goldberg ([Go]) building on his remarkable study of the internal relation. We believe that our proof of the weaker statement is of interest due to the fact that it only employs elementary concepts of the theory of extenders.

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[1] Beyond the possible (partial) ordering structure of the Mitchell order, the investigation can be further extended to non-transitive relations, as the Mitchell order need not be transitive in general (see [St93]). This direction is not developed in this paper.

[2] See Section 4 for a formulation of Steel's conjecture.

2. Basic definitions and observations

We start with fixing some notation for extenders. A (κ,λ)-extender is a sequence $E = (E_a \mid a \in [\lambda]^{<\omega})$ with $\mathrm{crit}(E) = \kappa$ and $\mathrm{lh}(E) = \lambda$. For $a \in [\lambda]^{<\omega}$, μ_a denotes the least μ such that $a \subseteq j(\mu)$, if we are deriving an extender from j. $i_E\colon V \to \mathrm{Ult}(V,E)$ (we identify $\mathrm{Ult}(V,E)$ with its transitive Mostowski collapse) denotes the ultrapower embedding by E. We also write M_E for $\mathrm{Ult}(V,E)$. For a rank-to-rank extender E we write λ^E for the least $\lambda > \mathrm{crit}(E)$ such that $i_E(\lambda) = \lambda$. Moreover, we write $\kappa_0^E = \mathrm{crit}(E)$ and $\kappa_{n+1}^E = i_E(\kappa_n^E)$ for $n<\omega$, and call $(\kappa_n^E \mid n<\omega)$ the *critical sequence of* E. Note that for any rank-to-rank extender E, $\lambda^E = \sup_{n<\omega}\kappa_n^E = \mathrm{lh}(E)$.
For every $n<\omega$ let $E \restriction \kappa_n^E$ be the cutback of E to the measures E_a, $a \in [\kappa_n^E]^{<\omega}$. We have that $\mathrm{Ult}(V,E) = \bigcup_n N_n$ where $N_n = \mathrm{Ult}(V, E\restriction\kappa_n^E)$. Moreover, for each $n<\omega$, N_{n+1} contains $V_{\kappa_{n+1}^E}$ and in particular $i_E``\kappa_n^E \in N_{n+1}$. It follows from a standard argument that ${}^{\kappa_n^E}N_{n+1} \subseteq N_{n+1}$.
The following observations will be useful in comparing extenders in different ultrapowers.

Lemma 2.1. *Assume that $M = \bigcup_{n<\omega} N_n$ is an increasing union of transitive classes N_n, and there exists a sequence of cardinals $\langle\kappa_n \mid n<\omega\rangle$, such that ${}^{\kappa_n}N_n \subseteq N_n$ for all $n<\omega$. Let $E \in M$ be a rank-to-rank extender of height $\mathrm{lh}(E) = \lambda = \bigcup_n \kappa_n$.*

(1) For every $a \in [\lambda]^{<\omega}$ and $f : [\mu_a]^{|a|} \to M$ a function in V, there exists a function $f_M \in M$ such that $\{\nu \in [\mu_a]^{|a|} \mid f(\nu) = f_M(\nu)\} \in E_a$.
(2) Let $i_E : V \to \mathrm{Ult}(V,E)$ and $i_E^M : M \to \mathrm{Ult}(M,E)$ be the ultrapower embeddings by E of V and M, respectively. For every set $x \in M$, $i_E(x) = i_E^M(x)$.

Proof. (1) $M_E = \bigcup_n N_n$ is an increasing union, and E_a is σ-complete, there exists some n such that the set $A_n = \{\nu \in [\mu_a]^{|a|} \mid f(\nu) \in N_n\}$ belongs to E_a and $\mu_a < \kappa_n^E$. Since N_{n+1} is closed under κ_n^E-sequences, it follows that $f \restriction A_n : [\mu_a]^{|a|} \to V_{\kappa_n^E}$ belongs to $N_{n+1} \in M$. The claim follows.
(2) For every $x \in M$, $i_E(x)$ ($i_E^M(x)$) consists of equivalence classes $[a,f]_E$ of $a \in [\lambda]^{<\omega}$ and $f : [\mu_a]^{|a|} \to x$ in V (in M, respectively). Since $M = \bigcup_n N_n$ and each N_n is transitive, $x \subseteq M$. Hence, by the first part of the Lemma, $i_E(x) = i_E^M(x)$.

□

Lemma 2.2. *Suppose E_2, E_1, E_0 are three rank-to-rank extenders of the same length $\lambda = \lambda^{E_i}$, $i = 0,1,2$, such that E_2 is Mitchell order below E_1, and both E_2, E_1 are Mitchell order below E_0. Then $\mathrm{Ult}(V,E_0)$ also witnesses that E_2 is Mitchell order below E_1.*

Proof. The fact $E_2 \lhd E_1$ means that E_2 is represented in the V-ultrapower by E_1, by a function f and a generator a of E_1. Take $k < \omega$ such that $a \in [\kappa^{E_1}_{k+1}]^{<\omega}$. Write $M_{E_0} = \mathrm{Ult}(V, E_0)$.
The reason it is not immediate that the relation $E_2 \lhd E_1$ also holds in M_{E_0} is that the function f need not belong to M_{E_0}. However, we can argue that M_{E_0} does see some witnessing function f^* by using approximations. Assume without loss of generality that $f(\nu)$ is an extender for every $\nu \in \mathrm{dom}(f)$. Indeed, notice that for every $n < \omega$, the function f_n with $\mathrm{dom}(f_n) = \mathrm{dom}(f)$ such that for every ν, $f_n(\nu)$ is the restriction of the extender $f(\nu)$ to length $\kappa^{E_1}_n$. Clearly, $[a, f_n]^V_{E_1}$ represents the cut back of E_2 to length $\kappa^{E_1}_{n+1}$. Moreover, f_n belongs to M_{E_0} for every $n < \omega$ since $V_\lambda \subseteq M_{E_0}$.
Let $\mathcal{E}(\eta)$ for some ordinal η denote the set of all (κ, η)-extenders. Now, working in M_{E_0} and utilizing the fact that both E_1, E_2 belong to the model, we consider the tree T of all pairs (τ, n) such that $\tau : [\kappa^{E_1}_k]^{|a|} \to \mathcal{E}(\kappa^{E_1}_n)$ satisfies that $[a, \tau]^{M_{E_0}}_{E_1}$ represents the restriction of E_2 to length $\kappa^{E_1}_{n+1}$. The tree order $<_T$ is given by $(\tau, n) <_T (\tau', n')$ if $n < n'$ and $\tau'(\nu)$ extends $\tau(\nu)$ for all $\nu \in \mathrm{dom}(\tau)$. It is clear that a cofinal branch in T translates to a function F for which $[a, F]_{E_1}$ represents E_2, and vice versa. Therefore $f \in V$ witnesses that T has a cofinal branch in V, and thus, by absoluteness of well-foundedness, it must also have one in M_{E_0}. $\square$

Steel gives in [St93] a folklore example that for rank-to-rank extenders the Mitchell order need not be well-founded. We recall it here because some of the ideas will be used later.

Proposition 2.3 (Folklore)**.** *Let E be a rank-to-rank extender. Then there is a strictly decreasing sequence of length ω in the Mitchell order on which $\lhd$ is transitive.*

Proof. Consider the following sequence of rank-to-rank extenders $(E_n : n < \omega)$. Let $E_0 = E$ and $E_{n+1} = i_{E_n}(E_n)$, where $i_{E_n} : V \to \mathrm{Ult}(V, E_n)$ is the canonical embedding associated to E_n. Then it is straightforward to check that every E_n is a V-extender and $E_{n+1} \lhd E_n$ for all $n < \omega$.

Claim 1. The Mitchell order is transitive on $(E_n : n < \omega)$.

Proof. Let $n < \omega$. We show that $E_{n+2} \lhd E_n$, the rest follows analogously. By construction $E_{n+2} \in \mathrm{Ult}(V, E_{n+1})$ and $E_{n+1} \in \mathrm{Ult}(V, E_n)$, we argue that $E_{n+2} \in \mathrm{Ult}(V, E_n)$. Write $M = \mathrm{Ult}(V, E_n)$ and argue that $i_{E_{n+1}}(E_{n+1}) = i^M_{E_{n+1}}(E_{n+1})$, where $i^M_{E_{n+1}} : M \to \mathrm{Ult}(M, E_{n+1})$. By Lemma 2.1, applied to $E = E_{n+1}$ and M, we see that $E_{n+2} = i_{E_{n+1}}(E_{n+1}) = i^M_{E_{n+1}}(E_{n+1})$, and hence $E_{n+2} \in M$. $\square$

$\square$

3. Countable decreasing sequences in the Mitchell order

We now prove Theorem 1.3 and show that there can be strictly decreasing transitive sequences in the Mitchell order of any countable length.

Proof of Theorem 1.3. Let E be a rank-to-rank extender with critical sequence $(\kappa_n \mid n < \omega)$ and $\lambda^E = \lambda$. In what follows, all extenders will have the same length λ. We start by introducing some notation for sequences of extenders as constructed in the proof of Proposition 2.3. For a rank-to-rank extender F, write $S^1(F) = i_F(F)$ and $S^{n+1}(F) = i_{S^n(F)}(S^n(F))$. Now, the decreasing sequences in the Mitchell order we construct will be of the following form.

Definition 3.1. Let $\vec{E} = (E_\alpha \mid \alpha < \gamma)$ be a sequence of rank-to-rank extenders. Then we say that $\vec{E}$ *is guided by an internal iteration* iff there are well-founded models $(M_\alpha \mid \alpha \leq \gamma)$, $(M^*_\alpha \mid \alpha \leq \gamma)$ with $M_0 = M^*_0 = V$ and elementary embeddings $j^*_{\alpha,\beta} \colon M^*_\alpha \to M^*_\beta$ for all $\alpha < \beta \leq \gamma$ such that

1. (1) $E_{\alpha+1} = S^n(E_\alpha)$ for some $n \geq 1$ and all $\alpha + 1 < \gamma$,
2. (2) $M_\nu = M^*_\nu$ for all limit ordinals $\nu \leq \gamma$ is given as the direct limit of the directed system $\langle M^*_\alpha, j^*_{\alpha,\beta} \mid \alpha \leq \beta < \nu \rangle$.
3. (3) $M_{\alpha+1} = \mathrm{Ult}(V, E_\alpha)$ and $M^*_{\alpha+1} = M^{M^*_\alpha}_{\alpha+1}$ for all $\alpha + 1 \leq \gamma$, where for a model N, $M^N_{\alpha+1}$ denotes $\mathrm{Ult}(N, E_\alpha)$,
4. (4) for all limit ordinals $\nu < \gamma$, $E_\nu = j^*_{0,\nu}(E_0)$ and $\mathrm{Ult}(V, E_\nu)$ is well-founded, and
5. (5) the following diagram commutes and all maps in the diagram are given by internal ultrapowers.

$$\begin{tikzcd}
V \arrow[r] & M^*_1 \arrow[r] & M^*_2 \arrow[r] & M^*_3 \arrow[r] & \cdots & M_\omega \arrow[r] & M^*_{\omega+1} \arrow[r] & M^*_{\omega+2} \arrow[r] & \cdots & M^*_\gamma \\
 & V \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & M_3^{M_2} \arrow[r] \arrow[u] & \cdots & & M_\omega \arrow[r] \arrow[u] & M^{M_\omega}_{\omega+2} \arrow[r] \arrow[u] & \cdots & \\
 & & V \arrow[r] \arrow[u] & M_3 \arrow[r] \arrow[u] & \cdots & & & M_\omega \arrow[r] \arrow[u] & \cdots & \\
 & & & V \arrow[u] & \ddots & & & & \ddots &
\end{tikzcd}$$

Before we prove the existence of sequences of extenders guided by internal iterations, we show an abstract claim which will allow us to extend any such sequence by one further element. This is shown as in the proof of Theorem 2.2 in [St93] (where this particular argument is attributed to Martin). We sketch the proof here for the reader's convenience.

Claim 2. Let M be an internal iterate of V via a rank-to-rank extender of length λ and $E \lhd F$ extenders of length λ with $E, F \in M$. Then the following diagram commutes and all maps are given by internal ultrapowers.

$$\begin{array}{ccc} \mathrm{Ult}(M,F) & \longrightarrow & \mathrm{Ult}(\mathrm{Ult}(M,F),E) = \mathrm{Ult}(\mathrm{Ult}(M,E), i_E(F)) \\ \uparrow & & \uparrow \\ M & \longrightarrow & \mathrm{Ult}(M,E) \end{array}$$

Proof. First, we note that $\mathrm{Ult}(\mathrm{Ult}(M,E), i_E(F)) = i_E(\mathrm{Ult}(M,F))$, where we consider $\mathrm{Ult}(M,F)$ as a class of M. Second, we assume $E \lhd F$ in V, which by Lemma 2.2 implies that $E \lhd F$ in M as well, i.e, $E \in \mathrm{Ult}(M,F)$. We can therefore form the internal ultrapower of $\mathrm{Ult}(M,F)$ by E. Let $i_E^{\mathrm{Ult}(M,F)} \colon \mathrm{Ult}(M,F) \to \mathrm{Ult}(\mathrm{Ult}(M,F),E)$ be the resulting ultrapower embedding. Finally, we have that every $x \in i_E(\mathrm{Ult}(M,F))$ can be indentified with equivalent classes $[a,f]_E$, where $a \in [\lambda]^{<\omega}$ and $f \colon [\mu_a]^{|a|} \to \mathrm{Ult}(M,F)$, and by Lemma 2.1, $[a,f]_E = [a,g]_E$ for some function $g \in \mathrm{Ult}(M,F)$. It follows that $i_E(\mathrm{Ult}(M,F))$ identifies with the internal ultrapower $\mathrm{Ult}(\mathrm{Ult}(M,F),E)$, and $i_E \restriction \mathrm{Ult}(M,F)$ with $i_E^{\mathrm{Ult}(M,F)}$. □

Next, we argue that sequences as in Definition 3.1 are in fact as desired.

Claim 3. If a sequence $\vec{E} = (E_\alpha \mid \alpha < \gamma)$ of extenders is guided by an internal iteration, the embeddings in the diagram witness that $E_\beta \lhd E_\alpha$ for all $\alpha < \beta < \gamma$. In particular, $\vec{E}$ is a decreasing transitive sequence in $\lhd$.

Proof. We start by showing the following subclaim.

Subclaim 1. For each $\alpha < \gamma$, $E_\alpha \in M^*_\alpha$.

Proof. The claim is immediate when α is a limit ordinal or $\alpha = 0$. Let $\alpha = \beta + 1$ be a successor ordinal and assume inductively that $E_\beta \in M^*_\beta$. Suppose first that $E_\alpha = S^n(E_\beta)$ for $n = 1$. Then $E_\alpha = i_{E_\beta}(E_\beta)$, where $i_{E_\beta} \colon V \to \mathrm{Ult}(V, E_\beta)$ is the V-ultrapower embedding given by E_β. Recall that $M^*_\alpha = \mathrm{Ult}(M^*_\beta, E_\beta)$. By applying Lemma 2.1 to $M = M^*_\beta$ and $E = E_\beta$, we conclude that $E_\alpha = i_{E_\beta}^{M^*_\beta}(E_\beta) \in \mathrm{Ult}(M^*_\beta, E_\beta) = M^*_\alpha$. For $n > 1$, the result is similarly obtained by applying the Lemma n many times. □

Using this subclaim and the fact that all maps in the diagram witnessing that $\vec{E}$ is guided by an internal iteration are internal ultrapowers, it is straightforward to verify that $E_\alpha \lhd E_n$ whenever $n < \alpha$, using the fact there are internal iterations from M_{n+1} (the ultrapower of V by E_n), and up to M^*_{n+1} (i.e., vertical maps), and from M^*_{n+1} to M^*_α. More generally, to see that $E_\alpha \lhd E_\beta$ for all $\beta < \alpha$, suppose $\beta = \eta + n$ for some limit ordinal η and

$$\begin{tikzcd}
\mathrm{Ult}(V, E_{\eta+n}) \arrow[r] & \mathrm{Ult}(M_1, E_{\eta+n}) \arrow[r] & \mathrm{Ult}(M_2^*, E_{\eta+n}) \arrow[r] & \cdots \ \mathrm{Ult}(M_\eta, E_{\eta+n}) \\
V \arrow[u] \arrow[r] & M_1 \arrow[u] \arrow[r] & M_2^* \arrow[u] \arrow[r] & \cdots \ M_\eta \arrow[u]
\end{tikzcd}$$

$n < \omega$. The proof of Claim 2 shows that the following diagram commutes and moreover that the top row is an internal iteration of $\mathrm{Ult}(V, E_{\eta+n})$.

This, in combination with the subclaim and the fact that all maps in the diagram witnessing that $\vec{E}$ is guided by an internal iteration are internal ultrapowers as well, suffices to argue that $E_\alpha \lhd E_{\eta+n}$. $\square$

We now turn to prove following claim, which immediately yields the theorem.

Claim 4. Assume there is a rank-to-rank extendere E. For every countable ordinal $\gamma < \omega_1$ and an ordinal $\kappa < \lambda = \lambda^E$, there is a sequence of rank-to-rank extenders $\vec{E} = (E_\alpha \mid \alpha \leq \gamma)$ which is guided by an internal iteration, such that the induced embedding $j^*_{0,\gamma} \colon V \to M^*_\gamma$ satisfies $\kappa < \mathrm{crit}(j^*_{0,\gamma}) < \lambda$ and $j^*_{0,\gamma}(\lambda) = \lambda$.

Proof. We prove the claim by induction on $\gamma < \omega_1$. There is nothing to show for $\gamma = 0$. Suppose that $\gamma = 1$. Let $\kappa < \lambda$ be arbitrary and fix some $n < \omega$ such that $\kappa < \kappa_n^E$. Then $E_0 = S^{n+1}(E)$ and $E_1 = S^1(E_0)$ giving rise to $j^*_{0,1} \colon V \to \mathrm{Ult}(V, E_0)$ are as desired.

Now, suppose $\gamma = \alpha + 1$ and the claim holds for α witnessed by $(E_\nu \mid \nu \leq \alpha)$. Let n be such that $\kappa < \kappa_n^{E_\alpha}$, the n-th element of E_α's critical sequence, and $E_{\alpha+1} = S^n(E_\alpha)$. Then $\mathrm{crit}(E_{\alpha+1}) \geq \kappa_n^\alpha$ and hence, using that inductively $\mathrm{crit}(j^*_{0,\alpha}) > \kappa$, we have $\mathrm{crit}(j^*_{0,\alpha+1}) > \kappa$. Using Claim 2 the extended sequence $(E_\nu \mid \nu \leq \alpha + 1)$ is as desired.

Finally, suppose $\gamma < \omega_1$ is a limit ordinal and fix an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$, cofinal in γ, with $\alpha_0 = 0$. We also fix a well-ordering $<_w$ of H_{λ^+} in V.

By the inductive hypothesis applied to α_1, there is a sequence $\vec{E}^0$ of rank-to-rank extenders which is guided by an internal iteration and an elementary embedding $j^*_{0,\alpha_1} \colon V \to M^*_{\alpha_1}$ with critical point $\nu_0 > \kappa$ and $j^*_{0,\alpha_1}(\lambda) = \lambda$. We pick $\vec{E}^0$ to be the minimal such sequence with respect to $<_w$. By elementarity of j^*_{0,α_1}, we can apply the inductive hypothesis again inside $M^*_{\alpha_1}$ to get a sequence $\vec{E}^1$ of rank-to-rank extenders which is guided by an internal iteration and an elementary embedding $j^*_{\alpha_1,\alpha_2} \colon M^*_{\alpha_1} \to M^*_{\alpha_2}$ with critical point $\nu_1 > j^*_{0,\alpha_1}(\nu_0)$ and $j^*_{\alpha_1,\alpha_2}(\lambda) = \lambda$. We take $\vec{E}^1$ to be the minimal such sequence in $M^*_{\alpha_1}$, with respect to $j^*_{0,\alpha_1}(<_w)$. Repeating this procedure yields a sequence $((\vec{E}^n, j^*_{\alpha_n,\alpha_{n+1}}) \mid n < \omega)$ of sequences of extenders together with elementary embeddings.

Let $(E_\alpha \mid \alpha < \gamma)$ be the concatenation of the sequences $\vec{E}^n$, $n < \omega$. The choice of $\vec{E}^{n+1}$ to be minimal in $M^*_{\alpha_n}$ with respect to the well ordering $j^*_{0,\alpha_n}(<_w)$ guarantees that the sequence $\langle \vec{E}^m \mid m > n\rangle$ belongs to $M^*_{\alpha_n}$, and thus also the tail of the iteration $\langle M^*_\alpha, j^*_{\alpha,\beta} \mid \alpha_n < \alpha \leq \beta < \gamma\rangle$. Note that $\mathrm{crit}(j^*_{0,\gamma}) > \kappa$ since $\mathrm{crit}(j^*_{\alpha_n,\alpha_{n+1}}) > \kappa$ for all $n < \omega$. Let $j^*_{0,\gamma} \colon V \to M^*_\gamma = M_\gamma$ be the direct limit embedding of the system.

The *reflecting a minimal counterexample* argument used to show that internal iterations by normal ultrafilters are well-founded (see e.g., [Je03, Theorem 19.7] for normal ultrafilters or [Di18, Proposition 5.8] for rank-to-rank extenders), can also be used to show that M_γ is well-founded.

Subclaim 2. $j^*_{0,\gamma}(\lambda) = \lambda$.

Proof. Suppose not. Then there is some $\eta < \lambda$ such that $j(\eta) \geq \lambda$. But for every $\eta < \lambda$, there is by choice of the embeddings $j^*_{\alpha_n,\alpha_{n+1}}$ some $n < \omega$ such that $\nu_n = \mathrm{crit}(j^*_{\alpha_n,\alpha_{n+1}}) > \eta$, i.e. $j^*_{\alpha_n,\gamma}(\eta) = \eta$. □

Moreover, it is clear from the construction that, letting $E_\gamma = j^*_{0,\gamma}(E)$, the resulting sequence $(E_\alpha \mid \alpha \leq \gamma)$ of rank-to-rank extenders is guided by an internal iteration. The only condition that needs a small argument is the following.

Subclaim 3. $\mathrm{Ult}(V, E_\gamma)$ is well-founded.

Proof. As M_γ is well-founded and E_γ is an extender in M_γ, $\mathrm{Ult}(M_\gamma, E_\gamma)$ is well-founded. We prove the subclaim by defining an elementary embedding $\pi\colon \mathrm{Ult}(V, E_\gamma) \to \mathrm{Ult}(M_\gamma, E_\gamma)$ as follows. For $[a, f]^V_{E_\gamma} \in \mathrm{Ult}(V, E_\gamma)$, let

$$\pi([a, f]^V_{E_\gamma}) = [a, j^*_{0,\gamma}(f) \circ j^*_{0,\gamma}]^{M_\gamma}_{E_\gamma}.$$

π is well-defined since for $a \in [\lambda]^{<\omega}$, $\mu_a < \lambda$. Therefore, $j^*_{0,\gamma} \restriction [\mu_a]^{|a|} \in V_\lambda \subseteq M_\gamma$. In addition, $j^*_{0,\gamma}(f) \circ j^*_{0,\gamma} = j^*_{0,\gamma} \circ f$, so π is elementary. □

□

This finishes the proof of Theorem 1.3. □

4. A bound on the length of decreasing sequences in the Mitchell order

Steel proved in [St93] that in a Mitchell order decreasing sequence of rank-to-rank extenders, the extenders cannot all have the same critical point. He conjectured the following stronger statement.

Conjecture 4.1 (Steel)**.** *Suppose that $(E_m \mid m < \omega)$ is a sequence of rank-to-rank extenders which is strictly decreasing in $\lhd$. Let λ be the unique ordinal such that $\lambda = \lambda^{E_m}$ for all sufficiently large m. Then* $\sup_{m<\omega} \mathrm{crit}(E_m) = \lambda$.

Theorem 4.2 below establishes Steel's Conjecture for the special case that the Mitchell order is *transitive* on the sequence $(E_m \mid m < \omega)$.

Theorem 4.2. *Suppose that $(E_m \mid m < \omega)$ is a sequence of rank-to-rank extenders, which is strictly decreasing and transitive in $\lhd$. Let λ be the unique ordinal such that $\lambda = \lambda^{E_m}$ for all sufficiently large m. Then* $\sup_{m<\omega} \operatorname{crit}(E_m) = \lambda$.

Proof. Suppose otherwise, and let γ_0 be the minimal ordinal for which there exists a $\lhd$-decreasing and transitive sequence $\vec{E} = (E_m \mid m < \omega)$ such that $\gamma_0 = \sup_{m<\omega} \kappa_0^{E_m} < \lambda^{\vec{E}} = \sup_{m<\omega} \lambda^{E_m}$. We assume without loss of generality that $\lambda^{\vec{E}} = \lambda^{E_m}$ for all $m < \omega$. Let $n < \omega$ be the integer for which $\kappa_n^{E_0} \leq \gamma_0 < \kappa_{n+1}^{E_0}$. We move to the ultrapower M_{E_0}. By our assumption, $E_m \in M_{E_0}$ for every $m > 0$, and by Lemma 2.2, M_{E_0} sees that E_m is Mitchell order below E_k for every $0 < k < m < \omega$. Since M_{E_0} is not closed under ω-sequences of its elements (in V) the sequence $(E_m \mid 1 \leq m < \omega)$ need not belong to M_{E_0}. Nevertheless, we may define in M_{E_0} the tree T^* of all finite sequences of rank-to-rank extenders $(E^*_m \mid m < N)$, which are strictly Mitchell order decreasing, transitive, have length $\lambda^{\vec{E}}$, and satisfy $\gamma_0 \geq \max_{m<N} \kappa_0^{E^*_m}$. The sequence $(E_m \mid 1 \leq m < \omega)$ witnesses that T^* has a cofinal branch in V. So by absoluteness of well-foundedness there is a cofinal branch in M_{E_0} as well. We can now reflect this from M_{E_0} back to V. Using the fact that $\kappa_n^{E_0} \leq \gamma_0 < \kappa_{n+1}^{E_0} = i_{E_0}(\kappa_n^{E_0})$, we conclude that in V, there exists some $\gamma_{-1} < \kappa_n^{E_0} \leq \gamma_0$, and a sequence $\vec{E}^* = (E^*_m \mid m < \omega)$ of rank-to-rank extenders which is strictly $\lhd$-decreasing and transitive such that $\sup_{m<\omega} \kappa_0^{E^*_m} \leq \gamma_{-1} < \lambda^{\vec{E}^*}$. This is a contradiction to the minimality of γ_0. □

Now we can obtain Theorem 1.4 as a corollary.

Proof of Theorem 1.4. Suppose otherwise. Let $\vec{E} = (E_\alpha \mid \alpha < \omega_1)$ be a sequence of extenders which is strictly decreasing and transitive in the Mitchell order. We may assume that all E_α are rank-to-rank extenders and that there exists some $\lambda^{\vec{E}}$ such that $\lambda^{E_\alpha} = \lambda^{\vec{E}}$ for all $\alpha < \omega_1$. In particular, $\operatorname{cf}(\lambda^{\vec{E}}) = \omega$ and we may choose a cofinal sequence $(\rho_n \mid n < \omega)$ in $\lambda^{\vec{E}}$. By a straightforward pressing down argument, we can find an uncountable set $I \subseteq \omega_1$ and some $n^* < \omega$ such that $\kappa_0^{E_\alpha} < \rho_{n^*}$ for all $\alpha \in I$. Taking $(\alpha_n \mid n < \omega)$ to be the first ω many ordinals of I, it follows that $(E_{\alpha_n} \mid n < \omega)$ is strictly decreasing and transitive in the Mitchell order, with $\sup_{n<\omega} \kappa_0^{E_{\alpha_n}} \leq \rho_{n^*} < \lambda^{\vec{E}}$. This contradicts Theorem 4.2. □

5. Questions

After studying the length of the Mitchell order for rank-to-rank extenders, a natural question that arises is about the structure this order can have.

Question 5.1. *Suppose there is a rank-to-rank extender. Can the tree order on the infinite binary tree $2^{<\omega}$ be realized by a Mitchell order?*

We can even ask the following more general question.

Question 5.2. *Suppose there is a rank-to-rank extender. Can any tree order on ω be realized by a Mitchell order?*

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