ON THE POWERSETS OF SINGULAR CARDINALS IN HOD

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Abstract. We prove that the assertion "There is a singular cardinal κ such that for all $x \subseteq \kappa$, HOD_x does not contain the entire powerset of κ" is equiconsistent with the assertion that there is a cardinal κ such that $\{o(\nu) \mid \nu < \kappa\}$ is unbounded in κ.

Shelah [9] proved that for every singular cardinal κ of uncountable cofinality there exists $x \subseteq \kappa$ such that $\mathcal{P}(\kappa) \subseteq \mathrm{HOD}_x$. In work by Cummings, Friedman, Magidor, Rinot, and Sinapova [2] it has been shown that the statement is consistently false for a singular κ of cofinality ω, starting from the large cardinal assumption of an inaccessible cardinal λ and an infinite sequence of $< \lambda$-supercompact cardinals below it. In fact they prove the stronger assertion that it is consistent that there is a singular cardinal κ of cofinality ω such that for all $x \subseteq \kappa$, κ^+ is inaccessible in HOD_x. Gitik and Merimovich ([6]) and Kafkoulis ([7]) have obtained the same result from a slightly weaker assumption (see Remarks 6,7 in [2]).

We isolate the exact strength of the failure of Shelah's theorem at a singular cardinal of cofinality ω. In particular, we prove the following theorems.

Theorem 1. *Assuming there is a cardinal κ of cofinality ω such that $\{o(\nu) \mid \nu < \kappa\}$ is unbounded in κ, there is a forcing extension in which for every subset x of κ, HOD_x does not contain the powerset of κ.*

Theorem 2. *Assuming that κ is a singular strong limit cardinal of cofinality ω such that $\{o^K(\nu) \mid \nu < \kappa\}$ is bounded in κ, there is $x \subseteq \kappa$ such that $\mathcal{P}(\kappa) \subseteq \mathrm{HOD}_x$.*

We also show that in Theorem 1, κ can be made into $\aleph_\omega$ using standard arguments.

Recently, there has been renewed interest in the extent of covering properties for HOD. In particular, we mention papers of Cummings, Friedman and Golshani [3], the first and fourth author [1], and Gitik and Merimovich [6]. The central idea in all of these works is the notion of homogeneity of forcing posets. Theorem 1 is a continuation of this study, but at a considerably lower consistency strength.

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The proof of Theorem 2 is a strong refinement of the proof of the Mitchell's covering lemma which is possible precisely under our anti-large cardinal assumption. The main idea of the proof is to give a uniform version of the Mitchell's covering lemma so that all the covering sets for subsets of κ can be defined relative to a single assignment of indiscernibles.

Theorems 1 and 2 are proved in Sections 1 and 2 respectively.

1. Upper bound

Suppose that in V, there exists a cardinal κ so that $\{o(\nu) \mid \nu < \kappa\}$ is unbounded in κ. Since the first such cardinal has countable cofinality, we may assume that $\mathrm{cf}(\kappa) = \omega$. We will define a "short extender" type forcing to add κ^+ many ω-sequences to a singular cardinal $\kappa = \sup_{n<\omega} \kappa_n$ and prove that the forcing has enough homogeneity to establish the conclusion. Let $\langle \kappa_n \mid n < \omega\rangle$ and $\langle \lambda_n \mid n < \omega\rangle$ be increasing sequences of regular cardinals with $\kappa_n < \lambda_n < \kappa_{n+1}$ and $o(\lambda_n) = \kappa_n$ for all $n < \omega$.

By arguments from [4], we can pass to a generic extension in which each λ_n carries a Rudin-Keisler increasing sequence of ultrafilters $\langle U_{n,\alpha} \mid \alpha < \kappa_n\rangle$. By Proposition 1.1.1 of [5] we can assume that each sequence $\langle U_{n,\alpha} \mid \alpha < \kappa_n\rangle$ is a "nice system". We remark that the notion of a nice system abstracts the details of deriving a Rudin-Keisler increasing sequence of ultrafilters from an elementary embedding by an extender, which is commonly used in the study of extender based forcing. Of course in this case we do not have such an extender embedding. For each $\alpha < \beta < \kappa_n$ let $\pi^n_{\beta,\alpha}$ be the Rudin-Keisler projection map from $U_{n,\beta}$ to $U_{n,\alpha}$. We drop the n on $\pi^n_{\beta,\alpha}$ when it is understood.

For each $n < \omega$, the sequence $\langle U_{n,\alpha} \mid \alpha < \kappa_n\rangle$ from [4] is shown to satisfiy the following property, which essentially asserts that that for each $\alpha < \beta < \kappa_n$, the measure $U_{n,\beta}$ is strictly above $U_{n,\alpha}$ in the Rudin-Keisler order.

Lemma 3. *Let $n < \omega$, $\beta < \kappa_n$, and $r \subseteq \beta$. For every $U_{n,\beta}$ measure one set A, there are distinct $\nu_1, \nu_2 \in A$ such that $\pi^n_{\beta,\alpha}(\nu_1) = \pi^n_{\beta,\alpha}(\nu_2)$ for every $\alpha \in r$.*

We are now ready to define the components of the main forcing.

1.1. **The Forcing.**

Definition 4. Let $\mathbb{Q}_{n1} = \{f \mid f$ is a partial function from κ^+ to λ_n with $|f| \leq \kappa\}$ ordered by extension.

Definition 5. Let $(a, A, f) \in \mathbb{Q}_{n0}$ if:

(1) $f \in \mathbb{Q}_{n1}$.
(2) a is a partial order preserving function from κ^+ to κ_n with $|\mathrm{dom}(a)| < \kappa_n$, $\mathrm{rng}(a)$ has a maximal element $\mathrm{mc}(a)$ and $\mathrm{dom}(a) \cap \mathrm{dom}(f) = \emptyset$.
(3) $A \in U_{n,\mathrm{mc}(a)}$.

(4) For all $\nu \in A$ and all $\alpha < \beta$ from $\mathrm{dom}(a)$, $\pi^n_{\mathrm{mc}(a),a(\alpha)}(\nu) < \pi^n_{\mathrm{mc}(a),a(\beta)}(\nu)$.
(5) For all $\alpha < \beta < \gamma$ from $\mathrm{dom}(a)$, $\pi^n_{a(\gamma),a(\alpha)}(\rho) = \pi^n_{a(\beta),a(\alpha)}(\pi^n_{a(\gamma),a(\beta)}(\rho))$ for all $\rho \in \pi^n_{\mathrm{mc}(a),a(\gamma)}$"$A$.

Define $(a, A, f) \leq (b, B, g)$ if:
(1) $f \leq g$ in $\mathbb{Q}_{n1}$.
(2) $b \subseteq a$.
(3) $\pi_{\mathrm{mc}(a),\mathrm{mc}(b)}$"$A \subseteq B$.

We use these as the components in our diagonal Prikry type forcing. Let $p = \langle p_n \mid n < \omega\rangle$ be in $\mathbb{P}$ if there is $l = \mathrm{lh}(p)$ such that for all $n < l$, $p_n \in \mathbb{Q}_{n1}$ and all $n \geq l$, $p_n \in \mathbb{Q}_{n0}$ where if we write $p_n = (a_n, A_n, f_n)$ then for $m \geq n \geq l$, $\mathrm{dom}(a_m) \supseteq \mathrm{dom}(a_n)$.

For ease of notation we write $p_n = f^p_n$ for $i < \mathrm{lh}(p)$ and $p_n = (a^p_n, A^p_n, f^p_n)$ for $n \geq \mathrm{lh}(p)$.

For $p \in \mathbb{P}$ and $\nu \in A^p_{\mathrm{lh}(p)}$ we define a *one point extension* $p \frown \nu$ to be the condition of length $\mathrm{lh}(p) + 1$ with $f^{p\frown\nu}_{\mathrm{lh}(p)}$ as $f^p_{\mathrm{lh}(p)} \cup \{(\gamma, \pi^{\mathrm{lh}(p)}_{\mathrm{mc}(a),a(\gamma)}(\nu) \mid \gamma \in \mathrm{dom}(a)\}$ and the rest of the condition unchanged. We define *n-step extensions* for $n > 1$ by iterating one point extensions. We write $p \leq^* q$ if $\mathrm{lh}(p) = \mathrm{lh}(q)$ and for all $n < \omega$, $p_n \leq q_n$ in the appropriate poset. The ordering we will force with is obtained by a combination of direct extensions and one point extensions.

The key property of the order is the following:

Lemma 6. *For $p, q \in \mathbb{P}$, $p \leq q$ if and only if there are $n < \omega$ and $\vec{\nu}$ of length n such that $p \leq^* q \frown \vec{\nu}$.*

For a condition $q \in \mathbb{P}$, let $\mathbb{P}/q$ denote the cone of conditions $p \in \mathbb{P}$ extending q. Next we sketch a proof of the Prikry property, since it is mostly standard.

Lemma 7. *Let $p \in \mathbb{P}$ and φ be a statement in the forcing language. There is a direct extension of p which decides φ.*

As a first step, we construct $p^* \leq^* p$ such that for all sequences $\vec{\nu}$ if there is a direct extension of $p^* \frown \vec{\nu}$ which decides φ, then $p^* \frown \vec{\nu}$ decides φ. This is done by constructing a sequence of direct extensions $\langle p^n \mid n \geq \mathrm{lh}(p)\rangle$ with $p^{\mathrm{lh}(p)} = p$ and for $m \geq 1$, constructing $p^{\mathrm{lh}(p)+m}$ by diagonalizing over possible m-step extensions of $p^{\mathrm{lh}(p)+m-1}$. Crucially, we use the fact that the order on the a_n parts of conditions in $\mathbb{Q}_{n0}$ is $\kappa_n > \lambda_{n-1}$ complete. Note that the Rudin-Kiesler ordering of the system of measures $\langle U_{n,\alpha} \mid \alpha < \kappa_n\rangle$ is $\mathrm{cf}(\kappa_n) = \kappa_n$ complete.

In the second step, we produce measure one sets so that for $m < \omega$, any two $m + 1$-sequences $\vec{\nu} \frown \rho$ and $\vec{\nu} \frown \rho'$ from these measure one sets give the same decision about φ. Here we use the fact that the completeness of the measures $U_{n,\alpha}$ for $\alpha < \kappa_n$ is greater than λ_{n-1}. Let p^{**} be the resulting direct extension.

At this point we can argue that there is a direct extension of p^{**} which decides φ. Otherwise, let q be an extension of p^{**} of minimal length which decides φ. By the first step there is a sequence $\vec{\nu} \frown \rho$ such that $p^{**} \frown \vec{\nu} \frown \rho$ decides φ. By the second step, every extension of the form $p^{**} \frown \vec{\nu} \frown \rho'$ gives the same decision about φ as $p^{**} \frown \vec{\nu} \frown \rho$. Hence $p^{**} \frown \vec{\nu}$ decides φ. This contradicts the minimality of the length of q.

In fact, $\mathbb{P}$ satisfies the following stronger property whose proof is similar.

Lemma 8. *For every condition $p \in \mathbb{P}$ and every dense open set D, there are a direct extension p^* of p and $n < \omega$ such that every n-step extension of p^* is in D.*

This finishes the proof of the Prikry lemma. Utilizing the Prikry property, and the fact that for each $n < \omega$ and $p \in \mathbb{P}$ with $lh(p) \geq n$. the direct extension of $\mathbb{P}/p$ is λ_n-closed, it is routine to verify that no new bounded subsets of κ are added. Moreover, building on Lemma 8, a standard argument shows that κ^+ is preserved.

Let G be $\mathbb{P}$-generic. For $n < \omega$ and $\alpha < \kappa^+$, we define $t_\alpha(n) = f^p_n(\alpha)$ for some (any) $p \in G$ for which $\alpha \in \mathrm{dom}(f^p_n)$. Standard arguments show that whenever there is $p \in G$ with $\alpha < \beta$ from a^p_n for some n, then $t_\alpha, t_\beta \notin V$ and $t_\alpha <^* t_\beta$. It follows that there are κ^+ many new ω-sequences in $V[G]$. We note that our construction simplifies the original short extenders based forcing of Gitik ([**?**]) and does not require using the weaker ordering $\to$ to replace our ordering $\leq$. The reason is that original forcing is designed add κ^{++} many ω-sequences, without collapsing κ^{++} to κ^+.

1.2. **Homogeneity.** In this section we prove Theorem 1 by showing that the forcing $\mathbb{P}$ from the previous section has a certain homogeneity property.

Definition 9.

(1) For a condition $q \in \mathbb{P}$, let $\mathbb{P}/q$ denote the cone of conditions $p \in \mathbb{P}$ extending q.
(2) For $q_1, q_2 \in \mathbb{P}$, a cone isomorphism of $\mathbb{P}/q_1, \mathbb{P}/q_2$ is an order preserving bijection $\sigma : \mathbb{P}/q_1 \to \mathbb{P}/q_2$.

Theorem 1 is an immediate consequence of the following result.

Theorem 10. *If $\dot{x}$ is a $\mathbb{P}$-name for a subset of κ, then it is forced by $\mathbb{P}$ that there is $\delta < \kappa^+$ such that $\dot{t}_\delta \notin \mathrm{HOD}_x$.*

Proof. Suppose otherwise. Let p be a condition which forces that $\dot{t}_\delta \in \mathrm{HOD}_x$ for all $\delta < \kappa^+$.

Since κ is strong limit, we let $\langle x_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of bounded subsets of κ. Since $\mathbb{P}$ does not add bounded subsets of κ, $\dot{x}$ is coded by the sequence of indices of the sets $\dot{x} \cap \kappa_n$ for $n < \omega$. So we may assume that $\dot{x}$ is a name for an ω-sequence of ordinals $\dot{\alpha}_m$ for $m < \omega$.

Using the strong version of the Prikry Lemma, we can find a direct extension q of p such that for all $m < \omega$ there is a natural number n_m such that all n_m-step extensions of p decide the value of $\dot{\alpha}_m$. Let $\delta < \kappa^+$ be an ordinal greater than every ordinal appearing in the domains of the partial functions in q. Now clearly $\dot{x}$ is determined by q and $\langle \dot{t}_\gamma \mid \gamma < \delta \rangle$.

Let r be a direct extension of q with $\delta \in \mathrm{dom}(a^q_k)$ for all $k \geq \mathrm{lh}(q)$. Now using our assumption and strengthening r if necessary, there are a formula ϕ and an ordinal γ such that r forces $\dot{t}_\delta = \{\beta < \kappa \mid \phi(\beta, \check{\gamma}, \dot{x})\}$.

Let $l = \mathrm{lh}(r)$. By Lemma 3 for $\langle U_{l,\alpha} \mid \alpha < \kappa_l \rangle$, we can find distinct $\nu_1, \nu_2 \in \pi^l_{\mathrm{mc}(a^r_l) a^r_l(\delta)}$ “A^r_l such that for all $\xi \in \mathrm{rng}(a^r_l)$ with $\xi < a^r_l(\delta)$, $\pi^l_{a^r_l(\delta)\xi}(\nu_1) = \pi^l_{a^r_l(\delta)\xi}(\nu_2)$.

Let $r_1 = r \frown \tau_1$ and $r_2 = r \frown \tau_2$ where for $i < 2$, $\pi_{\mathrm{mc}(a^r_l) a^r_l(\delta)}(\tau_i) = \nu_i$.

We define a cone isomorphism σ from $\mathbb{P}/r_1$ to $\mathbb{P}/r_2$. For a condition $w_1 \in \mathbb{P}/r_1$, we define $\sigma(w_1)$ to be the condition w_2 defined by replacing $f^{w_1}_l \restriction \mathrm{dom}(f^{r_1}_l)$ with $f^{r_2}_l$ and leaving the rest of the condition unchanged. It is straightforward to verify that this indeed defines a cone isomorphism.

By the choice of q and σ, it is clear that for every $\mathbb{P}$-generic filter G, with $r_1 \in G$, then $\dot{x}_G = \dot{x}_{\sigma[G]}$. It follows that

$$(\dot{t}_\delta)_G = \{\beta < \kappa \mid \phi(\beta, \check{\gamma}, \dot{x}_G)\} = \{\beta < \kappa \mid \phi(\beta, \check{\gamma}, \dot{x}_{\sigma[G]})\} = (\dot{t}_\delta)_{\sigma[G]}.$$

This is a contradiction since $(\dot{t}_\delta(l))_G = \nu_1$ and $(\dot{t}_\delta(l))_{\sigma[G]} = \nu_2$. $\square$

We now give a brief description of how to bring the result down to $\aleph_\omega$. For ease of notation we set $\lambda_{-1} = \omega_1$.

We define a forcing $\hat{\mathbb{P}}$ as follows. Conditions are of the form $\langle p_n \mid n < \omega \rangle$ where there is $\mathrm{lh}(p) < \omega$ such that for all $n < \mathrm{lh}(p)$, $p_n = (\rho_n, f_n, h_{<n}, h_{>n})$ where

(1) $\rho_n < \lambda_n$,
(2) $f_n \in \mathbb{Q}_{n1}$,
(3) $h_{<n} \in \mathrm{Coll}(\lambda^{+++}_{n-1}, < \rho_n)$ and
(4) $h_{>n} \in \mathrm{Coll}(\rho^{++}_n, < \lambda_n)$,

and for all $n \geq \mathrm{lh}(p)$, $p_n = (a_n, f_n, h_{<n}, C_n, A_n)$ such that

(1) a_n is as in the definition of $\mathbb{Q}_{n0}$ and $\mathrm{dom}(a_n) \subseteq \mathrm{dom}(a_{n+1})$,
(2) $f_n \in \mathbb{Q}_{n1}$,
(3) $h_{<n} \in \mathrm{Coll}(\lambda^{+++}_{n-1}, < \lambda_n)$,
(4) C_n is a function with domain $\pi_{\mathrm{mc}(a_n),0}$ “A_n where for all $\rho \in \mathrm{dom}(C_n)$, $C_n(\rho) \in \mathrm{Coll}(\rho^{++}, < \lambda_n)$ and
(5) $A_n \in U_{n,\mathrm{mc}(a_n)}$ with the properties listed in the definition of $\mathbb{Q}_{n0}$.

As before we indicate that different parts of the condition belong to p with a superscript, ρ^p_n, f^p_n, etc. We also write $\bar{p}$ for the natural condition in $\mathbb{P}$ derived from p. We write $p \leq^* q$ if $\mathrm{lh}(p) = \mathrm{lh}(q)$ and

(1) $\bar{p} \leq^* \bar{q}$,
(2) for all $n < \mathrm{lh}(p)$, $h^p_{<n} \leq h^q_{<n}$ and $h^p_{>n} \leq h^q_{>n}$,
(3) for all $n \geq \mathrm{lh}(p)$, $h^p_{<n} \leq h^q_{<n}$ and for all $\rho \in \mathrm{dom}(C^p_n)$, $C^p_n(\rho) \leq C^q_n(\rho)$.

For a condition p and $\nu \in A^p_{\mathrm{lh}(p)}$ with $\pi_{\mathrm{mc}(a^p_{\mathrm{lh}(p)}),0}(\nu) > \sup(\mathrm{rng}(h_{<\mathrm{lh}(p)}))$ we define $p \frown \nu$ to be the condition of length $\mathrm{lh}(p)+1$ determined by strengthening $\bar{p}$ to $\bar{p} \frown \nu$ and setting $\rho^{p\frown\nu}_{\mathrm{lh}(p)} = \pi_{\mathrm{mc}(a_{\mathrm{lh}(p)}),0}(\nu)$ and $h^{p\frown\nu}_{>\mathrm{lh}(p)} = C^p_{\mathrm{lh}(p)}(\rho^{p\frown\nu}_{\mathrm{lh}(p)})$ and leaving the rest of the condition unchanged. As before the ordering comes from a combination of one step extensions and direct extensions.

All of the previous claims remain valid here. The forcing $\hat{\mathbb{P}}$ satisfies the Prikry Lemma.

It follows that bounded subsets of κ are added by a finite product of the collapses and hence κ is preserved. The stronger version of the Prikry lemma is also true. This is used to show that κ^+ is preserved and it forms the basis for the homogeneity argument. The homogeneity argument is the same where we note that the cone isomorphism to be defined is the identity on the collapse parts of the condition. This is possible due to the fact that the new collapsing components depend only on the Prikry points associated with the least (normal) measures $U_{n,0}$, $n < \omega$, while our cone isomorphisms are defined to make changes to sequences t_δ for sufficiently large generators $\delta > 0$.

2. Lower bound

In this section we prove Theorem 2 which establishes the lower bound. Let κ be a strong limit singular cardinal of cofinality ω. We prove that if there is no inner model $L[E]$ with a singular limit α such that the set of $L[E]$ Mitchell orders $\{o^E(\nu) \mid \nu < \alpha\}$ is unbounded in α, then in V there exists a subset $x \subseteq \kappa$ such that HOD_x contains the power set of κ. $o^E(\nu)$ denotes the number of total extenders on the sequence E with critical point ν. Assuming $L[E]$ does not have a strong cardinal, every total extender on a cardinal ν in $L[E]$ is on the sequence E (see [10]). Since our large cardinal assumption is below $o(\kappa) = \kappa^{++}$ where all extenders are equivalent to their normal measures (i.e., extenders with a single generators), we can assume that $o^E(\nu)$ is the number of total measures on ν in $L[E]$.

Lemma 11. *Suppose that $x \subseteq \kappa$ is such that $\bigcup_{\alpha<\kappa} \mathcal{P}(\alpha)$ and $[\kappa]^\omega$ are contained in HOD_x then $\mathcal{P}(\kappa)^V \subseteq \mathrm{HOD}_x$.*

Proof. Working in HOD_x, let $\langle a_i \mid i < \kappa\rangle$ be an enumeration of the bounded subsets of κ. Consider the set $\{\bigcup_{\alpha \in z} a_\alpha \mid z \in [\kappa]^\omega\}$. Clearly this set is in HOD_x and it is straightforward to see that it is $\mathcal{P}(\kappa)$. □

Lemma 12. *Suppose that $x \subseteq \kappa$ satisfies that HOD_x contains $\bigcup_{\alpha<\kappa} \mathcal{P}(\alpha)$ and that for every $z \in [\kappa]^\omega$ there exists a set $y \in \mathrm{HOD}_x$ such that $z \subseteq y$ and $|y| < \kappa$. Then $\mathcal{P}(\kappa)^V \subseteq \mathrm{HOD}_x$.*

Proof. By the previous lemma, it is enough to show that $[\kappa]^\omega$ is contained in HOD_x. Let $z \in [\kappa]^\omega$. By assumption there is $y \in \mathrm{HOD}_x$ such that $z \subseteq y$ and $|y| < \kappa$. Let π be the transitive collapse map and $\bar{y} = \pi(y)$. Clearly

$\pi(z) \in \mathrm{HOD}_x$, since every bounded subset of κ is in HOD_x. It follows that $z \in \mathrm{HOD}_x$, which finishes the proof. □

It is well known that the core model K exists under our anti-large cardinal assumption and can be represented as an extender model $K = L[E]$, so that every extender E_ν that appears in the sequence E is equivalent to a full or partial normal measure U.

Our proof relies on the covering argument for sequences of measures (see Mitchell [8]). We commence with a brief description of the covering scenario. Suppose that $Y \prec H_\chi$ for some sufficiently large regular cardinal χ in V with $\kappa \in Y$ and $|Y| < \kappa$. Let $\bar{K}$ be the transitive collapse of $K \cap Y$, and $\sigma : \bar{K} \to Y \cap K \prec K||_\chi$ be the inverse of the transitive collapse map. Suppose that σ is continuous at every point of cofinality ω. Let $\bar{\kappa} \in \bar{K}$ be such that $\sigma(\bar{\kappa}) = \kappa$.

2.1. **Covering Scenarios.** It is a well-known fact from inner model theory that $\bar{K}$ is not moved in its coiteration with K (see [10] or [8]). Moreover, for every $\bar{K}$-cardinal $\bar{\alpha}$, in the course of the coiteration of K, $\bar{K}$ up to $\bar{\alpha}$, there are only finitely many drops, that is, stages where the ultrapower on the K-side structure by some measure of height $< \bar{\alpha}$ warrants moving to a proper initial segment. If M_0 is the structure obtained after the last drop on the K-side, then M_0 is a mouse, which is sound above $\bar{\rho} = \bar{\rho}^{M_0}$, and normally coiterates with $\bar{K}$ up to $\bar{\alpha}$ without drops.

Let $\langle M_i, \pi_{i,j} \mid i \leq j \leq \theta\rangle$ be the iteration on the M_0-side of the comparison with $\bar{K}$, $\langle \nu_i \mid i < \theta\rangle$ be the indices of the measures used in the coiteration, and $\bar{c} = \langle \bar{\kappa}_i \mid i < \theta\rangle$ be their corresponding critical points. Note $\bar{c} \subseteq \bar{\alpha} \setminus \bar{\rho}$. If $n < \omega$ is minimal such that $\bar{\rho} = \rho_n^{M_0}$, then for each $i \leq \theta$, $\bar{\rho} = \rho_n^{M_i}$, $p_n^{M_i} = \pi_{0,i}(p_n^{M_0})$ and $M_i = h_n^{M_i}[\bar{\rho} \cup p_n^{M_i} \cup \langle \bar{\kappa}_j \mid j < i\rangle]$.[1]

Let $M = M_\theta$, $\alpha = \sigma(\bar{\alpha})$, and $\sigma^* : M \to N$ be the $(\mathrm{cp}(\sigma), \alpha)$ long ultrapower embedding derived from σ. It is well-known that $N = K||_{\eta_\alpha}$ is a mouse of some height $\eta_\alpha < (\alpha^+)^K$. Furthermore, $\mathrm{rng}(\sigma) \subseteq h_n[p \cup \rho \cup c]$ where $h_n = h_n^{K||\eta_\alpha}$, $\rho = \sup(\sigma``\bar{\rho})$, $p = \sigma^*(p_n^M)$, and $c = \sigma^*[\bar{c}] = \langle \kappa_i \mid i < \theta\rangle$.

We refer to this description as the covering scenario of Y at α (or relative to (Y, α)). When necessary, we denote the relevant parameters $\bar{\rho}$, n, $\bar{p}_n^M$ and h_n, by $\bar{\rho}^{Y,\alpha}$, $n^{Y,\alpha}$, $\bar{p}^{Y,\alpha}$, and $h^{Y,\alpha}$, respectively. Similarly, ρ, p_n^N, and h_n^N, will be denoted by $\rho^{Y,\alpha}$, $p^{Y,\alpha}$, and $h^{Y,\alpha}$, respectively.

This description is part of Mitchell's covering argument. It shows that $Y \cap \alpha$ can be covered by the hull, over a level of K, using parameters in: (i) an ordinal $\rho < \alpha$; (ii) a finite set of ordinals p; and (iii) a small set $c \subseteq \alpha$ of indiscernibles. These objects, and in particular the set c, depend on $Y \cap \alpha$. The main work we do in this section is to obtain a substitute for c in a uniform way that is independent of Y up to a finite set. The precise

[1]We note that in the context of applying a Skolem function such as h_n, on a set X, $h[X]$ is always understood to mean $h[X^{<\omega}]$.

statement of our covering is the hypothesis of Lemma 16. We prove it under the anti-large cardinal assumption of Theorem 2.

Let τ be a cardinal of uncountable cofinality in M. Our proof of Theorem 2 relies on the ability to show that in many cases, we can find a finite set t of ordinals in M, such that when added to the standard domain of the Skolem hull of M (i.e., the projectum, and the standard parameter) the resulting hull is stationary in τ (in fact, it is unbounded in τ and countably closed).

The following folklore fact asserts that this is the case for every such τ, with the exception of a cardinal τ which is measurable in M, and a limit of a cofinal, Magidor-type generic sequence over M, consisting of critical points in $\{\kappa_i \mid i < \theta\}$. We sketch the proof for completeness.

Lemma 13. *Suppose that τ is a regular cardinal in M and has uncountable cofinality in V. If τ is not the limit of a closed unbounded increasing sequence of critical points $\langle \bar{\kappa}_{i_\nu} \mid \nu < \lambda\rangle$ of a limit length, for which $\bar{\kappa}_{i_{\nu+1}} = \pi_{i_\nu, i_{\nu+1}}(\bar{\kappa}_{i_\nu})$ or every $\nu < \lambda$, then there exists a finite set t of ordinals in M such that $\tau \cap h_n^M[p_n^M \cup \rho^M \cup t]$ is a stationary subset of τ in V.*

Proof. Since $M = M_\theta$ is the direct limit of the system of all finite sub-iterations of the iteration $\langle M_i, \pi_{i,j} \mid i \leq j \leq \theta\rangle$, $\tau \in M$ is represented in one of its finite sub-iterated ultrapowers. For notational simplicity, we argue first in the case where τ is represented in M_0. Let $\tau_0 = \pi_{0,\theta}^{-1}(\tau) \in M_0$.

If τ_0 does not represent any of the critical points $\bar{\kappa}_i$ (i.e., $\pi_{0,i}(\tau_0) \neq \bar{\kappa}_i$ for any i) then a straightforward induction on $i \leq \theta$ shows that $\pi_{0,i}``\tau_0 \subseteq \pi_{0,i}(\tau_0)$ is cofinal in $\pi_{0,i}(\tau_0)$ and closed under countable limits. I.e., since $\pi_{0,i}$ is continuous at countable cofinalities. The conclusion of the lemma follows as $\pi_{0,\theta}``\tau_0 \subseteq \mathrm{rng}(\pi_{0,\theta}) = \pi_{0,\theta}``h_n^{M_0}[\bar{\rho} \cup p_n^{M_0}] = h_n^{M_\theta}[\bar{\rho} \cup p_n^{M_\theta}]$ (i.e., $t = \emptyset$ suffices here).

Suppose now that there exists some i_0 such that $\pi_{0,i_0}(\tau_0) = \bar{\kappa}_{i_0}$ is a critical point of the iteration. Let $\langle \bar{\kappa}_{i_\nu} \mid \nu < \lambda\rangle$ be an increasing enumeration of all the critical points $\bar{\kappa}_i < \tau$ for which $\tau = \pi_{i,\theta}(\bar{\kappa}_i)$. In particular, $\bar{\kappa}_{i_{\nu+1}} = \pi_{i_\nu, i_{\nu+1}}(\bar{\kappa}_{i_\nu})$ for every $\nu < \lambda$. Let $i_\lambda = \sup_{\nu<\lambda} i_\nu$. Note that if λ is a limit ordinal then the normality of the iteration implies that the ordinal which represents τ in the iterated ultrapower M_{i_λ} is $\tau' = \sup_{i<\lambda} \bar{\kappa}_{i_\nu}$ and that the next critical points $\bar{\kappa_{i_\lambda}}$ is greater or equal than τ'. $\bar{\kappa}_{i_\lambda}$ cannot be τ' since this would imply $\bar{\kappa}_{i_\lambda} \in \langle \bar{\kappa}_{i_\nu} \mid \nu < \lambda\rangle$ which is an absurd. It follows that $\mathrm{cp}(\pi_{i_\lambda,\theta}) > \tau'$ and thus, that $\tau = \pi_{i_\lambda,\theta}(\tau') = \tau'$. But this contradicts our assumption that τ is not the limit of a closed unbounded increasing sequence of critical points $\langle \bar{\kappa}_{i_\nu} \mid \nu < \lambda\rangle$ of a limit length. We conclude that if $\langle \bar{\kappa}_{i_\nu} \mid \nu < \lambda\rangle$ is an increasing enumeration of all the critical points $\bar{\kappa}_i < \tau$ for which $\tau = \pi_{i,\theta}(\bar{\kappa}_i)$ then $\lambda = \lambda^- + 1$ is a successor ordinal.

Denoting i_{λ^-} by i^*, let $\tau_{i^*+1} = \pi_{i^*,i^*+1}(\bar{\kappa}_{i^*})$. Clearly, $\tau = \pi_{i^*+1,\theta}(\tau_{i^*+1})$, and by our choice of i^*, τ_{i^*+1} does not represent any of the critical points $\bar{\kappa}_i$, $i \geq i^* + 1$ (i.e., the critical points of the iteration from M_{i^*+1} to $M_\theta = M$).

As in the previous case we have that $\pi_{i^*+1,\theta}``\tau_{i^*+1} \subseteq \tau$ is cofinal and closed under countable limits. Hence, to complete the argument, it suffices to verify that for some finite t, the hull $h^{M_{i^*+1}}[p^{M_{i^*+1}} \cup \rho^{M_{i^*+1}} \cup t]$ is cofinal and is countably closed in τ_{i^*+1}. To this end, note that since $\tau_{i^*+1} = \pi_{i^*,i^*+1}(\bar{\kappa}_{i^*})$ the cofinality of τ_{i^*+1} in M_{i^*} is $(\bar{\kappa}_{i^*}^+)^{M_{i^*}} = (\bar{\kappa}_{i^*}^+)^{M_{i^*+1}}$. Let U_{i^*} be the normal measure on $\bar{\kappa}_{i^*}$,used to form the ultrapower M_{i^*+1} of M_{i^*}. Since $(\bar{\kappa}_{i^*}^+)^{M_{i^*}}$ and its images in the iteration are not a critical points, the argument of the first case implies that there is a finite set t so that $h^{M_{i^*}}[p^{M_{i^*}} \cup \rho^{M_{i^*}} \cup t] = X$ is cofinal in $\bar{\kappa}_{i^*}^+$ and is countably closed. Since M_{i^*} satisfies the GCH, X is cofinal and countably closed in ${}^{\bar{\kappa}_{i^*}}\bar{\kappa}_{i^*}$ with respect to the standard eventual domination order (i.e., modulo the bounded sets ideal). This, in turn, guarantees that the hull of $\pi_{i^*,i^*+1}(X) \cup \{\bar{\kappa}_{i^*}\}$ in M_{i^*+1} is countably closed and cofinal in τ_{i^*+1}.

Now, in general, $\tau \in M$ need not be represented in M_0, however it is represented in some finite sub-iterate M_0' of M_θ. As such, M_0' is sound up to an addition of a finite set of ordinls t' (i.e., the set of the critical points of the finite iteration from M_0 to M_0'). It follows that the same analysis and conclusions of the iterated ultrapower from M_0' to M_θ applies when adding t' or its appropriate image to the Skolem hulls of the relevant iterands. □

2.2. **Closure Procedures.**

2.2.1. *The standard closure procedure relative to a substructure.* For our purposes, it useful to describe the result of the covering scenario as a closure process. For this, we first need to describe the coiteration induced assignment of the critical points $\bar{\kappa}_i$, $i < \theta$ to $\bar{K}$ measurable cardinals $\mu \leq \alpha$.

For each $i < \theta$, let $\bar{\mu}_i = \pi_{i,\theta}(\bar{\kappa}_i)$. Note that since $\bar{\kappa}_i$ is definable in M_i from $p_n^{M_i}$, and a sequence of ordinals $a \subseteq \bar{\rho} \cup \langle \bar{\kappa}_j \mid j < i\rangle$, then the same holds for $\bar{\mu}_i$ in $M = M_\theta$ with respect to h_n^M, p_n^M, and $a = \pi_{i,\theta}(a) \subseteq \bar{\rho} \cup \langle \bar{\kappa}_j \mid j < i\rangle$. We define for each measurable cardinal $\mu \leq \alpha$, the set $\bar{c}_\mu = \langle \bar{\kappa}_i \mid \bar{\mu}_i = \mu\rangle$.

The sets $\bar{c}_\mu$ induce a natural partial ordering $<^*$ on θ, in which for two ordinals $i, i' < \theta$, we have that $i <^* i'$ if and only if $i < i'$ and $\bar{\kappa}_{i'} \in \bar{c}_\mu$ for some $\mu \in h_n^M[\rho \cup \{p_n^M\} \cup \{\kappa_j\}_{j \leq i}]$. The previous observation implies that the minimal values of $<^*$ are the ordinals $i < \theta$ for which κ_i belongs to the hull $h_n^M[\rho \cup \{p_n^M\}]$. Since the partial ordering $<^*$ is clearly well-founded, we can get from each $i' < \theta$ to a minimal value of $<^*$ in finitely many steps.

The description of the order $<^*$ allows us to present M as the closure $M = \cup_n \bar{y}_n$ of an ω-procedure which incorporates the Skolem function h_n^M and the assignment $\mu \mapsto \bar{c}_\mu$: We start from $\bar{y}_0 = \bar{\rho} \cup p_n^M$, and for each $m < \omega$ define $\bar{y}_{m+1}$ to be the closure of $\bar{y}_m \cup (\bigcup_{\mu \in \bar{y}_m} \bar{c}_\mu)$ under $h_n^M(p_n^M, \cdot)$. It is routine to verify that each $\bar{\kappa}_i$ is added to $\bar{y}_n$, where $n < \omega$ is the $<^*$-rank of i.

Similarly, for each N measurable cardinal $\mu \in Y \cap \alpha = \mathrm{rng}(\sigma^*) \cap \alpha$, $\mu = \sigma(\bar{\mu})$, we define $c_\mu = \sigma``[c_{\bar{\mu}}]$. It follows that $\mathrm{rng}(\sigma^*) \cap On$ is covered in a similar closure procedure, which starts from $\rho \cup p$ and incorporates $h_n^{K||\eta_\alpha}(p, \cdot)$ and the assignment $\mu \mapsto c_\mu$.

2.2.2. *The g-closure for a covering scenario.* Given a function $g : \kappa \to [\kappa]^{<\kappa}$, we consider an alternative closure procedure for a covering scenario, in which the sets c_μ are replaced with $g(\mu)$ for μ that appears in the closure process. As opposed to the standard closure procedure, described above, here we also allow an arbitrary initial finite (seed) set of ordinals s to be added to the closure process.

More precisely, we define the g-closure relative to (Y, α, s) for a finite set of ordinals $s \subseteq \alpha$ to be the set $y = \bigcup_{m<\omega} y_m$ where $y_0 = \rho \cup s$ and for each $m < \omega$, y_{m+1} is the closure of $y_m \cup (\bigcup_{\mu \in y_m} g(\mu))$ under $h_n^N(p, \cdot)$. Note that the structure $N = K||\eta_\alpha$ determines the Skolem function h_n^N which is used for computing the g-closure. N results from taking Y and running the covering argument to cover $Y \cap \alpha$. We refer to the resulting closure set $y_\omega = \cup_n y_n$ as the g-closure relative to (Y, α, s).

We see that to apply the g-closure relative to (Y, α, s) one needs access to the parameter p, the ordinal ρ and the finite set of ordinals s, function g and the Skolem function $h_n^{K||\eta_\alpha}(p, \cdot)$. Hence it is definable in HOD_g, or any sufficiently closed model containing these parameters.

2.3. **The Final Argument.** We would like to refine Lemma 12 to fit the covering scenario described above. To describe this refinement, we introduce a weak version of the δ-closed property. We note that since κ is a strong limit cardinal in V, there are unboundedly many $\delta < \kappa$ of uncountable cofinality with $\delta^{\aleph_0} = \delta$. Indeed, this is the case for every cardinal $\delta = \gamma^{\aleph_0}$ for some $\gamma < \kappa$.

Definition 14. Suppose that $\delta < \kappa$ is a cardinal of uncountable cofinality, with $\delta^{\aleph_0} = \delta$. Let $Y \prec H_{\kappa^+}$. We say that Y is δ-weakly-closed if it satisfies the following conditions:

(1) ${}^\omega Y \subseteq Y$;
(2) $|Y| = \delta^+$;
(3) $\delta^+ \subseteq Y$;
(4) For every $\alpha \in Y \cap \kappa$, if $\mathrm{cf}(\alpha) \leq \delta$ then $Y \cap \alpha$ contains a club subset of δ;
(5) For every $\alpha \in Y \cap \kappa$, if $\mathrm{cf}(\alpha) \geq \delta^+$ then $\mathrm{cf}(\sup(Y \cap \alpha)) = \delta^+$, and Y contains a closed unbounded subset of $\sup(Y \cap \alpha)$.

Remark 15. (1) *We note that assuming $\delta^{\aleph_0} = \delta$, there are stationarily many δ-weakly-closed structures Y. Indeed, for every function $F : [H_{\kappa^+}]^{<\omega} \to H_{\kappa^+}$, it is straightforward to construct a δ-weakly-closed structure Y which is closed under F, in δ^+ many steps: Let $Y_0 = \delta + 1$. Assuming that Y_γ has been defined, let $Y_{\gamma+1}$ be obtained from Y_γ by closing under F, closing under ω-sequences, closing under a Skolem function of H_{κ^+}, for each $\alpha \in Y_\gamma$ adding a club subset of*

α of minimal ordertype if $\mathrm{cf}(\alpha) \leq \delta$*, and adding the ordinal* $\alpha' = \sup(Y_\gamma \cap \alpha)$*, if* $\mathrm{cf}(\alpha) \geq \delta^+$*. At limit stages take unions. It is clear that $Y = Y_{\delta^+}$ is δ-weakly-closed.*

(2) *Let $\sigma : \bar{K} \to K||\kappa^+$ be the restriction of the inverse of the restriction of the transitive collapse map of Y, to $K \cap Y \prec K||\kappa^+$. We note that the fact $Y \cap \alpha$ is cofinal at each $\alpha \in Y$ with* $\mathrm{cf}(\alpha) \leq \delta$*, implies that σ is continuous at limits of cofinality ω. Therefore, the covering scenario analysis given at the outset of this section applies to Y and σ.*

Lemma 16. *Suppose there exist a cardinal $\delta < \kappa$ and a function $g : \kappa+1 \to [\kappa]^{\leq\delta}$ in V such that for every δ-weakly-closed structure $Y \prec (H_{\kappa^+}, g)$ there exists a finite set $s \subseteq Y \cap \kappa$ such that the g-closure set relative to (Y, κ, s) covers $Y \cap \kappa$. If κ is strong limit and $\delta^{\aleph_0} = \delta$ then there exists $x \subseteq \kappa$ such that $\mathcal{P}(\kappa)^V \subseteq \mathrm{HOD}_x$.*

Proof. Let x be a subset of κ which codes all bounded subsets of κ and the function g in a natural way. We prove that HOD_x satisfies the requirements of Lemma 12.

Let $z \in [\kappa]^\omega$. Let Y be an elementary substructure containing z as in the hypotheses of the lemma. By assumption if y is the g-closure relative to (Y, κ, s) for some finite set s, then y covers $Y \cap \kappa$ and hence z. We have that $y \in \mathrm{HOD}_x$, since the g-closure is definable from parameters in HOD_x. Finally y has size less than κ since $g(\mu)$ has size at most δ for all μ. So y satisfies the requirements of Lemma 12. □

The rest of this section is devoted to proving that the assumption of the previous lemma holds under our anti-large cardinal hypothesis. More precisely, our anti-large cardinal hypothesis implies that the set $\{o^K(\mu) \mid \mu < \kappa\}$ is bounded in κ.

Assumptions: Let $\delta \geq \sup(\{(2^{\aleph_0})^+\} \cup \{o^K(\mu) \mid \mu < \kappa\})$ be a cardinal below κ, of uncountable cofinality, such that $\delta^{\aleph_0} = \delta$. We construct a function $g : \kappa \to [\kappa]^{\leq\delta}$ satisfying the requirements of the previous lemma by induction on its restrictions $g \restriction \alpha + 1$.

To prove the main Lemma 16, we will show by induction on ordinals $\alpha \in (\kappa + 1) \setminus \delta^+$, that for every $Y \prec (H^V_{\alpha^+}, g \restriction \alpha + 1)$ which is δ-weakly-closed in V, there exists a finite set of ordinals $s \subseteq \alpha$, such that the g-closure relative to (Y, α, s) covers $Y \cap \alpha$.

Lemma 17. *Let $Y \prec (H_{\alpha^+}, g \restriction \alpha)$ and $z_0 \in [Y \cap \alpha]^{<\omega}$. Suppose that $y \subseteq \alpha \cap Y$ is contained in the g-closure relative to (Y, α, z_0). Then there exists some $\alpha_0 < \alpha$ such that for every $\beta \in (\alpha_0, \alpha) \cap y$, there is some finite $x_\beta \subseteq Y \cap \beta$, such that the g-closure relative to $(Y, \alpha, z_0 \cup x_\beta)$ covers $Y \cap \beta$.*

Proof. Recall that $\sigma : \bar{K} \to K \cap Y$ is (the restriction of) the inverse of the transitive collapse map of Y. Let $\bar{\beta} = \sigma^{-1}(\beta)$. Then the covering scenario for

(Y,β) corresponds to the coiteration of $\bar{K}$ with K up to $\bar{\beta}$. This coiteration is an initial segment of the coiteration of $\bar{K}$ with K up to $\bar{\alpha} = \sigma^{-1}(\alpha)$. Let $\bar{\alpha}_0 < \bar{\alpha}$ be an ordinal above the height of the last drop on the K-side of the coiteration. Then for every $\bar{\beta} > \bar{\alpha}_0$ the coiteration of K and $\bar{K}$, up to $\bar{\beta}$, agrees with the coiteration of up to $\bar{\alpha}$ past the last drop.

Let $\langle M_i^\beta, \pi_{i,j}^\beta \mid i \leq j \leq \theta^\beta \rangle$ denote the iteration of the K-side structures in the comparison with $\bar{K}$ up to $\bar{\beta}$ starting after the last drop. It follows that this iteration coincides with an initial segment of the iteration $\langle M_i, \pi_{i,j} \mid i \leq j \leq \theta \rangle$ of the K-side structures, in the comparison with $\bar{K}$ up to $\bar{\alpha}$. I.e. $\theta^\beta \leq \theta$, and $M_i^\beta = M_i$, $\pi_{i,j}^\beta = \pi_{i,j}$, for every $i \leq j \leq \theta^\beta$.

Let $M^\beta = M_{\theta^\beta}^\beta$ denote the last structure in the comparison up to $\bar{\beta}$, and $\pi^\beta = \pi_{\theta^\beta,\theta} : M^\beta \to M$. It follows at once that $\bar{\rho}^{Y,\alpha} = \bar{\rho}^{Y,\beta}$, $n^{Y,\alpha} = n^{Y,\beta}$, and $\bar{p}^{Y,\alpha} = \pi^\beta(\bar{p}^{Y,\beta})$.

Let $\sigma^\beta : M^\beta \to N^\beta$ be the $(\mathrm{cp}(\sigma),\beta)$ long ultrapower embedding derived by the extender E^β of height β, derived from σ. Recall that $\sigma^* : M \to N$ denotes the $(\mathrm{cp}(\sigma),\alpha)$ long ultrapower embedding derived from σ. Let $\hat{\pi}^\beta : N^\beta \to N$ be the embedding induced by π^β, σ^*, and σ^β: Recall that every element of N^β is of the form $x = \sigma^\beta(f)(a)$ for some $f \in M^\beta$ and a finite $a \subseteq \beta$. Set $\hat{\pi}^\beta(x) = (\sigma^* \circ \pi_\beta)\,(f)(a)$. Clearly, $\hat{\pi}^\beta(a) = a$ for every $a \in [\beta]^{<\omega}$, and thus, $cp(\hat{\pi}^\beta) \geq \beta$. It is routine to verify that $\hat{\pi}^\beta$ is $\Sigma_n^{(0)}$ elementary and takes $p^{Y,\beta}$ to $p^{Y,\alpha}$. It follows that for every $z \in [\alpha]^{<\omega}$, if β belongs to the g-closure relative to (Y,α,z) , then it further covers the g-closure relative to $(Y,\beta,z\cap\beta)$.

Suppose now that $z_0 \in [\alpha]^{<\omega}$ such that β belongs to the g-closure relative to (Y,α,z_0). By the inductive assumption applied to the covering scenario at (Y,β), there exists some $x_\beta \in [Y\cap\beta]^{<\omega}$ such that the g-closure relative to (Y,β,x_β) covers $Y\cap\beta$. It follows that the g-closure relative to $(Y,\alpha,z_0 \cup x_\beta)$ contains the g-closure relative to (Y,β,x_β), and in particular, covers $Y\cap\beta$. □

Remark 18. *Suppose that $g \restriction \alpha$ has been defined and satisfies the inductive assumption for all $\beta < \alpha$. If $\alpha \in Y$ is not a cardinal in K then it is straightforward to verify that the Skolem hull $h^{Y,\alpha}[\rho^{Y,\alpha} \cup p^{Y,\alpha}]$ has a surjection from $\beta = |\alpha|^K \in Y$ onto α, and therefore β belongs to the g-closure relative to (Y,α). The argument of the previous lemma implies that for every finite set s, the g-closure relative to (Y,α,s) covers the closure relative to $(Y,|\alpha|^K,s)$. We may therefore apply the inductive assumption to $\beta = |\alpha|^K$, and use it to conclude that the induction hypothesis holds at α with $g(\alpha) = \emptyset$.*
Therefore, for the rest of the proof we restrict our attention to ordinals $\alpha \leq \kappa$ which are cardinals in K.

Let α be a K-cardinal. The definition of $g(\alpha)$ and the proof that it works will be divided into three cases according to the V-cofinality of α.

2.3.1. *Case I:* $\mathrm{cf}^V(\alpha) > \delta$. We set $g(\alpha) = \emptyset$ and show that for every δ-weakly-closed $Y \prec (H_{\alpha^+}, g)$ there exists some finite $s \subseteq \alpha \cap Y$ such that the g-closure relative to (Y, α, s) covers $Y \cap \alpha$. Let $\alpha' = \sup(\alpha \cap Y) \leq \alpha$. Since Y is δ-weakly-closed, $\mathrm{cf}^V(\alpha') \geq \delta^+$ and $g(\beta) \subseteq Y$ for every $\beta \in Y \cap \alpha$.

Lemma 19. *There exists some $z \in [Y \cap \alpha]^{<\omega}$ such that the g-closure relative to (Y, α, z) contains a stationary subset of α' (in V).*

Proof. The fact Y is δ-weakly-closed guarantees $Y \cap \alpha$ contains a closed unbounded subset of α'. Let $\bar{\alpha} = \sigma^{-1}(\alpha)$. Since σ is continuous at all points of cofinality $\leq \delta$, it suffices to show that there exists a finite set $\bar{z} \subseteq \bar{\alpha}$ such that the h_n^M closure of $\bar{p}^{Y,\alpha} \cup \bar{\rho}^{Y,\alpha} \cup \bar{z}$ contains a stationary subset of $\bar{\alpha}$ in V.

To this end, note that $\mathrm{cp}(\sigma) > \delta^+$. It follows that $\tau = \mathrm{cf}^M(\bar{\alpha}) \geq \delta^+$ and $\bar{K}||\delta^+ + 1 = K||\delta^+ + 1$.

We see that in V, $\mathrm{cf}(\tau) = \mathrm{cf}(\bar{\alpha}) \geq \delta^+$. By the Mitchell Covering Theorem ([8]), if τ is the limit of a closed unbounded increasing sequence of critical points $\langle \bar{\kappa}_{i_\nu} \mid \nu < \lambda \rangle$ of a limit length, for which $\bar{\kappa}_{i_{\nu+1}} = \pi_{i_\nu, i_{\nu+1}}(\bar{\kappa}_{i_\nu})$, then $\lambda \leq \omega^{o^K(\tau)}$ (the ordinal exponent). Since the Mitchell order of τ in $\bar{K}$ is bounded by δ and $\mathrm{cf}(\tau) > \delta$, we conclude that τ satisfies the assumptions of Lemma 13. The Lemma implies in turn, that there a stationary subset S of $\bar{\alpha}$ (stationary in V) which is covered by the Skolem hull by h_n^M, of the ordinals below the projectum, the standard parameters, and a finite set of ordinals t.

□

Let $S \subseteq Y \cap \alpha'$ be the stationary subset of α' which is given by the previous lemma. Fix for each $\beta \in S$, a finite set $x_\beta \subseteq \beta \cap Y$ such that the g-closure relative to (Y, α, x_β) covers $Y \cap \beta$. This is possible by Lemma 17.

We would like to show that there is a stationary subset $S^* \subseteq S$ and a finite $s^* \subseteq \alpha \cap Y$, such that the g-closure relative to (Y, α, s^*) covers the g-closure relative to (Y, α, x_β) for every $\beta \in S^*$, as it would clearly imply that the g-closure relative to (Y, α, s^*) covers $Y \cap \alpha$.

For this, we press down on the map $\beta \mapsto \max(x_\beta) < \beta$. Since $\mathrm{cf}(\alpha') > \omega$ we can find some $\beta^* \in Y \cap \alpha$ and a stationary subset $S^* \subseteq S$, such that $x_\beta \subseteq \beta^*$ for each $\beta \in S^*$. Now, since there exists a finite set $s^* \subseteq Y$ such that the g-closure relative to (Y, α, s^*) covers $Y \cap \beta^*$. In particular, the g-closure relative to (Y, α, s^*) contains x_β for every $\beta \in S^*$, as required.

2.3.2. *Case II:* $\aleph_0 < \mathrm{cf}^V(\alpha) \leq \delta$. In this case, we take $g(\alpha)$ to be some closed unbounded subset of α of minimal ordertype. Note that for every δ-weakly-closed substructure $Y \prec H_{\alpha^+}$, $S = Y \cap g(\alpha)$ contains a closed and unbounded subset of α. Since $\mathrm{cf}(\alpha) > \aleph_0$, our pressing down argument for the case $\mathrm{cf}(\alpha) > \delta$ can be applied to S to show that there exists a seed s^* such that the g-closure relative to (Y, α, s^*) covers $Y \cap \alpha$.

2.3.3. *Case III:* $\mathrm{cf}^V(\alpha) = \aleph_0$. The construction and argument are different than the previous two cases. The idea is to choose $g(\alpha)$ which will guarantee covering up to α with respect to a sufficiently elementary structure $Z \prec H_{\alpha^+}$. I.e., a structure of the form $Z = Z^* \cap H_{\alpha^+}$ where $Z^* \prec (H_{\alpha^{++}}, g \restriction \alpha)$. We will then use a reflection argument to prove the set $g(\alpha)$ satisfies the induction hypothesis at α.

We work to define $g(\alpha)$. Let $\langle \beta_k \mid k < \omega \rangle \in Z$ be a cofinal sequence in α. By Lemma 17 and the induction hypothesis, we may assume that for every $k < \omega$ there exists a finite set $s_k \subseteq \beta_k$ such that the $g \restriction \alpha$-closure relative to $(Z, \alpha, s_k \cup \{\beta_k\})$ covers $Z \cap \beta_k$. We set $g(\alpha) = \bigcup_{k<\omega}(s_k \cup \{\beta_k\})$. It is immediate that the g-closure relative to $(Z, \alpha, \emptyset)$ covers $Z \cap \alpha$. Note that $g(\alpha) \subseteq Z = Z^* \cap H_{\alpha^+}$ and thus $Z^* \prec (H_{\alpha^{++}}, g)$.

Next, we claim that for every δ-weakly-closed $Y \prec (H_{\alpha^+}, g)$, there exists a finite $s \subseteq \alpha$ such that the g-closure relative to $(Y \cap \alpha, \alpha, s)$ covers $Y \cap \alpha$.

Suppose otherwise. Since the statement regarding a counterexample is definable in $H_{\alpha^{++}}$ in the parameter $g \restriction (\alpha+1)$, we may find a counterexample $Y \in Z^*$. In particular, for every finite $s \subseteq Y \cap \alpha$ the g-closure relative to $(Y \cap \alpha, \alpha, s)$ does not cover $Y \cap \alpha$. We work by induction to construct finite sets $\langle t_k \mid k < \omega \rangle$. Let $t_0 = \emptyset$. Suppose for some $k < \omega$ we have defined t_k. Let α_k be the least element of $Y \cap \alpha$ which is does not belong to the g-closure relative to $(Y \cap \alpha, \alpha, t_k)$. Let m_k be least such that $\beta_{m_k} > \alpha_k$, and $t_{k+1} \subseteq Y \cap \beta_{m_k}$ be such that the g-closure relative to $(Y \cap \alpha, \alpha, t_{k+1})$ covers $Y \cap \beta_{m_k}$. Note that if y is the above g-closure, then y covers the g-closure relative to $(Y \cap \alpha, \alpha, s)$ for any $s \in [Y \cap \beta_{m_k}]^{<\omega}$. In particular α_{k+1} is not in the g-closure relative to $(Y \cap \alpha, \alpha, s)$ for any $s \in [Y \cap \beta_{m_k}]^{<\omega}$. We will refer to this later as the key property of α_{k+1}. Note also that $\sup_{k<\omega} \alpha_k = \alpha$ and $\langle \alpha_n \mid n < \omega \rangle \in Y \cap Z$, since both Y and Z are closed under ω-sequences.

Recall that our choice of $g(\alpha)$ guarantees that the g-closure relative to $(Z, \alpha, \emptyset)$ covers $Z \cap \alpha$. This means there are $n = n^{Z,\alpha} < \omega$, $\eta = \eta^{Z,\alpha} < (\alpha^+)^K$, $\rho = \rho^{Z,\alpha} < \alpha$ and $p = p^{Z,\alpha} \in [\alpha]^{<\omega}$, such that the closure of $\rho \cup \{p\}$ under $h_n^{K||\eta}$ and g covers $\vec{\alpha}$. The last statement is true in H_{α^+}. Since $\vec{\alpha} \in Y \prec (H_{\alpha^+}, g)$, we can find some $n < \omega$, $\eta < (\alpha^+)^K$, $\rho < \alpha$, and $p \in [\alpha]^{<\omega} \cap Y$, such that Y satisfies the statement with respect to $\vec{\alpha}$ in these parameters. It follows that $\vec{\alpha}$ is covered by any g-closure relative to (Y, α, s), if this g-closure contains p, η, and covers $Y \cap \rho$. Moreover, since α is the maximal cardinal in the K-level in which the closure is taken, and the closure uses parameters below α which define a surjection of α onto the entire level, there exists some $\gamma \in Y \cap \alpha$ such that η belongs to the g-closure relative to $(Y, \alpha, \{\gamma\})$.

Let $k < \omega$ be large enough that $\max(p), \gamma, \rho < \beta_k$, and set $x = p \cup \{\gamma\} \cup t_k$. Then p, η belong to the g-closure relative to (Y, α, x), which further covers $Y \cup \rho$. Therefore, the closure covers $\vec{\alpha}$, and in particular contains α_{k+1}. However, $x \in [\beta_k]^{<\omega}$, thus, the last contradicts the key property of α_{k+1}.

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