ON THE POWERSETS OF SINGULAR CARDINALS IN HOD

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Abstract. We prove that the assertion “There is a singular cardinal $\kappa$ such that for all $x \subseteq \kappa$, $\text{HOD}_x$ does not contain the entire powerset of $\kappa$” is equiconsistent with the assertion that there is a cardinal $\kappa$ such that $\{o(\nu) \mid \nu < \kappa\}$ is unbounded in $\kappa$.

Shelah [9] proved that for every singular cardinal $\kappa$ of uncountable cofinality there exists $x \subseteq \kappa$ such that $\mathcal{P}(\kappa) \subseteq \text{HOD}_x$. In work by Cummings, Friedman, Magidor, Rinot, and Sinapova [2] it has been shown that the statement is consistently false for a singular $\kappa$ of cofinality $\omega$, starting from the large cardinal assumption of an inaccessible cardinal $\lambda$ and an infinite sequence of $< \lambda$-supercompact cardinals below it. In fact they prove the stronger assertion that it is consistent that there is a singular cardinal $\kappa$ of cofinality $\omega$ such that for all $x \subseteq \kappa$, $\kappa^+$ is inaccessible in $\text{HOD}_x$. Gitik and Merimovich ([6]) and Kafkoulis ([7]) have obtained the same result from a slightly weaker assumption (see Remarks 6,7 in [2]).

We isolate the exact strength of the failure of Shelah’s theorem at a singular cardinal of cofinality $\omega$. In particular, we prove the following theorems.

Theorem 1. Assuming there is a cardinal $\kappa$ of cofinality $\omega$ such that $\{o(\nu) \mid \nu < \kappa\}$ is unbounded in $\kappa$, there is a forcing extension in which for every subset $x$ of $\kappa$, $\text{HOD}_x$ does not contain the powerset of $\kappa$.

Theorem 2. Assuming that $\kappa$ is a singular strong limit cardinal of cofinality $\omega$ such that $\{o^K(\nu) \mid \nu < \kappa\}$ is bounded in $\kappa$, there is $x \subseteq \kappa$ such that $\mathcal{P}(\kappa) \subseteq \text{HOD}_x$.

We also show that in Theorem 1, $\kappa$ can be made into $\aleph_\omega$ using standard arguments.

Recently, there has been renewed interest in the extent of covering properties for HOD. In particular, we mention papers of Cummings, Friedman and Golshani [3], the first and fourth author [1], and Gitik and Merimovich [6]. The central idea in all of these works is the notion of homogeneity of forcing posets. Theorem 1 is a continuation of this study, but at a considerably lower consistency strength.

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The proof of Theorem 2 is a strong refinement of the proof of the Mitchell’s covering lemma which is possible precisely under our anti-large cardinal assumption. The main idea of the proof is to give a uniform version of the Mitchell’s covering lemma so that all the covering sets for subsets of \( \kappa \) can be defined relative to a single assignment of indiscernibles.

Theorems 1 and 2 are proved in Sections 1 and 2 respectively.

1. Upper Bound

Suppose that in \( V \), there exists a cardinal \( \kappa \) so that \( \{ o(\nu) \mid \nu < \kappa \} \) is unbounded in \( \kappa \). Since the first such cardinal has countable cofinality, we may assume that \( \text{cf}(\kappa) = \omega \). We will define a “short extender” type forcing to add \( \kappa^+ \) many \( \omega \)-sequences to a singular cardinal \( \kappa = \sup_{n<\omega} \kappa_n \) and prove that the forcing has enough homogeneity to establish the conclusion. Let \( \langle \kappa_n \mid n < \omega \rangle \) and \( \langle \lambda_n \mid n < \omega \rangle \) be increasing sequences of regular cardinals with \( \kappa_n < \lambda_n < \kappa_{n+1} \) and \( o(\lambda_n) = \kappa_n \) for all \( n < \omega \).

By arguments from [4], we can pass to a generic extension in which each \( \lambda_n \) carries a Rudin-Keisler increasing sequence of ultrafilters \( \langle U_{n,\alpha} \mid \alpha < \kappa_n \rangle \). By Proposition 1.1.1 of [5] we can assume that each sequence \( \langle U_{n,\alpha} \mid \alpha < \kappa_n \rangle \) is a “nice system”. We remark that the notion of a nice system abstracts the details of deriving a Rudin-Keisler increasing sequence of ultrafilters from an elementary embedding by an extender, which is commonly used in the study of extender based forcing. Of course in this case we do not have such an extender embedding. For each \( \alpha < \beta < \kappa_n \) let \( \pi_{\beta,\alpha}^{n} \) be the Rudin-Keisler projection map from \( U_{n,\beta} \) to \( U_{n,\alpha} \). We drop the \( n \) on \( \pi_{\beta,\alpha}^{n} \) when it is understood.

For each \( n < \omega \), the sequence \( \langle U_{n,\alpha} \mid \alpha < \kappa_n \rangle \) from [4] is shown to satisfy the following property, which essentially asserts that for each \( \alpha < \beta < \kappa_n \), the measure \( U_{n,\beta} \) is strictly above \( U_{n,\alpha} \) in the Rudin-Keisler order.

**Lemma 3.** Let \( n < \omega \), \( \beta < \kappa_n \), and \( r \subseteq \beta \). For every \( U_{n,\beta} \) measure one set \( A \), there are distinct \( \nu_1, \nu_2 \in A \) such that \( \pi_{\beta,\alpha}^{n}(\nu_1) = \pi_{\beta,\alpha}^{n}(\nu_2) \) for every \( \alpha \in r \).

We are now ready to define the components of the main forcing.

### 1.1. The Forcing

**Definition 4.** Let \( Q_{n1} = \{ f \mid f \text{ is a partial function from } \kappa^+ \text{ to } \lambda_n \text{ with } |f| \leq \kappa \} \) ordered by extension.

**Definition 5.** Let \( (a, A, f) \in Q_{n0} \) if:

1. \( f \in Q_{n1} \).
2. \( a \) is a partial order preserving function from \( \kappa^+ \) to \( \kappa_n \) with \( |\text{dom}(a)| < \kappa_n \), \( \text{rng}(a) \) has a maximal element \( \text{mc}(a) \) and \( \text{dom}(a) \cap \text{dom}(f) = \emptyset \).
3. \( A \in U_{n,\text{mc}(a)} \).
(4) For all $\nu \in A$ and all $\alpha < \beta$ from $\text{dom}(a)$, $\pi_{\text{mc}(a),a(\alpha)}(\nu) < \pi_{\text{mc}(a),a(\beta)}(\nu)$.  
(5) For all $\alpha < \beta < \gamma$ from $\text{dom}(a)$, $\pi_{\text{mc}(a),a(\alpha)}(\rho) = \pi_{a(\alpha),a(\alpha)}(\pi_{a(\gamma),a(\beta)}(\rho))$  
for all $\rho \in \pi_{\text{mc}(a),a(\gamma)}^a A$.

Define $(a, A, f) \leq (b, B, g)$ if:

1. $f \leq g$ in $Q_{n1}$.
2. $b \subseteq a$.
3. $\pi_{\text{mc}(a),mc(b)}a A \subseteq B$.

We use these as the components in our diagonal Prikry type forcing. Let $p = \langle p_n \mid n < \omega \rangle$ be in $P$ if there is $l = \text{lh}(p)$ such that for all $n < l$, $p_n \in Q_{n1}$ and all $n \geq l$, $p_n \in Q_{n0}$ where if we write $p_n = (a_n,A_n,f_n)$ then for $m \geq n \geq l$, $\text{dom}(a_m) \supseteq \text{dom}(a_n)$.

For ease of notation we write $p_n = f^p_n$ for $i < \text{lh}(p)$ and $p_n = (a^p_n,A^p_n,f^p_n)$ for $n \geq \text{lh}(p)$.

For $p \in P$ and $\nu \in A^p_{\text{lh}(p)}$ we define a one point extension $p \Join \nu$ to be the condition of length $\text{lh}(p) + 1$ with $f^p_{\text{lh}(p)} = f^p_{\text{lh}(p)} \cup \{ (\gamma, \pi^\text{lh}(p)) \mid \gamma \in \text{dom}(a) \}$ and the rest of the condition unchanged. We define $n$-step extensions for $n > 1$ by iterating one point extensions. We write $p \leq^* q$ if $\text{lh}(p) = \text{lh}(q)$ and for all $n < \omega$, $p_n \leq q_n$ in the appropriate poset. The ordering we will force with is obtained by a combination of direct extensions and one point extensions.

The key property of the order is the following:

**Lemma 6.** For $p, q \in P$, $p \leq q$ if and only if there are $n < \omega$ and $\bar{\nu}$ of length $n$ such that $p \leq^* q \Join \bar{\nu}$.

For a condition $q \in P$, let $P/q$ denote the cone of conditions $p \in P$ extending $q$. Next we sketch a proof of the Prikry property, since it is mostly standard.

**Lemma 7.** Let $p \in P$ and $\varphi$ be a statement in the forcing language. There is a direct extension of $p$ which decides $\varphi$.

As a first step, we construct $p^* \leq^* p$ such that for all sequences $\bar{\nu}$ there is a direct extension of $p^* \Join \bar{\nu}$ which decides $\varphi$, then $p^* \Join \bar{\nu}$ decides $\varphi$. This is done by constructing a sequence of direct extensions $\langle p^n \mid n \geq \text{lh}(p) \rangle$ with $p^\text{lh}(p) = p$ and for $m \geq 1$, constructing $p^{\text{lh}(p) + m}$ by diagonalizing over possible $m$-step extensions of $p^{\text{lh}(p) + m - 1}$. Crucially, we use the fact that the order on the $a_n$ parts of conditions in $Q_{n0}$ is $\kappa_n > \lambda_{n-1}$ complete. Note that the Rudin-Kiesler ordering of the system of measures $\langle U_{n,\alpha} \mid \alpha < \kappa_n \rangle$ is $\text{cf}(\kappa_n) = \kappa_n$ complete.

In the second step, we produce measure one sets so that for $m < \omega$, any two $m + 1$-sequences $\bar{\nu} \sim \rho$ and $\bar{\nu} \sim \rho'$ from these measure one sets give the same decision about $\varphi$. Here we use the fact that the completeness of the measures $U_{n,\alpha}$ for $\alpha < \kappa_n$ is greater than $\lambda_{n-1}$. Let $p^{**}$ be the resulting direct extension.
At this point we can argue that there is a direct extension of $p^{**}$ which decides $\varphi$. Otherwise, let $q$ be an extension of $p^{**}$ of minimal length which decides $\varphi$. By the first step there is a sequence $\bar{\nu} \sim \rho$ such that $p^{**} \sim \bar{\nu} \sim \rho$ decides $\varphi$. By the second step, every extension of the form $p^{**} \sim \bar{\nu} \sim \rho'$ gives the same decision about $\varphi$ as $p^{**} \sim \bar{\nu} \sim \rho$. Hence $p^{**} \sim \bar{\nu}$ decides $\varphi$. This contradicts the minimality of the length of $q$.

In fact, $\mathbb{P}$ satisfies the following stronger property whose proof is similar.

**Lemma 8.** For every condition $p \in \mathbb{P}$ and every dense open set $D$, there are a direct extension $p^*$ of $p$ and $n < \omega$ such that every $n$-step extension of $p^*$ is in $D$.

This finishes the proof of the Prikry lemma. Utilizing the Prikry property, and the fact that for each $n < \omega$ and $p \in \mathbb{P}$ with $lh(p) \geq n$. the direct extension of $\mathbb{P}/p$ is $\lambda_n$-closed, it is routine to verify that no new bounded subsets of $\kappa$ are added. Moreover, building on Lemma 8, a standard argument shows that $\kappa^+$ is preserved.

Let $G$ be $\mathbb{P}$-generic. For $n < \omega$ and $\alpha < \kappa^+$, we define $t_\alpha(n) = f^n_p(\alpha)$ for some (any) $p \in G$ for which $\alpha \in \text{dom}(f^n_p)$. Standard arguments show that whenever there is $p \in G$ with $\alpha < \beta$ from $d^n_p$ for some $n$, then $t_\alpha, t_\beta \notin V$ and $t_\alpha <^* t_\beta$. It follows that there are $\kappa^+$ many new $\omega$-sequences in $V[G]$. We note that our construction simplifies the original short extenders based forcing of Gitik ([?]) and does not require using the weaker ordering $\rightarrow$ to replace our ordering $\leq$. The reason is that original forcing is designed add $\kappa^{++}$ many $\omega$-sequences, without collapsing $\kappa^{++}$ to $\kappa^+$.

### 1.2. Homogeneity

In this section we prove Theorem 1 by showing that the forcing $\mathbb{P}$ from the previous section has a certain homogeneity property.

**Definition 9.**

1. For a condition $q \in \mathbb{P}$, let $\mathbb{P}/q$ denote the cone of conditions $p \in \mathbb{P}$ extending $q$.

2. For $q_1, q_2 \in \mathbb{P}$, a cone isomorphism of $\mathbb{P}/q_1, \mathbb{P}/q_2$ is an order preserving bijection $\sigma : \mathbb{P}/q_1 \rightarrow \mathbb{P}/q_2$.

**Theorem 10.** If $\dot{x}$ is a $\mathbb{P}$-name for a subset of $\kappa$, then it is forced by $\mathbb{P}$ that there is $\delta < \kappa^+$ such that $\dot{t_\delta} \notin \text{HOD}_x$.

**Proof.** Suppose otherwise. Let $p$ be a condition which forces that $\dot{t_\delta} \in \text{HOD}_x$ for all $\delta < \kappa^+$.

Since $\kappa$ is strong limit, we let $\langle x_\alpha | \alpha < \kappa \rangle$ be an enumeration of bounded subsets of $\kappa$. Since $\mathbb{P}$ does not add bounded subsets of $\kappa$, $\dot{x}$ is coded by the sequence of indices of the sets $\dot{x} \cap \kappa_n$ for $n < \omega$. So we may assume that $\dot{x}$ is a name for an $\omega$-sequence of ordinals $\alpha_m$ for $m < \omega$.

Using the strong version of the Prikry Lemma, we can find a direct extension $q$ of $p$ such that for all $m < \omega$ there is a natural number $n_m$ such
that all $n_m$-step extensions of $p$ decide the value of $\dot{\alpha}_m$. Let $\delta < \kappa^+$ be an ordinal greater than every ordinal appearing in the domains of the partial functions in $q$. Now clearly $\dot{x}$ is determined by $q$ and $\{\dot{t}_\gamma : \gamma < \delta\}$.

Let $r$ be a direct extension of $q$ with $\delta \in \text{dom}(a^\delta_q)$ for all $k \geq \text{lh}(q)$. Now using our assumption and strengthening $r$ if necessary, there are a formula $\phi$ and an ordinal $\gamma$ such that $r$ forces $\dot{t}_\delta = \{\beta < \kappa \mid \phi(\beta, \gamma, \dot{x})\}$.

Let $l = \text{lh}(r)$. By Lemma 3 for $(U_{l, \alpha} : \alpha < \kappa_l)$, we can find distinct $\nu_1, \nu_2 \in \pi^1_{\text{mc}(\dot{a}_l^\gamma)}\dot{a}_l^\gamma(\delta)$ such that for all $\xi \in \text{rng}(a^\xi_l)$ with $\xi < a^\gamma_l(\delta)$, $\pi^1_{\dot{a}_l^\gamma(\delta)}(\nu_1) = \pi^1_{\dot{a}_l^\gamma(\delta)}(\nu_2)$.

Let $r_1 = r \land \tau_1$ and $r_2 = r \land \tau_2$ where for $i < 2$, $\pi_{\text{mc}(\dot{a}_l^\gamma)}\dot{a}_l^\gamma(\delta)(\tau_i) = \nu_i$.

We define a cone isomorphism $\sigma$ from $\mathbb{P}/r_1$ to $\mathbb{P}/r_2$. For a condition $w_1 \in \mathbb{P}/r_1$, we define $\sigma(w_1)$ to be the condition $w_2$ defined by replacing $f_{\tau_1}^{l_1}$ with $f_{\tau_2}^{l_2}$ and leaving the rest of the condition unchanged. It is straightforward to verify that this indeed defines a cone isomorphism.

By the choice of $q$ and $\sigma$, it is clear that for every $\mathbb{P}$-generic filter $G$, with $r_1 \in G$, then $\dot{x}_G = \dot{x}_\sigma[G]$. It follows that

$$(\dot{t}_\delta)_G = \{\beta < \kappa \mid \phi(\beta, \gamma, \dot{x}_G)\} = \{\beta < \kappa \mid \phi(\beta, \gamma, \dot{x}_\sigma[G])\} = (\dot{t}_\delta)_{\sigma[G]}.$$

This is a contradiction since $(\dot{t}_\delta(l))_{G} = \nu_1$ and $(\dot{t}_\delta(l))_{\sigma[G]} = \nu_2$. $\square$

We now give a brief description of how to bring the result down to $\mathbb{H}_\omega$. For ease of notation we set $\lambda_1 = \omega_1$.

We define a forcing $\mathbb{P}$ as follows. Conditions are of the form $\langle p_n \mid n < \omega \rangle$ where there is $\text{lh}(p) < \omega$ such that for all $n < \text{lh}(p)$, $p_n = (\rho_n, f_n, h_{<n}, h_{\geq n})$ where

1. $\rho_n < \lambda_n$,  
2. $f_n \in \mathbb{Q}_{n1}$,  
3. $h_{<n} \in \text{Coll}(\lambda_{n-1}^{n+} +, \rho_n)$ and  
4. $h_{\geq n} \in \text{Coll}(\rho_n^{++}, < \lambda_n)$,

and for all $n \geq \text{lh}(p)$, $p_n = (a_n, f_n, h_{<n}, h_{<n}, C_n, A_n)$ such that

1. $a_n$ is as in the definition of $\mathbb{Q}_{n0}$ and $\text{dom}(a_n) \subseteq \text{dom}(a_{n+1})$,  
2. $f_n \in \mathbb{Q}_{n1}$,  
3. $h_{<n} \in \text{Coll}(\lambda_{n-1}^{n+} +, \lambda_n)$,  
4. $C_n$ is a function with domain $\pi_{\text{mc}(a_n), 0} a_n$ where for all $\rho \in \text{dom}(C_n)$, $C_n(\rho) \in \text{Coll}(\rho^{++}, < \lambda_n)$ and  
5. $A_n \in U_{n, \text{mc}(a_n)}$ with the properties listed in the definition of $\mathbb{Q}_{n0}$.

As before we indicate that different parts of the condition belong to $p$ with a superscript, $\rho_n^p$, $f_n^p$, etc. We also write $\bar{p}$ for the natural condition in $\mathbb{P}$ derived from $p$. We write $p \leq^* q$ if $\text{lh}(p) = \text{lh}(q)$ and

1. $\bar{p} \leq^* \bar{q}$,  
2. for all $n < \text{lh}(p)$, $h_{<n}^p \leq h_{<n}^q$ and $h_{\geq n}^p \leq h_{\geq n}^q$,  
3. for all $n \geq \text{lh}(p)$, $h_{<n}^p \leq h_{<n}^q$ and for all $\rho \in \text{dom}(C_n^p)$, $C_n^p(\rho) \leq C_n^q(\rho)$.  

For a condition $p$ and $\nu \in A^p_{lh(p)}$ with $\pi_{mc}(a^p_{lh(p)}), 0(\nu) > \sup(rng(h_{<lh(p)}))$ we define $p \sim \nu$ to be the condition of length $lh(p) + 1$ determined by strengthening $\bar{p}$ to $\bar{p} \sim \nu$ and setting $\bar{p}^p_{lh(p)} = \pi_{mc}(a^p_{lh(p)}), 0(\nu)$ and $h^p_{<lh(p)} = C^p_{lh(p)}(\rho^p_{lh(p)})$ and leaving the rest of the condition unchanged. As before the ordering comes from a combination of one step extensions and direct extensions.

All of the previous claims remain valid here. The forcing $\hat{\mathbb{P}}$ satisfies the Prikry Lemma.

It follows that bounded subsets of $\kappa$ are added by a finite product of the collapses and hence $\kappa$ is preserved. The stronger version of the Prikry lemma is also true. This is used to show that $\kappa^+$ is preserved and it forms the basis for the homogeneity argument. The homogeneity argument is the same where we note that the cone isomorphism to be defined is the identity on the collapse parts of the condition. This is possible due to the fact that the new collapsing components depend only on the Prikry points associated with the least (normal) measures $U_n, n < \omega$, while our cone isomorphisms are defined to make changes to sequences $t_\delta$ for sufficiently large generators $\delta > 0$.

2. LOWER BOUND

In this section we prove Theorem 2 which establishes the lower bound. Let $\kappa$ be a strong limit singular cardinal of cofinality $\omega$. We prove that if there is no inner model $L[E]$ with a singular limit $\alpha$ such that the set of $L[E]$ Mitchell orders $\{o^E(\nu) \mid \nu < \alpha\}$ is unbounded in $\alpha$, then in $V$ there exists a subset $x \subseteq \kappa$ such that HOD$_x$ contains the power set of $\kappa$. $o^E(\nu)$ denotes the number of total extenders on the sequence $E$ with critical point $\nu$. Assuming $L[E]$ does not have a strong cardinal, every total extender on a cardinal $\nu$ in $L[E]$ is on the sequence $E$ (see [10]). Since our large cardinal assumption is below $o(\kappa) = \kappa^{++}$ where all extenders are equivalent to their normal measures (i.e., extenders with a single generators), we can assume that $o^E(\nu)$ is the number of total measures on $\nu$ in $L[E]$.

Lemma 11. Suppose that $x \subseteq \kappa$ is such that $\bigcup_{\alpha < \kappa} \mathcal{P}(\alpha)$ and $[\kappa]^\omega$ are contained in HOD$_x$ then $\mathcal{P}(\kappa)^V \subseteq$ HOD$_x$.

Proof. Working in HOD$_x$, let $\langle a_i \mid i < \kappa \rangle$ be an enumeration of the bounded subsets of $\kappa$. Consider the set $\{\bigcup_{a \in z} a \mid z \in [\kappa]^\omega\}$. Clearly this set is in HOD$_x$ and it is straightforward to see that it is $\mathcal{P}(\kappa)$. □

Lemma 12. Suppose that $x \subseteq \kappa$ satisfies that HOD$_x$ contains $\bigcup_{\alpha < \kappa} \mathcal{P}(\alpha)$ and that for every $z \in [\kappa]^\omega$ there exists a set $y \in$ HOD$_x$ such that $z \subseteq y$ and $|y| < \kappa$. Then $\mathcal{P}(\kappa)^V \subseteq$ HOD$_x$.

Proof. By the previous lemma, it is enough to show that $[\kappa]^\omega$ is contained in HOD$_x$. Let $z \in [\kappa]^\omega$. By assumption there is $y \in$ HOD$_x$ such that $z \subseteq y$ and $|y| < \kappa$. Let $\pi$ be the transitive collapse map and $\bar{y} = \pi(y)$. Clearly
\[ \pi(z) \in \text{HOD}_v, \text{ since every bounded subset of } \kappa \text{ is in } \text{HOD}_v. \] It follows that \[ z \in \text{HOD}_v, \] which finishes the proof. \[ \square \]

It is well known that the core model \( K \) exists under our anti-large cardinal assumption and can be represented as an extender model \( K = L[E] \), so that every extender \( E_\upsilon \) that appears in the sequence \( E \) is equivalent to a full or partial normal measure \( U \).

Our proof relies on the covering argument for sequences of measures (see Mitchell [8]). We commence with a brief description of the covering scenario. Suppose that \( Y \prec \text{H}_\chi \) for some sufficiently large regular cardinal \( \chi \) in \( V \) with \( \kappa \in Y \) and \( |Y| < \kappa \). Let \( \tilde{K} \) be the transitive collapse of \( K \cap Y \), and \( \sigma : \tilde{K} \to Y \cap K \prec K||_\chi \) be the inverse of the transitive collapse map. Suppose that \( \sigma \) is continuous at every point of cofinality \( \omega \). Let \( \tilde{\kappa} \in \tilde{K} \) be such that \( \sigma(\tilde{\kappa}) = \kappa \).

### 2.1. Covering Scenarios

It is a well-known fact from inner model theory that \( \tilde{K} \) is not moved in its coiteration with \( K \) (see [10] or [8]). Moreover, for every \( \tilde{K} \)-cardinal \( \tilde{\alpha} \), in the course of the coiteration of \( K; \tilde{K} \) up to \( \tilde{\alpha} \), there are only finitely many drops, that is, stages where the ultrapower on the \( K \)-side structure by some measure of height \( < \tilde{\alpha} \) warrants moving to a proper initial segment. If \( M_0 \) is the structure obtained after the last drop on the \( K \)-side, then \( M_0 \) is a mouse, which is sound above \( \tilde{\rho} = \tilde{\rho}^{M_0} \), and normally coiterates with \( \tilde{K} \) up to \( \tilde{\alpha} \) without drops.

Let \( \langle M_i, \pi_{i,j} \mid i \leq j \leq \theta \rangle \) be the iteration on the \( M_0 \)-side of the comparison with \( \tilde{K} \), \( \langle \pi_i \mid i < \theta \rangle \) be the indices of the measures used in the coiteration, and \( \tilde{\kappa} = \langle \kappa_i \mid i < \theta \rangle \) be their corresponding critical points. Note \( \tilde{\kappa} \subseteq \tilde{\alpha} \setminus \tilde{\rho} \).

If \( n < \omega \) is minimal such that \( \tilde{\rho} = \tilde{\rho}^{M_0}_n \), then for each \( i \leq \theta \), \( \tilde{\rho} = \rho^{M_i}_n \), \( \rho^{M_i}_n = \pi_{0,i}(\rho^{M_0}_n) \) and \( M_i = h^{M_i}_n[\tilde{\rho} \cup \rho^{M_i}_n \cup \{ \kappa_j \mid j < i \}] \).

Let \( M = M_0, \alpha = \sigma(\tilde{\kappa}) \), and \( \sigma^* : M \to N \) be the \( (\text{cp}(\sigma), \alpha) \) long ultrapower embedding derived from \( \sigma \). It is well-known that \( N = K||_{\eta_{\alpha}} \) is a mouse of some height \( \eta_{\alpha} < (\alpha^+)^K \). Furthermore, \( \text{rng}(\sigma) \subseteq h_n[p \cup \rho \cup \epsilon] \)

where \( h_n = h^{K||_{\eta_{\alpha}}, \rho = \text{sup}(\sigma^*(\tilde{\rho})), p = \sigma^*(\rho^{M_i}_n), \text{ and } c = \sigma^*[\epsilon] = \langle \kappa_i \mid i < \theta \rangle \).

We refer to this description as the covering scenario of \( Y \) at \( \alpha \) (or relative to \( (Y, \alpha) \)). When necessary, we denote the relevant parameters \( \tilde{\rho}, n, \tilde{\rho}^{M_i}_n, h_n, \) by \( \rho^{X,\alpha}, n^{X,\alpha}, \tilde{\rho}^{Y,\alpha}, \) and \( h^{Y,\alpha}, \) respectively. Similarly, \( p_n^N, \) and \( h_n^N, \) will be denoted by \( p^{X,\alpha}, \rho^{X,\alpha}, \) and \( h^{Y,\alpha}, \) respectively.

This description is part of Mitchell’s covering argument. It shows that \( Y \cap \alpha \) can be covered by the hull, over a level of \( K \), using parameters in: (i) an ordinal \( \rho < \alpha \); (ii) a finite set of ordinals \( p \); and (iii) a small set \( c \subseteq \alpha \) of indiscernibles. These objects, and in particular the set \( c \), depend on \( Y \cap \alpha \). The main work we do in this section is to obtain a substitute for \( c \) in a uniform way that is independent of \( Y \) up to a finite set. The precise

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1We note that in the context of applying a Skolem function such as \( h_n \), on a set \( X \), \( h[X^\prec] \) is always understood to mean \( h[X^\prec] \).
statement of our covering is the hypothesis of Lemma 16. We prove it under the anti-large cardinal assumption of Theorem 2.

Let \( \tau \) be a cardinal of uncountable cofinality in \( M \). Our proof of Theorem 2 relies on the ability to show that in many cases, we can find a finite set \( t \) of ordinals in \( M \), such that when added to the standard domain of the Skolem hull of \( M \) (i.e., the projectum, and the standard parameter) the resulting hull is stationary in \( \tau \) (in fact, it is unbounded in \( \tau \) and countably closed).

The following folklore fact asserts that this is the case for every such \( \tau \), with the exception of a cofinal, Magidor-type generic sequence over \( M \), consisting of critical points in \( \{ \kappa_i \mid i < \theta \} \). We sketch the proof for completeness.

**Lemma 13.** Suppose that \( \tau \) is a regular cardinal in \( M \) and has uncountable cofinality in \( V \). If \( \tau \) is not the limit of a closed unbounded increasing sequence of critical points \( \{ \kappa_\nu \mid \nu < \lambda \} \) of a limit length, for which \( \kappa_{\nu+1} = \pi_{\nu+1}(\kappa_\nu) \) or every \( \nu < \lambda \), then there exists a finite set \( t \) of ordinals in \( M \) such that \( \tau \cap h^M_n[p^M_n \cup \rho^M_n \cup t] \) is a stationary subset of \( \tau \) in \( V \).

**Proof.** Since \( M = M_\theta \) is the direct limit of the system of all finite sub-iterations of the iteration \( \langle M_i, \pi_{i,j} \mid i < j < \theta \rangle \), \( \tau \in M \) is represented in one of its finite sub-iterated ultrapowers. For notational simplicity, we argue first in the case where \( \tau \) is represented in \( M_0 \). Let \( \tau_0 = \pi^{-1}_{0,\theta}(\tau) \in M_0 \).

If \( \tau_0 \) does not represent any of the critical points \( \kappa_i \) (i.e., \( \pi_{0,i}(\tau_0) \neq \kappa_i \) for any \( i \)) then a straightforward induction on \( i \leq \theta \) shows that \( \pi_{0,i}(\tau_0) \subseteq \pi_{0,i}(\tau_0) \) is cofinal in \( \pi_{0,i}(\tau_0) \) and closed under countable limits. I.e., since \( \pi_{0,i} \) is continuous at countable cofinalities. The conclusion of the lemma follows as \( \pi_{0,\theta}(\tau_0) \subseteq \operatorname{rng}(\pi_{0,\theta}) = \pi_{0,\theta}^{-1}(h^M_n[p^M_n \cup \rho^M_n] = h^M_n[p^M_n \cup \rho^M_n] \) (i.e., \( t = \emptyset \) suffices here).

Suppose now that there exists some \( i_0 \) such that \( \pi_{0,i_0}(\tau_0) = \kappa_{i_0} \) is a critical point of the iteration. Let \( \langle \kappa_\nu \mid \nu < \lambda \rangle \) be an increasing enumeration of all the critical points \( \kappa_i \) less than \( \tau \) for which \( \tau = \pi_{i,\theta}(\kappa_i) \). In particular, \( \kappa_{i+1} = \pi_{i,i+1}(\kappa_i) \) for every \( \nu < \lambda \). Let \( i_\lambda = \sup_{\nu < \lambda} i_\nu \). Note that if \( \lambda \) is a limit ordinal then the normality of the iteration implies that the ordinals which represent \( \tau \) in the iterated ultrapower \( M_{i_\lambda} \) is \( \tau' = \sup_{i < \lambda} \kappa_i \), and that the next critical points \( \kappa_{i_\lambda} \) is greater or equal than \( \tau' \). \( \kappa_{i_\lambda} \) cannot be \( \tau' \) since this would imply \( \kappa_{i_\lambda} \in \langle \kappa_i \mid \nu < \lambda \rangle \) which is an absurd. It follows that \( \tau = \pi_{i_\lambda,\theta}(\kappa_{i_\lambda}) \) so that \( \tau = \pi_{i_\lambda,\theta}(\tau') \). But this contradicts our assumption that \( \tau \) is the limit of a closed unbounded increasing sequence of critical points \( \{ \kappa_\nu \mid \nu < \lambda \} \) of a limit length. We conclude that if \( \langle \kappa_\nu \mid \nu < \lambda \rangle \) is an increasing enumeration of all the critical points \( \kappa_i \) then \( \lambda = \lambda^+ + 1 \) is a successor ordinal.

Denoting \( i_{\lambda^+} \) by \( i^* \), let \( \tau_{i^*+1} = \pi_{i^*,i^*+1}(\kappa_{i^*}) \). Clearly, \( \tau = \pi_{\tau_{i^*+1},\theta}(\tau_{i^*+1}) \), and by our choice of \( i^* \), \( \tau_{i^*+1} \) does not represent any of the critical points \( \kappa_i \), \( i \geq i^* + 1 \) (i.e., the critical points of the iteration from \( M_{i^*+1} \) to \( M_\theta = M \)).
As in the previous case we have that $\pi^{\ast+1}_i \rho^{\ast} \tau^{\ast+1}_i \subseteq \tau$ is cofinal and closed under countable limits. Hence, to complete the argument, it suffices to verify that for some finite $t$, the hull $h_{\tau^{\ast+1}}{[p^{M^{\ast+1}}_{\pi^{\ast+1}} \cup \rho^{M^{\ast+1}}_{\pi^{\ast+1}} \cup t]}$ is cofinal and is countably closed in $\tau^{\ast+1}$. To this end, note that since $\tau^{\ast+1} = \pi^{\ast}, \tau^{\ast+1}(\kappa^{\ast+1})$ the cofinality of $\tau^{\ast+1}$ in $M^{\ast}$ is $(\kappa^{\ast+1})^{M^{\ast}} = (\kappa^{\ast+1})^{M^{\ast+1}}$. Let $U_{\tau^{\ast}}$ be the normal measure on $\kappa^{\ast+1}$, used to form the ultrapower $M^{\ast+1}$ of $M^{\ast}$. Since $(\kappa^{\ast+1})^{M^{\ast}}$ and its images in the iteration are not at critical points, the argument of the first case implies that there is a finite set $t$ so that $h_{\tau^{\ast+1}}[p^{M^{\ast}} \cup \rho^{M^{\ast}} \cup t] = X$ is cofinal in $\kappa^{\ast+1}$ and is countably closed. Since $M^{\ast}$ satisfies the GCH, $X$ is cofinal and countably closed in $\kappa^{\ast+1}$ with respect to the standard eventual domination order (i.e., modulo the bounded sets ideal). This, in turn, guarantees that the hull of $\pi^{\ast}, \tau^{\ast+1}(X) \cup \{\kappa^{\ast+1}\}$ in $\tau^{\ast+1}$ is countably closed and cofinal in $\tau^{\ast+1}$.

Now, in general, $\tau \in M$ need not be represented in $M_0$, however it is represented in some finite sub-iterate $M'_0$ of $M_0$. As such, $M'_0$ is sound up to an addition of a finite set of ordinals $t'$ (i.e., the set of the critical points of the finite iteration from $M_0$ to $M'_0$). It follows that the same analysis and conclusions of the iterated ultrapower from $M'_0$ to $\theta$ applies when adding $t'$ or its appropriate image to the Skolem hulls of the relevant iterates. □

2.2. Closure Procedures.

2.2.1. The standard closure procedure relative to a substructure. For our purposes, it useful to describe the result of the covering scenario as a closure process. For this, we first need to describe the coiteration induced assignment of the critical points $\bar{\kappa}_i$, $i < \theta$ to $\bar{K}$ measurable cardinals $\mu \leq \alpha$.

For each $i < \theta$, let $\bar{\mu}_i = \pi_{i,0}(\bar{\kappa}_i)$. Note that since $\bar{\kappa}_i$ is definable in $M_i$ from $p^M_n$, and a sequence of ordinals $a \subseteq \bar{\rho} \cup \langle \bar{\kappa}_j \mid j < i \rangle$, then the same holds for $\bar{\mu}_i$. We define for each measurable cardinal $\mu \leq \alpha$, the set $\bar{\mu} = \langle \bar{\kappa}_i \mid \mu_i = \mu \rangle$.

The sets $\bar{\mu}$ induce a natural partial ordering $\ast$ on $\theta$, in which for two ordinals $i, i' < \theta$, we have that $i \ast i'$ if and only if $i < i'$ and $\bar{\kappa}_i \in \bar{\mu}$ for some $\mu \in h^M_n[\rho \cup \{p^M_n\} \cup \{\kappa_j\}_{j \leq i}]$. The previous observation implies that the minimal values of $\ast$ are the ordinals $i < \theta$ for which $\kappa_i$ belongs to the hull $h^M_n[\rho \cup \{p^M_n\}]$. Since the partial ordering $\ast$ is clearly well-founded, we can get from each $i' < \theta$ to a minimal value of $\ast$ in finitely many steps.

The description of the order $\ast$ allows us to present $M$ as the closure $M = \cup_n \bar{y}_n$ of an $\omega$-procedure which incorporates the Skolem function $h^M_n$ and the assignment $\mu \mapsto \bar{\mu}_i$: We start from $\bar{y}_0 = \bar{\rho} \cup p^M_n$, and for each $m < \omega$ define $\bar{y}_{m+1}$ to be the closure of $\bar{y}_m \cup (\cup_{\mu \in \bar{y}_m} \bar{\mu})$ under $h^M_n(p^M_n, \cdot)$. It is routine to verify that each $\bar{\kappa}_i$ is added to $\bar{y}_n$, where $n < \omega$ is the $\ast$-rank of $i$.

Similarly, for each $N$ measurable cardinal $\mu \in Y \cap \alpha = \operatorname{rng}(\sigma^*) \cap \alpha$, $\mu = \sigma(\bar{\mu})$, we define $\bar{\mu} = \sigma[\bar{\mu}]$. It follows that $\operatorname{rng}(\sigma^*) \cap On$ is covered
in a similar closure procedure, which starts from $\rho \cup p$ and incorporates $h^{K[\eta_\alpha]}_n(p, \cdot)$ and the assignment $\mu \mapsto c_\mu$.

2.2.2. The $g$-closure for a covering scenario. Given a function $g : \kappa \to [\kappa]^{<\kappa}$, we consider an alternative closure procedure for a covering scenario, in which the sets $c_\mu$ are replaced with $g(\mu)$ for $\mu$ that appears in the closure process. As opposed to the standard closure procedure, described above, here we also allow an arbitrary initial finite (seed) set of ordinals $s$ to be added to the closure process.

More precisely, we define the $g$-closure relative to $(Y, \alpha, s)$ for a finite set $s \subseteq \alpha$ to be the set $y = \bigcup_{m<\omega} y_m$ where $y_0 = \rho \cup s$ and for each $m < \omega$, $y_{m+1}$ is the closure of $y_m \cup \bigcup_{\mu \in y_m} g(\mu)$ under $h^{N}_n(p, \cdot)$. Note that the structure $N = K[\eta_\alpha]$ determines the Skolem function $h^{N}_n$ which is used for computing the $g$-closure. $N$ results from taking $Y$ and running the covering argument to cover $Y \cap \alpha$. We refer to the resulting closure set $y_\omega = \bigcup_n y_n$ as the $g$-closure relative to $(Y, \alpha, s)$.

We see that to apply the $g$-closure relative to $(Y, \alpha, s)$ one needs access to the parameter $p$, the ordinal $\rho$ and the finite set of ordinals $s$, function $g$ and the Skolem function $h^{K[\eta_\alpha]}_n(p, \cdot)$. Hence it is definable in HOD$_g$, or any sufficiently closed model containing these parameters.

2.3. The Final Argument. We would like to refine Lemma 12 to fit the covering scenario described above. To describe this refinement, we introduce a weak version of the $\delta$-closed property. We note that since $\kappa$ is a strong limit cardinal in $V$, there are unboundedly many $\delta < \kappa$ of uncountable cofinality with $\delta^{<\delta} = \delta$. Indeed, this is the case for every cardinal $\delta = \gamma^{<\delta}$ for some $\gamma < \kappa$.

**Definition 14.** Suppose that $\delta < \kappa$ is a cardinal of uncountable cofinality, with $\delta^{<\delta} = \delta$. Let $Y \prec H_{\kappa^+}$. We say that $Y$ is $\delta$-weakly-closed if it satisfies the following conditions:

1. $\omega Y \subseteq Y$;
2. $|Y| = \delta^+$;
3. $\delta^+ \subseteq Y$;
4. For every $\alpha \in Y \cap \kappa$, if $\text{cf}(\alpha) \leq \delta$ then $Y \cap \alpha$ contains a club subset of $\delta$;
5. For every $\alpha \in Y \cap \kappa$, if $\text{cf}(\alpha) \geq \delta^+$ then $\text{cf}(\sup(Y \cap \alpha)) = \delta^+$, and $Y$ contains a closed unbounded subset of $\sup(Y \cap \alpha)$.

**Remark 15.**

1. We note that assuming $\delta^{<\delta} = \delta$, there are stationarily many $\delta$-weakly-closed structures $Y$. Indeed, for every function $F : [H_{\kappa^+}]^{<\omega} \to H_{\kappa^+}$, it is straightforward to construct a $\delta$-weakly-closed structure $Y$ which is closed under $F$, in $\delta^+$ many steps: Let $Y_0 = \delta + 1$. Assuming that $Y_\gamma$ has been defined, let $Y_{\gamma+1}$ be obtained from $Y_{\gamma}$ by closing under $F$, closing under $\omega$-sequences, closing under a Skolem function of $H_{\kappa^+}$, for each $\alpha \in Y_\gamma$ adding a club subset of
\( \alpha \) of minimal ordertype if \( \text{cf}(\alpha) \leq \delta \), and adding the ordinal \( \alpha' = \sup(Y_\gamma \cap \alpha) \), if \( \text{cf}(\alpha) \geq \delta^+ \). At limit stages take unions. It is clear that \( Y = Y_\delta \) is \( \delta \)-weakly-closed.

(2) Let \( \sigma : K \rightarrow K||\kappa^+ \) be the restriction of the inverse of the restriction of the transitive collapse map of \( Y \), to \( K \cap Y \prec K||\kappa^+ \). We note that the fact \( Y \cap \alpha \) is cofinal at each \( \alpha \in Y \) with \( \text{cf}(\alpha) \leq \delta \), implies that \( \sigma \) is continuous at limits of cofinality \( \omega \). Therefore, the covering scenario analysis given at the outset of this section applies to \( Y \) and \( \sigma \).

**Lemma 16.** Suppose there exist a cardinal \( \delta < \kappa \) and a function \( g : \kappa + 1 \rightarrow [\kappa]^{\leq \delta} \) in \( V \) such that for every \( \delta \)-weakly-closed structure \( Y \prec (H_{\kappa^+}, g) \) there exists a finite set \( s \subseteq Y \cap \kappa \) such that the \( g \)-closure set relative to \( (Y, \kappa, s) \) covers \( Y \cap \kappa \). If \( \kappa \) is strong limit and \( \delta^{\text{Sd}} = \delta \) then there exists \( x \subseteq \kappa \) such that \( P(\kappa)^V \subseteq \text{HOD}_x \).

**Proof.** Let \( x \) be a subset of \( \kappa \) which codes all bounded subsets of \( \kappa \) and the function \( g \) in a natural way. We prove that \( \text{HOD}_x \) satisfies the requirements of Lemma 12.

Let \( z \in [\kappa]^{\omega} \). Let \( Y \) be an elementary substructure containing \( z \) as in the hypotheses of the lemma. By assumption if \( y \) is the \( g \)-closure relative to \( (Y, \kappa, s) \) for some finite set \( s \), then \( y \) covers \( Y \cap \kappa \) and hence \( z \). We have that \( y \in \text{HOD}_x \), since the \( g \)-closure is definable from parameters in \( \text{HOD}_x \). Finally, \( y \) has size less than \( \kappa \) since \( g(\mu) \) has size at most \( \delta \) for all \( \mu \). So \( y \) satisfies the requirements of Lemma 12. \( \square \)

The rest of this section is devoted to proving that the assumption of the previous lemma holds under our anti-large cardinal hypothesis. More precisely, our anti-large cardinal hypothesis implies that the set \( \{o^K(\mu) \mid \mu < \kappa\} \) is bounded in \( \kappa \).

**Assumptions:** Let \( \delta \geq \sup\{\{(2^{\omega_0})^+\} \cup \{o^K(\mu) \mid \mu < \kappa\}\} \) be a cardinal below \( \kappa \), of uncountable cofinality, such that \( \delta^{\text{Sd}} = \delta \). We construct a function \( g : \kappa \rightarrow [\kappa]^{\leq \delta} \) satisfying the requirements of the previous lemma by induction on its restrictions \( g \upharpoonright \alpha + 1 \).

To prove the main Lemma 16, we will show by induction on ordinals \( \alpha \in (\kappa + 1) \setminus \delta^+ \), that for every \( Y \prec (H_{\alpha^+}, g \upharpoonright \alpha + 1) \) which is \( \delta \)-weakly-closed in \( V \), there exists a finite set of ordinals \( s \subseteq \alpha \), such that the \( g \)-closure relative to \( (Y, \alpha, s) \) covers \( Y \cap \alpha \).

** Lemma 17.** Let \( Y \prec (H_{\alpha^+}, g \upharpoonright \alpha) \) and \( z_0 \in [Y \cap \alpha]^{<\omega} \). Suppose that \( y \subseteq \alpha \cap Y \) is contained in the \( g \)-closure relative to \( (Y, \alpha, z_0) \). Then there exists some \( \alpha_0 < \alpha \) such that for every \( \beta \in (\alpha_0, \alpha) \cap y \), there is some finite \( x_\beta \subseteq Y \cap \beta \), such that the \( g \)-closure relative to \( (Y, \alpha, z_0 \cup x_\beta) \) covers \( Y \cap \beta \).

**Proof.** Recall that \( \sigma : K \rightarrow K \cap Y \) is (the restriction of) the inverse of the transitive collapse map of \( Y \). Let \( \beta = \sigma^{-1} (\beta) \). Then the covering scenario for
(\(Y, \beta\)) corresponds to the coiteration of \(K\) with \(K\) up to \(\bar{\beta}\). This coiteration is an initial segment of the coiteration of \(K\) with \(K\) up to \(\bar{\alpha}\) = \(\sigma^{-1}(\alpha)\). Let \(\alpha_{i} < \bar{\alpha}\) be an ordinal above the height of the last drop on the \(K\)-side of the coiteration. Then for every \(\beta > \alpha_{0}\) the coiteration of \(K\) and \(\bar{K}\), up to \(\bar{\beta}\), agrees with the coiteration of up to \(\bar{\alpha}\) past the last drop.

Let \(<M_{i}, \pi_{i,j} \mid i, j \leq \theta^{\beta}\rangle\) denote the iteration of the \(K\)-side structures in the comparison with \(\bar{K}\) up to \(\bar{\beta}\) starting after the last drop. It follows that this iteration coincides with an initial segment of the iteration \(<M_{i}, \pi_{i,j} \mid i \leq j \leq \theta^{\beta}\rangle\) of the \(K\)-side structures, in the comparison with \(\bar{K}\) up to \(\bar{\alpha}\). I.e. \(\theta^{\beta} \leq \theta\), and \(M_{1}^{\beta} = M_{1}, \pi_{i,j}^{\beta} = \pi_{i,j}\), for every \(i, j \leq \theta^{\beta}\).

Let \(M^{\beta} = M^{\beta}_{\emptyset}\) denote the last structure in the comparison up to \(\bar{\beta}\), and \(\pi^{\beta} = \pi_{\emptyset, \beta} : M^{\beta} \to M\). It follows at once that \(\hat{\rho}^{Y, \alpha} = \hat{\rho}^{Y, \beta}, n^{Y, \alpha} = n^{Y, \beta}\), and \(\hat{p}^{Y, \alpha} = \pi^{\beta}(\hat{p}^{Y, \beta})\).

Let \(\sigma^{\beta} : M^{\beta} \to N^{\beta}\) be the \((\text{cp}(\sigma), \beta)\) long ultrapower embedding derived by the extender \(E^{\beta}\) of height \(\beta\), derived from \(\sigma\). Recall that \(\sigma^{\beta} : M \to N\) denotes the \((\text{cp}(\sigma), \alpha)\) long ultrapower embedding derived from \(\sigma\). Let \(\hat{\pi}^{\beta} : N^{\beta} \to N\) be the embedding induced by \(\pi^{\beta}\), \(\sigma^{\beta}\), and \(\sigma^{\beta}\). Recall that every element of \(N^{\beta}\) is of the form \(x = \sigma^{\beta}(f)(a)\) for some \(f \in M^{\beta}\) and a finite \(a \subseteq \beta\). Set \(\hat{\pi}^{\beta}(x) = (\sigma^{*} \circ \pi^{\beta})(f)(a)\). Clearly, \(\hat{\pi}^{\beta}(a) = a\) for every \(a \in [\beta]^{<\omega}\), and thus, \(\text{cp}(\hat{\pi}^{\beta}) \geq \beta\). It is routine to verify that \(\hat{\pi}^{\beta}\) is \(\Sigma_{0}^{(0)}\) elementary and takes \(p^{Y, \beta}\) to \(p^{Y, \alpha}\). It follows that for every \(z \in [\alpha]^{<\omega}\), if \(\beta\) belongs to the \(g\)-closure relative to \((Y, \alpha, z)\), then it further covers the \(g\)-closure relative to \((Y, \beta, z \cap \beta)\).

Suppose now that \(z_{0} \in [\alpha]^{<\omega}\) such that \(\beta\) belongs to the \(g\)-closure relative to \((Y, \alpha, z_{0})\). By the inductive assumption applied to the covering scenario at \((Y, \beta)\), there exists some \(x_{\beta} \in [Y \cap \beta]^{<\omega}\) such that the \(g\)-closure relative to \((Y, \beta, x_{\beta})\) covers \(Y \cap \beta\). It follows that the \(g\)-closure relative to \((Y, \alpha, z_{0} \cup x_{\beta})\) contains the \(g\)-closure relative to \((Y, \beta, x_{\beta})\), and in particular, covers \(Y \cap \beta\).

\(\square\)

**Remark 18.** Suppose that \(g \restriction \alpha\) has been defined and satisfies the inductive assumption for all \(\beta < \alpha\). If \(\alpha \in Y\) is not a cardinal in \(K\) then it is straightforward to verify that the Skolem hull \(h^{Y, \alpha}[p^{Y, \alpha} \cup p^{Y, \alpha}]\) has a surjection from \(\beta = [\alpha]^{K} \in Y\) onto \(\alpha\), and therefore \(\beta\) belongs to the \(g\)-closure relative to \((Y, \alpha)\). The argument of the previous lemma implies that for every finite set \(s\), the \(g\)-closure relative to \((Y, \alpha, s)\) covers the closure relative to \((Y, [\alpha]^{K}, s)\). We may therefore apply the inductive assumption to \(\beta = [\alpha]^{K}\), and use it to conclude that the induction hypothesis holds at \(\alpha\) with \(g(\alpha) = \emptyset\).

**Therefore, for the rest of the proof we restrict our attention to ordinals \(\alpha \leq \kappa\) which are cardinals in \(K\).**

Let \(\alpha\) be a \(K\)-cardinal. The definition of \(g(\alpha)\) and the proof that it works will be divided into three cases according to the \(V\)-cofinality of \(\alpha\).
2.3.1. Case I: cf\(^V\)(\(\alpha\)) > \(\delta\). We set \(g(\alpha) = \emptyset\) and show that for every \(\delta\)-weakly-closed \(Y \prec (H_{\alpha+}, g)\) there exists some finite \(s \subseteq \alpha \cap Y\) such that the \(g\)-closure relative to \((Y, \alpha, s)\) covers \(Y \cap \alpha\). Since \(Y\) is \(\delta\)-weakly-closed, \(cf(Y) \geq \delta^+\) and \(g(\beta) \subseteq Y\) for every \(\beta \in Y \cap \alpha\).

**Lemma 19.** There exists some \(z \in [Y \cap \alpha]^{<\omega}\) such that the \(g\)-closure relative to \((Y, \alpha, z)\) contains a stationary subset of \(\alpha^\prime\) (in \(V\)).

*Proof.* The fact \(Y\) is \(\delta\)-weakly-closed guarantees \(Y \cap \alpha\) contains a closed unbounded subset of \(\alpha^\prime\). Let \(\bar{\alpha} = \sigma^{-1}(\alpha)\). Since \(\sigma\) is continuous at all points of cofinality \(\leq \delta\), it suffices to show that there exists a finite set \(\bar{\alpha} \subseteq \bar{\alpha}\) such that the \(h_n^M\) closure of \(\bar{p}^{Y,\alpha} \cup \bar{\rho}^{Y,\alpha} \cup \bar{\pi}\) contains a stationary subset of \(\alpha\) in \(V\).

To this end, note that \(cp(\bar{\alpha}) > \delta^+\). It follows that \(\tau = cf(M, \bar{\alpha}) \geq \delta^+\) and \(K||\delta^+ + 1 = K||\delta^+ + 1\).

We see that in \(V\), \(cf(\tau) = cf(\bar{\alpha}) \geq \delta^+\). By the Mitchell Covering Theorem ([8]), if \(\tau\) is the limit of a closed unbounded increasing sequence of critical points \(\langle \check{\kappa}_\nu \mid \nu < \lambda \rangle\) of a limit length, for which \(\check{\kappa}_{\nu+1} = \pi_{\nu, \nu+1}(\check{\kappa}_\nu)\), then \(\lambda \leq \omega^{\check{\kappa}(\tau)}\) (the ordinal exponent). Since the Mitchell order of \(\tau\) in \(K\) is bounded by \(\delta\) and \(cf(\tau) > \delta\), we conclude that \(\tau\) satisfies the assumptions of Lemma 13. The Lemma implies in turn, that there is a stationary subset \(S\) of \(\bar{\alpha}\) (stationary in \(V\)) which is covered by the Skolem hull by \(h_n^M\), of the ordinals below the projectum, the standard parameters, and a finite set of ordinals \(t\).

\[\square\]

Let \(S \subseteq Y \cap \alpha^\prime\) be the stationary subset of \(\alpha^\prime\) which is given by the previous lemma. Fix for each \(\beta \in S\), a finite set \(x_\beta \subseteq \beta \cap Y\) such that the \(g\)-closure relative to \((Y, \alpha, x_\beta)\) covers \(Y \cap \beta\). This is possible by Lemma 17.

We would like to show that there is a stationary subset \(S^* \subseteq S\) and a finite \(s^* \subseteq \alpha \cap Y\), such that the \(g\)-closure relative to \((Y, \alpha, s^*)\) covers the \(g\)-closure relative to \((Y, \alpha, x_\beta)\) for every \(\beta \in S^*\), as it would clearly imply that the \(g\)-closure relative to \((Y, \alpha, s^*)\) covers \(Y \cap \alpha\).

For this, we press down on the map \(\beta \mapsto \max(x_\beta) < \beta\). Since \(cf(\alpha^\prime) > \omega\) we can find some \(\beta^* \in Y \cap \alpha\) and a stationary subset \(S^* \subseteq S\), such that \(x_\beta \subseteq \beta^*\) for each \(\beta \in S^*\). Now, since there exists a finite set \(s^* \subseteq Y\) such that the \(g\)-closure relative to \((Y, \alpha, s^*)\) covers \(Y \cap \beta^*\). In particular, the \(g\)-closure relative to \((Y, \alpha, s^*)\) contains \(x_\beta\) for every \(\beta \in S^*\), as required.

2.3.2. Case II: \(\aleph_0 < cf^V (\alpha) \leq \delta\). In this case, we take \(g(\alpha)\) to be some closed unbounded subset of \(\alpha\) of minimal ordertype. Note that for every \(\delta\)-weakly-closed substructure \(Y \prec H_{\alpha+}, S = Y \cap g(\alpha)\) contains a closed and unbounded subset of \(\alpha\). Since \(cf(\alpha) > \aleph_0\), our pressing down argument for the case \(cf(\alpha) > \delta\) can be applied to \(S\) to show that there exists a seed \(s^*\) such that the \(g\)-closure relative to \((Y, \alpha, s^*)\) covers \(Y \cap \alpha\).

2.3.3. Case III: \(cf^V (\alpha) = \aleph_0\). The construction and argument are different than the previous two cases. The idea is to choose \(g(\alpha)\) which will guarantee
covering up to $\alpha$ with respect to a sufficiently elementary structure $Z \prec H_{\alpha^+}$. I.e., a structure of the form $Z = Z^* \cap H_{\alpha^+}$ where $Z^* \prec (H_{\alpha^+}, g \upharpoonright \alpha)$. We will then use a reflection argument to prove the set $g(\alpha)$ satisfies the induction hypothesis at $\alpha$.

We work to define $g(\alpha)$. Let $\langle \beta_k \mid k < \omega \rangle \in Z$ be a cofinal sequence in $\alpha$. By Lemma 17 and the induction hypothesis, we may assume that for every $k < \omega$ there exists a finite set $s_k \subseteq \beta_k$ such that the $g \upharpoonright \alpha$-closure relative to $(Z, \alpha, s_k \cup \{\beta_k\})$ covers $Z \cap \beta_k$. We set $g(\alpha) = \bigcup_{k<\omega} (s_k \cup \{\beta_k\})$. It is immediate that the $g$-closure relative to $(Z, \alpha, \emptyset)$ covers $Z \cap \alpha$. Note that $g(\alpha) \subseteq Z = Z^* \cap H_{\alpha^+}$ and thus $Z^* \prec (H_{\alpha^+}, g)$.

Next, we claim that for every $\delta$-weakly-closed $Y \prec (H_{\alpha^+}, g)$, there exists a finite $s \subseteq \alpha$ such that the $g$-closure relative to $(Y \cap \alpha, \alpha, s) \prec Y \cap \alpha$.

Suppose otherwise. Since the statement regarding a counterexample is definable in $H_{\alpha^+}$ in the parameter $g \upharpoonright (\alpha + 1)$, we may find a counterexample $Y \in Z^*$. In particular, for every finite $s \subseteq Y \cap \alpha$ the $g$-closure relative to $(Y \cap \alpha, \alpha, s)$ does not cover $Y \cap \alpha$. We work by induction to construct finite sets $\langle t_k \mid k < \omega \rangle$. Let $t_0 = \emptyset$. Suppose for some $k < \omega$ we have defined $t_k$. Let $\alpha_k$ be the least element of $Y \cap \alpha$ which is not belongs to the $g$-closure relative to $(Y \cap \alpha, \alpha, t_k)$. Let $m_k$ be least such that $\beta_{m_k} > \alpha_k$, and $t_{k+1} \subseteq Y \cap \beta_{m_k}$ be such that the $g$-closure relative to $(Y \cap \alpha, \alpha, t_{k+1})$ covers $Y \cap \beta_{m_k}$. Note that if $y$ is the above $g$-closure, then $y$ covers the $g$-closure relative to $(Y \cap \alpha, \alpha, s)$ for any $s \in [Y \cap \beta_{m_k}]^{<\omega}$. In particular $\alpha_{k+1}$ is not in the $g$-closure relative to $(Y \cap \alpha, \alpha, s)$ for any $s \in [Y \cap \beta_{m_k}]^{<\omega}$. We will refer to this later as the key property of $\alpha_{k+1}$. Note also that $\sup_{k<\omega} \alpha_k = \alpha$ and $\langle \alpha_n \mid n < \omega \rangle \in Y \cap Z$, since both $Y$ and $Z$ are closed under $\omega$-sequences.

Recall that our choice of $g(\alpha)$ guarantees that the $g$-closure relative to $(Z, \alpha, \emptyset)$ covers $Z \cap \alpha$. This means there are $n = n^{Z,\alpha} < \omega$, $\eta = \eta^{Z,\alpha} < (\alpha^+)^K$, $\rho = \rho^{Z,\alpha} < \alpha$ and $p = p^{Z,\alpha} \in [\alpha]^{<\omega}$, such that the closure of $\rho \cup \{p\}$ under $h_n^{K|\eta}$ and $g$ covers $\bar{\alpha}$. The last statement is true in $H_{\alpha^+}$.

Since $\bar{\alpha} \in Y \prec (H_{\alpha^+}, g)$, we can find some $n < \omega$, $\eta < (\alpha^+)^K$, $\rho < \alpha$, and $p \in [\alpha]^{<\omega} \cap Y$, such that $Y$ satisfies the statement with respect to $\bar{\alpha}$ in these parameters. It follows that $\bar{\alpha}$ is covered by any $g$-closure relative to $(Y, \alpha, s)$, if this $g$-closure contains $p, \eta$, and covers $Y \cap \rho$. Moreover, since $\alpha$ is the maximal cardinal in the $K$-level in which the closure is taken, and the closure uses parameters below $\alpha$ which define a surjection of $\alpha$ onto the entire level, there exists some $\gamma \in Y \cap \alpha$ such that $\eta$ belongs to the $g$-closure relative to $(Y, \alpha, \{\gamma\})$.

Let $k < \omega$ be large enough that $\max(p), \gamma, \rho < \beta_k$, and set $x = p \cup \{\gamma\} \cup t_k$. Then $p, \eta$ belong to the $g$-closure relative to $(Y, \alpha, x)$, which further covers $Y \cup \rho$. Therefore, the closure covers $\bar{\alpha}$, and in particular contains $\alpha_{k+1}$. However, $x \in [\beta_k]^{<\omega}$, thus, the last contradicts the key property of $\alpha_{k+1}$.

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