

Compactness and Guessing Principles in the Radin Extensions

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Abstract

We investigate the interaction between compactness principles and guessing principles in the Radin forcing extensions [Rad82]. In particular, we show that in any Radin forcing extension with respect to a measure sequence on κ , if κ is weakly compact, then $\diamond(\kappa)$ holds, answering a question raised in [BN19]. This provides contrast with a well-known theorem of Woodin [Cum], who showed that in a certain Radin extension over a suitably prepared ground model relative to the existence of large cardinals, the diamond principle fails at a strongly inaccessible Mahlo cardinal. Refining the analysis of the Radin extensions, we consistently demonstrate a scenario where a compactness principle, stronger than the diagonal stationary reflection principle, holds yet the diamond principle fails at a strongly inaccessible cardinal, improving a result from [BN19].

1 Introduction

This paper contributes to the study of the interaction between compactness principles and guessing principles, specifically, in the context of Radin forcing [Rad82]. Recall that for a regular uncountable cardinal κ , $\diamond(\kappa)$ asserts the existence of a sequence $\langle S_\alpha \subset \alpha : \alpha < \kappa \rangle$ such that for any $X \subset \kappa$, $\{\alpha < \kappa : X \cap \alpha = S_\alpha\}$ is stationary. An old open problem in this area asks if $\diamond(\kappa)$ must hold at a weakly compact cardinal κ . We prove that this is indeed the case in the Radin forcing extensions.

Theorem 1.1. *Let $R_{\bar{U}}$ be Radin forcing by a measure sequence \bar{U} on κ .*

$$\Vdash_{R_{\bar{U}}} (\kappa \text{ is weakly compact} \implies \diamond(\kappa) \text{ holds})$$

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The special attention given to Radin forcing in this context, originates in results of Woodin [Cum], who used Radin forcing to establish the consistency of a large cardinal κ , such as strongly inaccessible, Mahlo, and greatly Mahlo, with $\neg\Diamond(\kappa)$. In fact, Radin forcing is the only known method for producing models where the diamond principle fails fully on any large cardinals.

The history of the relation between the diamond and compactness principles, goes back to the work of Kunen and Jensen [JK69], who showed that $\Diamond(\kappa)$ must hold at every subtle cardinal. In fact, they prove that the stronger property $\Diamond(Reg_\kappa)$ holds at such cardinals, where $\Diamond(Reg_\kappa)$ asserts that there exists a diamond sequence supported on regulars. The consistency of $\neg\Diamond(Reg_\kappa)$ on weak compact cardinals was shown by Woodin, and improved by Hauser [Hau92] to indescribable cardinals, and by Džamonja and Hamkins [DH06] to strongly unfoldable cardinals. Each of these consistency results concerning the failure of the diamond principle on the regulars is established from its minimal corresponding large cardinal assumption. In contrast, $\neg\Diamond(\kappa)$ at relatively small large cardinals, such as Mahlo cardinals, is known to have a significantly stronger consistency strength. Jensen [Jen69] has shown that $\neg\Diamond(\kappa)$ at a Mahlo cardinal κ implies the existence of $0^\#$. Zeman [Zem00] improved the lower bound to the existence of an inner model with a cardinal κ , such that for every $\gamma < \kappa$, the set $\{\alpha < \kappa \mid o(\alpha) \geq \gamma\}$ is stationary in κ . The last large cardinal assumption is quite close to the hypermeasurability (large cardinal) assumptions used by Woodin to force $\neg\Diamond(\kappa)$ at a greatly Mahlo cardinal.

In [BN19] the first author studied Radin forcing $R_{\bar{U}}$, and the connection between properties of measure sequence \bar{U} on κ and large cardinal properties of κ in generic extensions by $R_{\bar{U}}$. It is shown that Woodin's construction of $\neg\Diamond(\kappa)$ can be extended to large cardinal properties such as stationary reflection principles. A question on whether we can extend the analysis to get a model of κ being weakly compact and $\neg\Diamond(\kappa)$ was asked in [BN19]. Theorem 1.1 answers this question in the negative by showing that this approach cannot yield significantly stronger results. Two properties of measure sequences \bar{U} which were isolated in [BN19], are the Weak Repeat Property (WRP) and Local Repeat Property (LRP). It is shown in [BN19] that \bar{U} satisfies WRP if and only if κ is weakly compact in generic extensions by $R_{\bar{U}}$, and asks if the stronger property of LRP has a similar characterization. We answer this question in Proposition 3.5, showing that LRP characterizes the large cardinal property of almost ineffability. In the rest of the paper we extend the study of compactness and guessing principles in Radin forcing extensions. The organization of the paper is as follows:

1. In Section 2, we provide some background on the type of forcing notions that we will work with for the rest of the paper.
2. In Section 3, we study characterizations of almost ineffable cardinals and weakly compact cardinals in the Radin extension based on certain properties of the measure sequence used to defined the forcing.
3. In Section 4, we isolate scenarios when a variety of guessing principles can

hold in the Radin extensions.

4. In section 5, we demonstrate a scenario where a compactness principle, stronger than the diagonal stationary reflection principle, holds but the diamond principle fails at a strongly inaccessible cardinal.

2 Preliminaries

2.1 Measure sequences

A *measure sequence* is a sequence $\bar{w} = \langle \kappa(\bar{w}) \rangle \frown \langle w(\tau) \mid \tau < lh(\bar{w}) \rangle$, where each $w(\tau)$ is a $\kappa(\bar{w})$ -complete ultrafilter on $V_{\kappa(\bar{w})}$, and \bar{w} is derived from an elementary embedding $j = j_{\bar{w}} : V \rightarrow M$ with $\text{crit}(j) = \kappa(\bar{w})$ in the sense that for $A \subseteq V_{\kappa(\bar{w})}$ and $\tau < lh(\bar{w})$,

$$A \in w(\tau) \iff \bar{w} \upharpoonright \tau \in j(A).$$

In particular, $w(0)$ is equivalent to the normal measure derived from j and $lh(\bar{w}) \leq j(\kappa(\bar{w}))$. The class of all measure sequences is denoted by \mathcal{MS} . For a set $A \subseteq \mathcal{MS}$, we denote the set of critical points $\kappa(\bar{w})$ for $\bar{w} \in A$ by $O(A)$.

All measure sequences \bar{w} in our constructions are assumed to satisfy $\mathcal{MS} \cap V_{\kappa(\bar{w})} \in \bigcap \bar{w}$ (see the discussion involving the set \bar{A} in [Git10, page 1402], for further details).

Definition 2.1. Let \bar{U} be a measure sequence on κ constructed by $j : V \rightarrow M$, and $A \subset \mathcal{MS} \cap V_{\kappa}$ and \bar{U} a measure sequence on $\kappa = \kappa(\bar{U})$. We say

1. A is \bar{U} -*measure-one* if $A \in \bigcap \bar{U}$.
2. A is \bar{U} -*tail-measure-one* if there is some $\gamma < lh(\bar{U})$ such that $A \in U(i)$ for all $\gamma < i < lh(\bar{U})$.
3. A is \bar{U} -*positive* if $\{i < lh(\bar{U}) : A \in U(i)\}$ is cofinal in $lh(\bar{U})$.
4. A is \bar{U} -*stationarily-positive* if $\{i < lh(\bar{U}) : A \in U(i)\}$ is a stationary subset of $lh(\bar{U})$.
5. A is \bar{U} -*non-null* if there is some $\gamma < lh(\bar{U})$, $A \in U(\gamma)$.

Definition 2.2. We say a function $b : \mathcal{MS} \cap V_{\kappa} \rightarrow V_{\kappa}$

1. is a *measure function* if $\text{dom}(b) = \mathcal{MS} \cap V_{\kappa}$ and for any $\bar{w} \in \text{dom}(b)$, $b(\bar{w}) \in \bigcap \bar{w}$.
2. is a *tail measure function* if $\text{dom}(b) = \mathcal{MS} \cap V_{\kappa}$ and for any $\bar{w} \in \text{dom}(b)$, $b(\bar{w})$ is \bar{w} -tail-measure-one.

2.2 Radin forcing

Let $R_{\bar{U}}$ be the Radin forcing defined using \bar{U} , constructed by $j : V \rightarrow M$. This forcing was first invented in [Rad82]. The notations regarding the measure sequence and its constructing embedding are fixed for the rest of the paper unless otherwise stated.

Our notations and presentation follow [BN19] or [Git10] for the most part, with the exception of using the forcing convention by which for two conditions p, q , $p \geq q$ means that q extends p (i.e., q is more informative). We refer the readers to the above for the definition of this forcing.

Notation 2.3. 1. For any $A \in \bigcap \bar{U}$, there exists another $A' \subset A$ in $\bigcap \bar{U}$ satisfying that for any $\bar{u} \in A'$, $A' \cap V_{\bar{w}} \in \bigcap \bar{w}$. We may without loss of generality assume all the measure one sets satisfy this property for the rest of this paper.

2. Conditions $p \in R_{\bar{U}}$ are finite sequences $p = \langle d_i \mid i \leq k \rangle$ where each d_i is either of the form $d_i = \langle \kappa_i \rangle$ for some $\kappa_i < \kappa$, or of the form $d_i = \langle \bar{\mu}_i, a_i \rangle$ where $\bar{\mu}_i$ is a measure sequence of length $lh(\bar{\mu}_i) > 0$ and $a_i \in \bigcap \bar{\mu}_i$. We denote $\bar{\mu}(d_i) = \langle \kappa_i \rangle$ in the former case and $\mu(d_i) = \bar{\mu}_i$ in the latter. We require that the top component $\bar{\mu}(d_k) = \bar{U}$. We also write $p = p_0 \frown d_k$, where $p_0 = \langle d_i \mid i < k \rangle$ denotes the *lower part* of p , and $d_k = (\bar{U}, A^p)$ denotes its *top part*.
3. Given a measure sequence \bar{w} and a \bar{w} -measure-one set c , we say a finite sequence of measure sequences with increasing critical points $\vec{\eta}$ is *addable* to (\bar{w}, c) , or $\vec{\eta} \ll (\bar{w}, c)$, if for each $\bar{u} \in \vec{\eta}$, $\bar{u} \in c$ and $c \cap V_{\kappa(\bar{u})} \in \bigcap \bar{u}$.

Remark 2.4. We clarify the following abuse of notations: \frown could mean concatenation or one-step extension depending on the context. However, our usage is without ambiguity:

1. If the object after \frown is a pair, for example $p \frown (\bar{w}, A)$ where \bar{w} is a measure sequence and A is \bar{w} -measure-one, then \frown means concatenation. In this case, $\kappa(\bar{w}) > \kappa(\bar{u})$ for any measure sequence \bar{u} appearing on p .
2. If the object after \frown is a measure sequence, for example $p \frown \bar{w}$, then this means it is a one-step extension. In this case, $\kappa(\bar{w})$ belongs to the measure one set A^{p_i} in one of the components p_i of p .

Definition 2.5. \bar{U} satisfies the *Repeat Property* (RP) if there exists $\gamma < lh(\bar{U})$ such that $\bigcap \bar{U} \upharpoonright \gamma = \bigcap \bar{U}$. We say γ is a repeat point for \bar{U} and γ witnesses \bar{U} satisfies RP.

Definition 2.6. We say $\gamma < lh(\bar{U})$ is a *weak repeat point* if γ witnesses $\bar{U} \upharpoonright \gamma + 1$ satisfies RP. If γ is not a weak repeat point, then it is *novel*.

Mitchell [Mit82] and Cummings-Woodin [Cum] independently proved that κ is measurable in $V^{R_{\bar{U}}}$ if and only if $\bigcap \bar{U} = \bigcap \bar{U}'$ for some measure sequence \bar{U}' satisfying the RP.

For the rest of the paper, we may assume \bar{U} does not satisfy RP. The reason is that suppose γ is the first repeat point of \bar{U} , then $R_{\bar{U}}$ is forcing equivalent to $R_{\bar{U} \upharpoonright \gamma}$ and $\bar{U} \upharpoonright \gamma$ does not satisfy RP. Notice that if \bar{U} does not satisfy RP, then there are unboundedly many $\gamma < lh(\bar{U})$ that is not a weak repeat point.

Remark 2.7. *If \bar{U} does not satisfy RP and $cf(lh(\bar{U})) \leq \kappa$, then in $V^{R_{\bar{U}}}$, κ becomes singular. This follows from the arguments in [Git10].*

Remark 2.8. *Suppose $|2^\kappa|^M$ does not divide $lh(\bar{U})$, then there exists some $\gamma < lh(\bar{U})$, such that for all $\gamma' \in (\gamma, lh(\bar{U}))$, γ is not a weak repeat point. We sketch this when $lh(\bar{U}) < |2^\kappa|^M$. Fix some function f on κ such that for each $\alpha < \kappa$, $f(\alpha)$ outputs a well ordering of 2^α of length $|2^\alpha|$. For any $\gamma < lh(\bar{U})$, let $A \subset \kappa$ be such that $j(f)(\kappa)(\gamma) = A$. Consider the set $B_\gamma = \{\bar{w} \in V_\kappa \cap \mathcal{MS} : f(\kappa(\bar{w}))(lh(\bar{w})) = A \cap \kappa(\bar{w})\}$. It is easy to see that $B_\gamma \in U(\gamma) - \bigcup_{\gamma' < \gamma} U(\gamma')$.*

Definition 2.9. For any generic $G \subset R_{\bar{U}}$ over V , let

$$\mathcal{MS}_G = \{\bar{w} \in \mathcal{MS} \mid \exists p \in G, p = \langle d_i : i \leq k \rangle, \bar{w} = \bar{\mu}(d_i) \text{ for some } i < k\}.$$

$$C_G = \{\kappa(\bar{w}) \mid \bar{w} \in \mathcal{MS}_G\}$$

We say $A \subset \kappa$ is *generated by a set in V* if there exists $B \subset \mathcal{MS}$ such that $A = O(B \cap \mathcal{MS}_G)$.

The following theorem asserts that any club in the Radin extensions $V[G]$, $G \subseteq R_{\bar{U}}$, where the regularity of κ is preserved, contains a club generated by a \bar{U} -tail-measure-one set in V .

Theorem 2.10 ([BN19]). *If \bar{U} satisfies $cf(lh(\bar{U})) \geq \kappa^+$, where $\kappa = \kappa(\bar{U})$, then given $p_0^\frown(\bar{U}, A) = p \Vdash \dot{\tau}$ is a club subset of κ , there exists a measure one set $A' \subset A$ and a \bar{U} -tail-measure-one set Γ such that $p_0^\frown(\bar{U}, A')$ forces $O(\Gamma \cap G) \subset \dot{\tau}$.*

The converse is also true:

Theorem 2.11 ([BN19]). *Suppose \bar{U} is a measure sequence not satisfying RP and $B \subset \mathcal{MS}$ is a \bar{U} -tail-measure-one set. Then in $V[G]$ where $G \subset R_{\bar{U}}$ is generic over V , $O(B \cap \mathcal{MS}_G)$ contains a club subset of κ .*

The following result shows that Theorem 2.10 about clubs in Radin generic extensions does not extend to stationary sets.

Proposition 2.12. *Suppose $lh(\bar{U}) \leq |2^\kappa|^M$ and $cf(lh(\bar{U})) \geq \kappa^+$, then in $V^{R_{\bar{U}}}$, there exists a partition $c : \kappa \rightarrow \kappa$, such that for any unbounded $A \subset C_G$ generated by a set in V , it is the case that $c''O(A) = \kappa$. In particular, $O(B)$ is not c -homogeneous for any \bar{U} -non-null set B .*

Proof. Let $(x_l^\alpha : l < |2^\alpha|)$ be an injective enumeration of 2^α for $\alpha < \kappa$. Let $(x_l^\kappa : l < |2^\kappa|^M) = j(\langle (x_l^\alpha : l < |2^\alpha|) : \alpha < \kappa \rangle)(\kappa)$. For $\alpha \leq \kappa$ and $\tau < 2^\alpha$, define $A_\tau^\alpha = \{\bar{v} : x_{lh(\bar{v})}^{\kappa(\bar{v})} = x_\tau^\alpha \cap \kappa(\bar{v})\}$. Note that $A_{lh(\bar{w})}^{\kappa(\bar{w})} \notin \bigcup \bar{w}$, for \bar{U} -measure-one many \bar{w} .

Define $A_{\geq lh(\bar{w})}^{\kappa(\bar{w})}$ as follows: $\bar{v} \in A_{\geq lh(\bar{w})}^{\kappa(\bar{w})}$ iff $\bar{v} \in A_{lh(\bar{w})}^{\kappa(\bar{w})}$ or some initial segment of \bar{v} is in $A_{lh(\bar{w})}^{\kappa(\bar{w})}$. In $V[G]$ where $G \subset R_{\bar{U}}$ is generic over V , define a mapping $c' : \mathcal{MS}_G \rightarrow \mathcal{MS}_G$ such that \bar{w} is mapped by c' to the maximal element in $G \cap A_{\geq lh(\bar{w})}^{\kappa(\bar{w})}$ if it exists.

We show, using a density argument, that c' has the following property: for any $A \in U(\gamma)$ and $B \in U(\tau)$ where $\gamma \leq \tau$, $c'[A \cap \mathcal{MS}_G] \cap B \neq \emptyset$.

Given $p = p_0 \frown (\bar{U}, E) \in R_{\bar{U}}$, we find \bar{w} from $(E \cap B \cap A_{\geq \gamma}^{\kappa} - V_{\kappa(\max(p_0))+1})$ and extend p to $p \frown \bar{w}$. The reason why such \bar{w} exists is that $A_{\geq \gamma}^{\kappa} \in \bigcap_{\gamma \leq i} U(i)$. Then we find some $\bar{u} \in A \cap E - V_{\kappa(\bar{w})+1}$ such that $A_{\geq \gamma}^{\kappa} \cap V_{\bar{u}} = A_{\geq lh(\bar{u})}^{\kappa(\bar{u})}$, and then we extend to $p_0 \frown (\bar{w}, E \cap V_{\kappa(\bar{w})}) \frown (\bar{u}, E - A_{\geq lh(\bar{u})}^{\kappa(\bar{u})}) \frown (\bar{U}, E)$, which forces $c'(\bar{u}) = \bar{w}$.

Finally, in V , if we let $* : \mathcal{MS} \rightarrow \kappa$ be such that for any $\alpha \in \kappa$, $(*)^{-1}(\alpha)$ is \bar{U} -positive, then in $V[G]$, $c =_{def} * \circ c'$ is as desired, namely, for any $A \subset \mathcal{MS}$ that is \bar{U} -non-null, $c[A \cap \mathcal{MS}_G] = \kappa$. \square

The next lemma is a well-known fact that concerns getting nice representations for certain sets in the Radin forcing extension.

Lemma 2.13. *Suppose $p \in R_{\bar{U}}$ and a sequence of names $\langle \dot{x}_\alpha \subset \alpha : \alpha < \kappa \rangle$ are given. Then there exists an extension $q \leq p$ with $q_0 = p_0$ and a function $f \in V$, such that for any $\bar{w} \in A^q$ with $lh(\bar{w}) > 0$, $f(\bar{w})$ is a $R_{\bar{w}}$ -name for a subset of $\kappa(\bar{w})$ and $q \frown \bar{w} \Vdash f(\bar{w}) = \dot{x}_{\kappa(\bar{w})}$.*

Proof. For each $\bar{w} \in A^p$ with $lh(\bar{w}) > 0$, we can find a \bar{U} -measure-one $A_{\bar{w}} \subset A^p$ and a $R_{\bar{w}}$ -name $f(\bar{w})$ such that $p_0 \frown (\bar{w}, A^p \cap V_{\kappa(\bar{w})}) \frown (\bar{U}, A_{\bar{w}})$ forces $f(\bar{w}) = \dot{x}_{\kappa(\bar{w})}$. To see this, consider

$$D =_{def} \{t \in R_{\bar{w}} : t \leq_{R_{\bar{w}}} p_0 \frown (\bar{w}, A^p \cap V_{\kappa(\bar{w})}),$$

$$\exists B_t \in \bigcap \bar{U}, \exists R_{\bar{w}}\text{-name } \dot{s}_t, t \frown (\bar{U}, B_t) \Vdash \dot{s}_t = \dot{x}_{\kappa(\bar{w})}\}.$$

Observe that D is a dense subset of $R_{\bar{w}}$ below $p_0 \frown (\bar{w}, A^p \cap V_{\kappa(\bar{w})})$: given any $s \in R_{\bar{w}}$ extending $p_0 \frown (\bar{w}, A^p \cap V_{\kappa(\bar{w})})$, since $R_{\bar{U}}/p \frown \bar{w} \simeq R_{\bar{w}}/p_0 \frown (\bar{w}, A^p \cap V_{\kappa(\bar{w})}) \times R_{\bar{U}}/(\bar{U}, A^p - V_{\kappa(\bar{w})+1})$ and $(R_{\bar{U}}/(\bar{U}, A^p - V_{\kappa(\bar{w})+1}), \leq^*)$ is $(2^{|R_{\bar{w}}|})^+$ -closed, we can find $t \leq_{R_{\bar{w}}} s$ and an $R_{\bar{w}}$ -name \dot{s}_t as well as some $B_t \in \bigcap \bar{U}$, such that $(\bar{U}, B_t) \Vdash_{R_{\bar{U}}} t \Vdash_{R_{\bar{w}}} \dot{x}_{\kappa(\bar{w})} = \dot{s}_t$. This shows that D is dense in $R_{\bar{w}}$.

Let $D' \subset D$ be a maximal antichain below $p_0 \frown (\bar{w}, A^p \cap V_{\kappa(\bar{w})})$. Then we can cook up a $R_{\bar{w}}$ -name $f(\bar{w})$ for a subset of $\kappa(\bar{w})$ such that $t \Vdash f(\bar{w}) = \dot{s}_t$ for any $t \in D'$. Now it is immediate that $f(\bar{w})$ and $A_{\bar{w}} = \bigcap_{t \in D'} B_t$ satisfy the requirement.

Finally, we let $q \leq p$ be such that $q_0 = p_0$ and $A^q = \Delta_{\bar{w} \in A^p} A_{\bar{w}}$, which is the desired extension of p as sought. \square

Unless otherwise stated, all the measure sequences relevant to our discussion in the following are assume to have length > 0 . An immediate consequence of the previous lemma is:

Corollary 2.14. *Let $G \subset R_{\bar{U}}$ be generic over V . Then in $V[G]$, for any $X \subset \kappa$, there exists $f \in V$ such that for every $\bar{w} \in \text{dom}(f) \cap \mathcal{MS}$, $f(\bar{w})$ is an $R_{\bar{w}}$ -name for a subset of $\kappa(\bar{w})$ and there exists $\alpha < \kappa$, $(f(\bar{w}))^{G \upharpoonright R_{\bar{w}}} = X \cap \kappa(\bar{w})$ for each $\bar{w} \in \mathcal{MS}_G - V_\alpha$.*

We finish this section with stating the following theorem.

Theorem 2.15 ([CW]). *Let $G \subseteq R_{\bar{U}}$ be a V -generic filter. If κ is regular in $V[G]$, then in $V[G]$, for every subset $X \subseteq \kappa$, if $X \cap \alpha \in V$ for all $\alpha < \kappa$ then $X \in V$.*

For a published account of the proof, the reader can consult [BN19].

3 Mild large cardinals in Radin extensions

In this section, we revisit two weakenings of the Repeat Property about the measure sequence \bar{U} , as considered in [BN19]. In particular, each property corresponds to κ being a certain large cardinal in the generic extension by $R_{\bar{U}}$.

3.1 Almost ineffable cardinals

Definition 3.1. \bar{U} satisfies the *Local Repeat Property* (LRP) if for any measure function b , there exist $A \in \bigcap \bar{U}$ and $\gamma < \text{lh}(\bar{U})$ such that $j(b)(\bar{U} \upharpoonright \gamma) = A$.

Since \bar{U} does not contain a repeat point, it is easy to check that if \bar{U} satisfies LRP, then $\text{lh}(\bar{U})$ is a limit ordinal.

Remark 3.2. *One obtains an equivalent definition by replacing “... $\gamma < \text{lh}(\bar{U})$ such that...” with “...cofinally many $\gamma < \text{lh}(\bar{U})$ such that...” in Definition 3.1.*

In [BN19], it was shown that if $\text{lh}(\bar{U}) < 2^\kappa$, then \bar{U} does not satisfy LRP, thus making LRP incompatible with Woodin’s argument for the failure of $\diamond(\kappa)$ in $R_{\bar{U}}$ -generic extension (see Theorem 4.5). [BN19] asks if LRP is equivalent to a natural large cardinal property of κ in $R_{\bar{U}}$ generic extensions. We show next that this is the case. This shows that it is not an accident that LRP is incompatible with the argument for the failure of $\diamond(\kappa)$.

Definition 3.3 ([JK69]). Fix a regular cardinal κ .

1. κ is *almost ineffable* if for any sequence $\langle t_\alpha \subset \alpha : \alpha < \kappa \rangle$, there is some $t \subset \kappa$ such that $\{\alpha : t \cap \alpha = t_\alpha\}$ is unbounded in κ .
2. κ is *subtle* if for any sequence $\langle t_\alpha \subset \alpha : \alpha < \kappa \rangle$ and any club $C \subset \kappa$, there are $\alpha < \beta \in C$ such that $t_\alpha \sqsubset t_\beta$.

Remark 3.4. *It is easy to check that κ is almost ineffable iff for any sequence $\langle t_\alpha \subset \alpha : \alpha < \kappa \rangle$ and any club $C \subset \kappa$, there is some $t \subset \kappa$ such that $\{\alpha \in C : t \cap \alpha = t_\alpha\}$ is unbounded in κ . As a result, if κ is almost ineffable, then κ is subtle, which in turn implies $\diamond(\kappa)$ by [JK69].*

Proposition 3.5. $V^{R\bar{U}} \models \kappa$ is almost ineffable iff \bar{U} satisfies LRP.

Proposition 3.5 follows from the following two lemmas.

Lemma 3.6. If \bar{U} satisfies the LRP, then κ is almost ineffable in $V^{R\bar{U}}$

Proof. Let $p' \in R_{\bar{U}}$ and a name $\langle \dot{s}_\alpha \subset \alpha : \alpha < \kappa \rangle$ be given. By Lemma 2.13, we can get $p = p_0 \widehat{\ } (\bar{U}, A) \leq^* p'$ and $\langle \dot{t}_{\bar{w}} : \bar{w} \in \mathcal{MS} \rangle$ such that $\dot{t}_{\bar{w}}$ is an $R_{\bar{w}}$ -name and for any $\bar{w} \in A$ with $lh(\bar{w}) > 0$, $p \widehat{\ } \bar{w} \Vdash \dot{t}_{\bar{w}} = \dot{s}_{\kappa(\bar{w})}$.

For each $\bar{w} \in A$, apply Lemma 2.13 to $p_0 \widehat{\ } (\bar{w}, A \cap V_{\kappa(\bar{w})})$ and $R_{\bar{w}}$. As a result, we can find $b(\bar{w}) \in \bigcap \bar{w}$, $b(\bar{w}) \subset A$, such that for any $\bar{v} \in b(\bar{w})$ with $lh(\bar{v}) > 0$, it is the case that $p_0 \widehat{\ } (\bar{v}, b(\bar{w}) \cap V_{\kappa(\bar{v})}) \widehat{\ } (\bar{w}, b(\bar{w})) \Vdash_{R_{\bar{w}}} \dot{t}_{\bar{w}} \cap \kappa(\bar{v}) = h_{\bar{w}}(\bar{v})$, where $h_{\bar{w}}$ is a function defined on $\mathcal{MS} \cap V_{\kappa(\bar{w})}$ with $h_{\bar{w}}(\bar{v})$ being a $R_{\bar{v}}$ -name for each $\bar{v} \in \text{dom}(h_{\bar{w}})$ with $lh(\bar{v}) > 0$. Let \bar{h} denote $\langle h_{\bar{w}} : \bar{w} \in \mathcal{MS} \cap V_{\kappa} \rangle$.

By the hypothesis that \bar{U} satisfies LRP, we can find $\tau < lh(\bar{U})$ such that $B = j(b)(\bar{U} \upharpoonright \tau) \in \bigcap \bar{U}$. Note $B \subset A$. We may assume $\tau \geq 2$ by Remark 3.2. Let $h = j(\bar{h})(\bar{U} \upharpoonright \tau)$, then h is a function from $\mathcal{MS} \cap V_{\kappa}$ to V_{κ} , taking \bar{v} to $h(\bar{v})$, which is an $R_{\bar{v}}$ -name for a subset of $\kappa(\bar{v})$.

Consider the set $C = \{ \bar{w} \in B : lh(\bar{w}) > 1, B \cap V_{\kappa(\bar{w})} \subset b(\bar{w}), h \upharpoonright V_{\kappa(\bar{w})} = h_{\bar{w}} \} \in U(\tau)$. Let G be a generic filter for $R_{\bar{U}}$ over V containing $p_0 \widehat{\ } (\bar{U}, B)$ and in $V[G]$, we define a sequence of bounded subsets of κ as follows: for each $\bar{w} \in C \cap \mathcal{MS}_G$, let $d_{\bar{w}}$ be the union of the interpretation $(h(\bar{v}))^G$ for all $\bar{v} \in \mathcal{MS}_G \cap V_{\kappa(\bar{w})}$.

Claim 3.7. $d_{\bar{w}}$ is well-defined, and it is equal to $t_{\bar{w}} =_{def} (\dot{t}_{\bar{w}})^G$.

Proof of the Claim. For any $\bar{v} < \bar{v}'$ both in $\mathcal{MS}_G \cap V_{\kappa(\bar{w})} \cap B - V_{rank(p_0)+1}$, we note that

$$(h_{\bar{w}}(\bar{v}))^G, (h_{\bar{w}}(\bar{v}'))^G \sqsubset t_{\bar{w}}.$$

This is because the statement is forced by $p_0 \widehat{\ } (\bar{v}, b(\bar{w}) \cap V_{\kappa(\bar{v})}) \widehat{\ } (\bar{w}, b(\bar{w}))$ (respectively, by $p_0 \widehat{\ } (\bar{v}', b(\bar{w}) \cap V_{\kappa(\bar{v}')}) \widehat{\ } (\bar{w}, b(\bar{w}))$), extended by $p_0 \widehat{\ } (\bar{v}, B \cap V_{\kappa(\bar{v})}) \widehat{\ } (\bar{w}, B \cap V_{\kappa(\bar{w})})$ by the fact that $\bar{w} \in C$. \square

Claim 3.8. For $\bar{w} < \bar{w}' \in \mathcal{MS}_G \cap C$, it is the case that $d_{\bar{w}} \sqsubset d_{\bar{w}'}$.

Proof of the Claim. Reason as above to check that for any $\mathcal{MS}_G \cap V_{\kappa(\bar{w})} \cap B - V_{rank(p_0)+1}$, $(h(\bar{v}))^G = (h_{\bar{w}}(\bar{v}))^G = (h_{\bar{w}'}(\bar{v}))^G$ is an initial segment of both $d_{\bar{w}}$ and $d_{\bar{w}'}$. \square

Let $X = \bigcup_{\bar{w} \in \mathcal{MS}_G \cap C} d_{\bar{w}}$. Then we are done. Namely, in $V[G]$, $\{ \alpha < \kappa : X \cap \alpha = (\dot{s}_\alpha)^G \}$ is unbounded in κ . \square

Lemma 3.9. If $V^{R\bar{U}}$, κ is almost ineffable, then \bar{U} satisfies LRP.

Proof. Fix a measure function b . Let $G \subset R_{\bar{U}}$ be generic over V . In $V[G]$, for each $\bar{w} \in G$, let $\gamma_{\bar{w}} < \kappa(\bar{w})$ be some ordinal such that $\mathcal{MS}_G - V_{\gamma_{\bar{w}}} \subset b(\bar{w})$. Consider the sequence $\bar{X} = \langle X_{\bar{w}} =_{def} \{\gamma_{\bar{w}}\} \cup (b(\bar{w}) - V_{\gamma_{\bar{w}+1}) : \bar{w} \in \mathcal{MS}_G \rangle$. Note that $\bar{X} \subset V$. As κ is almost ineffable, by Remark 3.4, we can find X such that $\{\bar{w} \in \mathcal{MS}_G : X \cap V_{\kappa(\bar{w})} = X_{\bar{w}}\}$ has size κ . In particular, X is fresh. By Theorem 2.15, we know that $X \in V$. As a result, there is some γ , such that there are unboundedly many \bar{w} with $\gamma_{\bar{w}} = \gamma$ and $X \cap V_{\kappa(\bar{w})} = \{\gamma_{\bar{w}}\} \cup (b(\bar{w}) - V_{\gamma_{\bar{w}+1})$. As a result, X contains a tail of \mathcal{MS}_G , which in turn implies $X \in \bigcap \bar{U}$.

Finally, we need to check that there is some $\tau < lh(\bar{U})$, such that $j(b)(\bar{U} \upharpoonright \tau) =^* X$. But this immediately follows from the fact that in $V[G]$, $\{\bar{w} \in \mathcal{MS}_G : X \cap V_{\kappa(\bar{w})} =^* b(\bar{w})\}$ is unbounded in κ . \square

3.2 Weakly compact cardinals

Definition 3.10 ([BN19], Definition 21). We say \bar{U} satisfies the *Weak Repeat Property* (WRP) if every measure function b has a *repeat filter* W with respect to \bar{U} in the following sense:

1. W is a κ -complete filter extending the co-bounded filter on \mathcal{MS} ,
2. any $X \in W$ is \bar{U} -non-null,
3. W measures b , namely for each \bar{u} , if we let $X_{b,\bar{u}} =_{def} \{\bar{v} : \bar{u} \in b(\bar{v})\}$, then either $X_{b,\bar{u}} \in W$ or $X_{b,\bar{u}}^c \in W$ and
4. $[b]_W =_{def} \{\bar{u} : X_{b,\bar{u}} \in W\} \in \bigcap \bar{U}$.

In [BN19], it was shown that κ is weakly compact in $V^{R_{\bar{U}}}$ iff \bar{U} satisfies WRP. We supply more equivalent characterizations of weak compactness in Radin extensions, which will be useful later on.

Proposition 3.11. *The following are equivalent:*

1. \bar{U} satisfies WRP.
2. $V^{R_{\bar{U}}} \models \kappa$ is weakly compact.
3. $V^{R_{\bar{U}}} \models \kappa$ is inaccessible and the V -tree property holds. Namely, any κ -tree T satisfying that $T \subset V$ admits a cofinal branch.
4. for any measure function b , there exists $A \in \bigcap \bar{U}$ such that for any $\alpha < \kappa$, there exists some $\beta < lh(\bar{U})$, with $A \cap V_\alpha \subset j(b)(\bar{U} \upharpoonright \beta)$.

Proof. The equivalence between 1 and 2 is proved in [BN19] and it is immediate that 2 implies 3.

- 3 implies 4: fix a measure function b . Let $G \subset R_{\bar{U}}$ be generic over V . In $V[G]$, the function $\bar{w} \in \mathcal{MS}_G \mapsto \gamma_{\bar{w}}$, where $\gamma_{\bar{w}}$ is the least $\gamma < \kappa(\bar{w})$ such that $\mathcal{MS}_G \cap V_{\kappa(\bar{w})} - V_\gamma \subset b(\bar{w})$, is regressive. Therefore, there exists some γ , such that $B = \{\bar{w} \in G : \gamma_{\bar{w}} = \gamma\}$ is stationary. More precisely,

$O(B \cap \mathcal{MS}_G)$ is stationary in κ . Define a κ -tree T such that $t \in T$ iff there exists some $\bar{w} \in B$ such that $t \sqsubseteq b(\bar{w})$. The order of T is end extension. Notice that $T \subset V$. Apply the hypothesis, we can get a branch d through T . By Theorem 2.15, $d \in V$.

Observe that $d \in \bigcap \bar{U}$. To see this, it suffices to see d contains $G - V_\gamma$. For any $\bar{u} \in G - V_\gamma$, $d \cap V_{\kappa(\bar{u})+lh(\bar{u})+1} \sqsubset b(\bar{w})$ for some $\bar{w} \in B$. By the choice of γ , we know that $\bar{u} \in b(\bar{w})$. Hence $\bar{u} \in d$.

Finally, it remains to show that for any $\alpha < \kappa$, we need to show that $E =_{def} \{\bar{w} : d \cap V_\alpha \subset b(\bar{w})\}$ is \bar{U} -non-null. But this follows immediately from the fact that $O(E \cap \mathcal{MS}_G)$ is unbounded in κ .

- 4 implies 1: given a measure function b , apply 4 to get $A \in \bigcap \bar{U}$ with the property. We first show a strengthening of the conclusion at 4 is possible.

Claim 3.12. We can find an A as in the conclusion of 4 that satisfies for any $\alpha < \kappa$, there exists some $\beta < lh(\bar{U})$, with $A \cap V_\alpha \sqsubset j(b)(\bar{U} \upharpoonright \beta)$.

Proof of the Claim. Note the difference is “ \subset ” is replaced with “ \sqsubset ”. Let A' be given by the conclusion of 4. For each $\alpha < \kappa$, let $\beta_\alpha < lh(\bar{U})$ be the least witnessing ordinal that $A' \cap V_\alpha \subset j(b)(\bar{U} \upharpoonright \beta_\alpha)$. Consider the tree on V_κ defined by $T = \{j(b)(\bar{U} \upharpoonright \beta_\alpha) \upharpoonright V_\gamma : \gamma \leq \alpha < \kappa\}$, ordered by end extension. By the weak compactness of κ in V , there exists a branch A through the tree T . Clearly, for each $\alpha < \kappa$, there is some $\beta < lh(\bar{U})$, such that $A \cap V_\alpha \sqsubset j(b)(\bar{U} \upharpoonright \beta)$ by the definition of the tree. Note also that $A \in \bigcap \bar{U}$, as evidenced by the fact that $A' \subset A$. To see the latter, given $\alpha < \kappa$, we can find some $\alpha' \geq \alpha$ such that $A \cap V_\alpha \sqsubset j(b)(\bar{U} \upharpoonright \beta_{\alpha'})$. In particular, $A \cap V_\alpha \supset A' \cap V_\alpha$. \square

Define $W' \subset \mathcal{MS}$ such that $B \in W'$ iff there is some $\bar{u} \in A$ with $B = X_{b, \bar{u}}$ or there is some $\bar{u} \notin A$ with $B = X_{b, \bar{u}}^c$. Let $\mathcal{F} \subset \mathcal{MS}$ be the co-bounded filter. Let W be the upward closure of $\{\bigcap_{B \in u} B : u \in [W' \cup \mathcal{F}]^{< \kappa}\}$. To check W as defined is as desired, we verify that $W \subset \bigcup \bar{U}$ is a κ -complete filter. It suffices to show that for any $\langle B_i : i < \mu \rangle \subset W'$ where $\mu < \kappa$, it is the case that $\bigcap_{i < \mu} B_i$ is \bar{U} -non-null, since $\mathcal{F} \subset \bigcap \bar{U}$. Depend on how B_i gets into W' , we can divide them into two groups with a function $g : \mu \rightarrow 2$ and elements $\langle \bar{u}_i : i < \mu \rangle$ such that $g(i) = 0$ iff $B_i = X_{b, \bar{u}_i}$ where $\bar{u}_i \in A$ and $g(i) = 1$ iff $B_i = X_{b, \bar{u}_i}^c$ where $\bar{u}_i \notin A$. Let $\alpha > \sup_{i < \mu} \kappa(\bar{u}_i)$. Then Claim 3.12 implies that $\{\bar{v} : A \cap V_\alpha \sqsubset b(\bar{v})\}$, as a subset of $\bigcap_{i < \mu} B_i$, is \bar{U} -non-null. \square

Remark 3.13. In item 4 above, we get an equivalent statement by replacing “...there exists some $\beta < lh(\bar{U})$...” with “...there exist unboundedly many $\beta < lh(\bar{U})$...”.

Lemma 3.14. *If $|2^\kappa|^M$ does not divide $lh(\bar{U})$ as ordinals, then $V^{R\bar{U}} \models \kappa$ is not weakly compact.*

Proof. Assume first that $lh(\bar{U}) < |2^\kappa|^M$. We will indicate later how to deal with the general case. Define g such that $g(\alpha) = \langle x_l^\alpha : l < |2^\alpha| \rangle$ is an injective enumeration of subsets of α . Denote $j(g)(\kappa) = \langle x_l^\kappa : l < |2^\kappa|^M \rangle$. Let $A = x_{lh(\bar{U})}^\kappa$. In general for any $\beta < \kappa$ and any set $B \subset \beta$, denote $index_g(B)$ to be the least l such that $B = x_l^\beta$. Note for a fixed β and B as above, $index_g(B)$ only depends on $g(\beta)$.

Define a measure function b such that

$$b(\bar{w}) = \{\bar{u} \in V_{\kappa(\bar{w})} : x_{lh(\bar{w})}^{\kappa(\bar{w})} \cap \kappa(\bar{u}) = x_\beta^{\kappa(\bar{u})}, \text{ where } lh(\bar{u}) < \beta < index_g(A \cap \kappa(\bar{u}))\},$$

whenever the set is in $\bigcap \bar{w}$ and otherwise $b(\bar{w}) = V_{\kappa(\bar{w})}$.

Let us call the second case in the definition of b *vacuous*. Observe that $\{\bar{w} \in V_\kappa : b(\bar{w}) \text{ is not vacuously defined}\} \in \bigcap \bar{U}$. To see this, note that $j(b)(\bar{U} \upharpoonright \tau) = \{\bar{u} \in V_\kappa : x_\tau^\kappa \cap \kappa(\bar{u}) = x_\beta^{\kappa(\bar{u})}, \tau < \beta < index_{j(g)}(j(A) \cap \kappa(\bar{u})) = index_g(A \cap \kappa(\bar{u}))\}$. Denote the set as E , then we see that $E \in \bigcap \bar{U} \upharpoonright \tau$. Given $\gamma < \tau$, we know that $\bar{U} \upharpoonright \gamma \in j(E)$ iff $x_\tau^\kappa = x_\beta^\kappa$, where $\gamma < \beta < index_{j(g)}(A) = lh(\bar{U})$, but the latter is true as evidenced by $\beta = \tau$.

Suppose for the sake of contradiction that κ is weakly compact in $V^{R\bar{U}}$. Apply 4 in Proposition 3.11 to b to get $B \in \bigcap \bar{U}$ such that:

for any $\alpha < \kappa$, there is $\beta < lh(\bar{U})$ with $B \cap V_\alpha \subset j(b)(\bar{U} \upharpoonright \beta)$.

Apply j again to the statement above, we get (in M):

for any $\alpha < j(\kappa)$, there is $\beta < lh(j(\bar{U}))$ with $j(B) \cap V_\alpha \subset j(j(b))(j(\bar{U}) \upharpoonright \beta)$.

In particular, working in M , if we consider $\alpha = (2^\kappa)^+M < j(\kappa)$, then we can find some $\tau < j(lh(\bar{U}))$ such that $j(B) \cap (V_\alpha)^M \subset j(j(b))(j(\bar{U}) \upharpoonright \tau) =$

$$\{\bar{u} \in j(V_\kappa) : x_\tau^{j(\kappa)} \cap \kappa(\bar{u}) = x_\beta^{\kappa(\bar{u})}, lh(\bar{u}) < \beta < index_{j(j(g))}(j(j(A)) \cap \kappa(\bar{u}))\}. \quad (1)$$

For the sake of consistency, we use $\langle x_l^{j(\kappa)} : l < dom[j(j(g))(j(\kappa))] \rangle$ to denote the enumeration of $j(j(g))(j(\kappa))$. Since $lh(\bar{U}) < dom(j(g)(\kappa))$, the elementarity of j implies that $j(lh(\bar{U})) < dom(j(j(g))(j(\kappa)))$. Thus the sentence above makes sense.

We show first that there exists some $\gamma < lh(\bar{U})$ such that $x_\tau^{j(\kappa)} \cap \kappa = x_\gamma^\kappa$. To see this, fix some $\xi < lh(\bar{U})$. Since $\bar{U} \upharpoonright \xi \in j(B) \cap (V_\alpha)^M$, by (1), we know $x_\tau^{j(\kappa)} \cap \kappa = x_\gamma^\kappa$, for some $\gamma > \xi$ and $\gamma < index_{j(j(g))}(j(j(A)) \cap \kappa) = index_{j(j(g))}(A) = index_{j(j(g) \upharpoonright \kappa)}(A) = index_{j(g)}(A) = lh(\bar{U})$. The second last equation is because the computation of $index_{j(j(g))}(A)$ only depends on $j(j(g))(\kappa)$, which in turn is decided by $j(j(g) \upharpoonright \kappa)(\kappa) = j(g)(\kappa)$.

Finally, fix some $\xi' > \gamma$ and $\xi' < lh(\bar{U})$ such that $\bar{U} \upharpoonright \xi' \in j(B) \cap (V_\alpha)^M$. Reasoning as above, we can find $\gamma' > \xi' > \gamma$ with $\gamma' < lh(\bar{U})$ such that $x_\tau^{j(\kappa)} \cap \kappa = x_{\gamma'}^\kappa$. But $x_\gamma^\kappa \neq x_{\gamma'}^\kappa$ by the injectivity of the enumeration. This is a contradiction.

In general without assuming $lh(\bar{U}) < |2^\kappa|^M$, we modify as follows. First notice that $lh(\bar{U}) \leq j(\kappa)$ and $j(\kappa)$ is divisible by $|2^\kappa|^M$. Hence, we may assume $lh(\bar{U}) < j(\kappa)$. By Remark 2.8, there exists some $\mu < lh(\bar{U})$ so that $lh(\bar{U}) = \mu + \delta$ with $\delta < |2^\kappa|^M$, and $U(\mu)$ is novel. It follows that there is some $D \subseteq \mathcal{MS} \cap V_\kappa$ so that $D \in U(i)$ if and only if $i < \mu$. For each $\bar{w} \in \mathcal{MS} \cap V_\kappa$ let $i_D(\bar{w})$ be least value $i < lh(\bar{w})$ so that $D \cap V_{\kappa(\bar{w})} \not\subseteq \bar{w}(i)$ if such i exists, and leave $i_D(\bar{w})$ undefined otherwise. If $i_D(\bar{w})$ is defined then clearly $i_D(\bar{w}) < lh(\bar{w})$, and we set $\delta_D(\bar{w})$ be the unique δ for which $i_D(\bar{w}) + \delta = lh(\bar{w})$.

Define g as before and $A = x_\delta^\kappa$. Define b such that

$$b(\bar{w}) = (D \cap V_{\kappa(\bar{w})}) \cup \{\bar{u} \in V_{\kappa(\bar{w})} \cap D^c : x_{\delta_D(\bar{w})}^{\kappa(\bar{w})} = x_\beta^{\kappa(\bar{u})}\}$$

$$\text{where } \delta_D(\bar{u}) < \beta < \text{index}_g(A \cap \kappa(\bar{u})),$$

whenever the set is in $\bigcap \bar{w}$ and otherwise $b(\bar{w}) = V_{\kappa(\bar{w})}$.

Again observe that $\{\bar{w} \in V_\kappa : b(\bar{w}) \text{ is not vacuously defined}\} \in \bigcap \bar{U}$. Apply 4 in Proposition 3.11 to b to get $B \in \bigcap \bar{U}$ as before. If we let $\alpha = (lh(\bar{U}) + 2^\kappa)^+ + M$, then we can find some $\tau < j(lh(\bar{U}))$ such that $j(B) \cap (V_\alpha)^M \subset j(j(b))(j(\bar{U}) \upharpoonright \tau)$. The choice of α ensures that $\{\bar{U} \upharpoonright \nu : \nu < lh(\bar{U})\} \subset (V_\alpha)^M$. Fix some $\xi < lh(\bar{U})$ such that $\xi > \mu$. Since $\bar{U} \upharpoonright \xi \in j(B) \cap (V_\alpha)^M$, it follows that $\bar{U} \upharpoonright \xi \in j(j(b))(j(\bar{U}) \upharpoonright \tau)$. Since $j(j(D)) \cap j(V_\kappa) = j(j(D) \cap V_\kappa) = j(D)$, we know that $\bar{U} \upharpoonright \xi \notin j(j(D))$ since $\bar{U} \upharpoonright \xi \notin j(D)$ and $\bar{U} \upharpoonright \xi \in j(V_\kappa)$. As a result, by looking at the definition of b , we must have

$$x_{j(j(\delta_D))(j(\bar{U}) \upharpoonright \tau)}^{j(\kappa)} \cap \kappa = x_\gamma^\kappa, \text{ where } j(\delta_D)(\bar{U} \upharpoonright \xi) = \xi - \mu < \gamma < \delta.$$

Fix another $\xi' < lh(\bar{U})$ such that $\xi' - \mu > \gamma$. Repeating the argument above, we get some $\gamma' > \gamma$ such that $x_{j(j(\delta_D))(j(\bar{U}) \upharpoonright \tau)}^{j(\kappa)} \cap \kappa = x_{\gamma'}^\kappa$. This contradicts with the fact that $x_\gamma^\kappa \neq x_{\gamma'}^\kappa$. \square

Remark 3.15. *The proof in Lemma 3.14 actually produces a measure function b such that there is no \bar{U} -positive A , such that for any $\alpha < \kappa$, there is $\beta < lh(\bar{U})$, $A \cap V_\alpha \subset j(b)(\bar{U} \upharpoonright \beta)$.*

Remark 3.16. *Similar proof as in Lemma 3.14 shows κ is not weakly compact in $V^{R_{\bar{U}}}$ if there is some ν such that $lh(\bar{U}) = \nu + |2^\kappa|^M$.*

4 Guessing principles in Radin extensions

Definition 4.1. Let $V \subseteq W$ be two transitive models of set theory, with κ being a regular cardinal in W and $S \subset \kappa$ being a stationary subset. We say

- $V\text{-}\diamond(S)$ holds (in W) if there exists $\langle s_\alpha \subset \alpha : \alpha < \kappa \rangle$ such that for all $X \subset \kappa$ in V , it is the case that $\{\alpha \in S : X \cap \alpha = s_\alpha\}$ is stationary.

- $V\text{-}\diamond^-(S)$ holds (in W) if there exists $\langle \mathcal{S}_\alpha \in [\mathcal{P}(\alpha)]^{\leq |\alpha|} : \alpha < \kappa \rangle$ such that for all $X \subset \kappa$ in V , it is the case that $\{\alpha \in S : X \cap \alpha \in \mathcal{S}_\alpha\}$ is stationary.

Cummings and Magidor [Cum13] showed that V -diamonds are equivalent to diamonds at κ in Radin generic extensions of V preserving the regularity of κ .

Theorem 4.2 (Cummings-Magidor, [Cum13]). *Let $G \subset R_{\bar{U}}$ be generic over V and assume that κ is regular in $V[G]$. Then in $V[G]$, for any stationary $S \subset \kappa$, $V\text{-}\diamond^-(S)$ is equivalent to $\diamond^-(S)$.*

Observation 4.3. Suppose $cf(lh(\bar{U})) \geq \kappa^+$. Then $V^{R_{\bar{U}}} \models \diamond(\kappa)$ holds iff there exists a function $f \in V$ such that $\text{dom}(f) = \mathcal{MS} \cap V_\kappa$ and $f(\bar{w}) \in [2^{\kappa(\bar{w})}]^{\leq \kappa(\bar{w})}$, satisfying that for any $X \subset \kappa$, $\{\bar{w} : X \cap \kappa(\bar{w}) \in f(\bar{w})\}$ is \bar{U} -positive.

Proof. By Theorem 4.2 and a well-known result of Kunen, we know $V^{R_{\bar{U}}} \models \diamond(\kappa)$ iff $V^{R_{\bar{U}}} \models \diamond^-(\kappa)$ iff $V^{R_{\bar{U}}} \models V\text{-}\diamond^-(\kappa)$ iff $V^{R_{\bar{U}}} \models V\text{-}\diamond(\kappa)$.

- Suppose there exists some $p \in R_{\bar{U}}$ forcing that $\langle \dot{s}_\alpha : \alpha < \kappa \rangle$ is a $V\text{-}\diamond(\kappa)$ -sequence. By Lemma 2.13, we may find $q \leq p$ and g such that for any $\bar{w} \in A^q$, $g(\bar{w})$ is an $R_{\bar{w}}$ -name for a subset of $\kappa(\bar{w})$ in V and $q \restriction \bar{w} \Vdash g(\bar{w}) = \dot{s}_{\kappa(\bar{w})}$. Since $R_{\bar{w}}$ is $\kappa(\bar{w})^+$ -c.c., we can find $f(\bar{w}) \in [2^{\kappa(\bar{w})}]^{\leq \kappa(\bar{w})}$ such that $1 \Vdash_{R_{\bar{w}}} g(\bar{w}) \in f(\bar{w})$. Suppose $X \subset \kappa$, then we know that we can find some $r \leq q$ such that $r \Vdash \{\kappa(\bar{w}) : \bar{w} \in G, X \cap \kappa(\bar{w}) = g(\bar{w}) \in f(\bar{w})\}$ is stationary". This implies that $\{\bar{w} : X \cap \kappa(\bar{w}) \in f(\bar{w})\}$ is \bar{U} -positive by Theorem 2.11.
- For the other direction, suppose there is one such f , then it is easy to verify that if $G \subset R_{\bar{U}}$ is generic over V , then for $\{f(\bar{w}) : \bar{w} \in G\}$ is the desired $V\text{-}\diamond^-(\kappa)$ -sequence. □

The last observation allows us to reproduce the following result of Cummings and Magidor.

Theorem 4.4 (Cummings-Magidor, [Cum13]). *If $|2^\kappa|^M$ divides $lh(\bar{U})$, then $V^{R_{\bar{U}}} \models \diamond(\kappa)$.*

Proof of Theorem 1.1. Let \bar{U} be a measure sequence on κ constructed by $j : V \rightarrow M$ and suppose $V^{R_{\bar{U}}} \models \kappa$ is weakly compact. By Lemma 3.14, $|2^\kappa|^M$ divides $lh(\bar{U})$. Hence, Theorem 4.4 implies $\diamond(\kappa)$ holds in $V^{R_{\bar{U}}}$. □

The following is a well-known theorem of Woodin (see [Cum] or [BN19] for published proofs).

Theorem 4.5 (Woodin). *If $|2^\kappa|$ does not divide $lh(\bar{U})$ and $cf(lh(\bar{U})) \geq \kappa^+$, then $V^{R_{\bar{U}}} \models \neg \diamond(\kappa)$.*

Remark 4.6. *If $j : V \rightarrow M$ constructs \bar{U} , and if $\bar{U} \restriction lh(\bar{U}) \in M$, then $V^{R_{\bar{U}}} \models \neg \diamond(\kappa)$ whenever $|2^\kappa|^M$ does not divide $lh(\bar{U})$. This is done by apply Woodin's theorem in M .*

Under certain circumstances, we can get stronger guessing principles. Recall the following characterization of stationary sets in the Radin extensions.

Theorem 4.7 ([BN19]). *If \dot{S} is an $R_{\bar{U}}$ -name for a stationary subset of κ and $p \in R_{\bar{U}}$, then there exists an extension $e = e_0 \frown (\bar{U}, B) \leq p$ and a measure function b such that*

1. $Z_{e_0} =_{def} \{\bar{u} : \exists A \in \bigcap \bar{U}, e_0 \frown (\bar{u}, b(\bar{u})) \frown (\bar{U}, A) \Vdash \kappa(\bar{u}) \in \dot{S}\}$ is \bar{U} -positive,
2. for any $\vec{\eta} \in \mathcal{MS}^{<\omega}$ and $\vec{\eta} \subset B$, $Z_e \restriction \vec{\eta} =_{def} \{\bar{u} \in Z_{e_0} : \vec{\eta} \ll (\bar{u}, b(\bar{u}))\}$ is \bar{U} -positive.

Proposition 4.8. *If \bar{U} is a measure sequence constructed by $j : V \rightarrow M$ and $M \models cf(lh(\bar{U})) \geq 2^\kappa$, then in $V^{R_{\bar{U}}}$, $\diamond(S)$ holds for all stationary $S \subset \kappa$.*

Proof. Let \dot{S} be a name for a stationary subset of κ and $p = p_0 \frown (\bar{U}, A)$ be a condition forcing this, which furthermore satisfies the conclusion of Theorem 4.7. We will construct a $V\text{-}\diamond^-(S)$ -sequence in $V^{R_{\bar{U}}}$. By Lemma 4.2 and a theorem of Kunen, we know $\diamond(S)$ holds in $V^{R_{\bar{U}}}$.

For each α , we fix $\langle x_l^\alpha : l < |2^\alpha| \rangle$ an enumeration of subsets of α with cofinal repetitions. For each $A \subset \mathcal{MS}$, we define $f_A(\bar{w}) = otp(\{\xi < lh(\bar{w}) : \bar{w} \restriction \xi \in A\})$.

In $V[G]$, we define a $V\text{-}\diamond^-(S)$ -sequence as follows: if $\bar{w} \in \mathcal{MS}_G$ and $\kappa(\bar{w}) \in S$, then define \mathcal{S}_α to consist of $x_l^{\kappa(\bar{w})}$ for those l such that there is some $\vec{\eta} \in A^{<\omega} \cap V_{\kappa(\bar{w})}$ such that $\bar{w} \in Z_{p_0} \restriction \vec{\eta}$ and $l = (f_{Z_{p_0} \restriction \vec{\eta}}(\bar{w}) \bmod 2^{\kappa(\bar{w})}) < 2^{\kappa(\bar{w})}$. Note that by the definition for each $\bar{w} \in \mathcal{MS}_G$ with $\alpha = \kappa(\bar{w})$, it is true that $|\mathcal{S}_\alpha| \leq |\alpha|$.

Let's check that this is as desired. Suppose not, then there exists an extension q of p and $X \subset \kappa$ and a tail measure one Γ as witnessed by $\nu < lh(\bar{U})$ such that q forces that X is not guessed at any point at $\dot{S} \cap O(\mathcal{MS}_G \cap \Gamma)$.

Write q as $p'_0 \frown (\vec{\eta}, \vec{A}_{\vec{\eta}}) \frown (\bar{U}, B')$, where there is some \bar{v} with $p_0, p'_0 \in R_{\bar{v}}$ and $p'_0 \leq_{R_{\bar{v}}} p_0$. Here $\vec{\eta}$ is an increasing sequence of measure sequence and $\vec{A}_{\vec{\eta}}$ is a sequence of measure one sets with respect to the measure sequences on $\vec{\eta}$.

By the choice of p , we know that $Z_{p_0} \restriction \vec{\eta}$ is \bar{U} -positive. Let $l < 2^\kappa$ be such that $x_l^\kappa = X$ and $j(f_{Z_{p_0} \restriction \vec{\eta}})(\bar{U} \restriction \xi) = l \pmod{2^\kappa}$ for some $\bar{U} \restriction \xi \in j(\Gamma \cap Z_{p_0} \restriction \vec{\eta})$. By elementarity, there is some \bar{u} reflecting $\bar{U} \restriction \xi$, namely, there is some $\bar{u} \in \Gamma \cap Z_{p_0} \restriction \vec{\eta}$ such that if $l = f_{Z_{p_0} \restriction \vec{\eta}}(\bar{u}) \pmod{2^{\kappa(\bar{u})}}$, then $x_l^{\kappa(\bar{u})} = X \cap \kappa(\bar{u})$. Since $\vec{\eta} \ll (\bar{u}, b(\bar{u}))$, we know that $p'_0 \frown (\vec{\eta}, \vec{A}_{\vec{\eta}}) \frown (\bar{u}, b(\bar{u})) \frown (\bar{U}, B')$ is compatible with $p_0 \frown (\bar{u}, b(\bar{u})) \frown (\bar{U}, A)$. Hence a common extension will force $\kappa(\bar{u}) \in \dot{S}$, as well as $\kappa(\bar{u}) \in \Gamma$ and $X \cap \kappa(\bar{u}) \in \dot{S}_\alpha$, contradicting with our assumption. \square

Remark 4.9. *Proposition 4.8 is optimal in the following sense: if $M \models cf(lh(\bar{U})) < 2^\kappa$ and $\bar{U} \in M$, then in $V^{R_{\bar{U}}}$ there exists a stationary set S such that $\diamond(S)$ fails. It suffices to show there exists a \bar{U} -positive set B such that $M \models \{\tau : B \in U(\tau)\}$ has size $< 2^\kappa$. Then we can finish with the argument of Theorem 4.5.*

Recall $j : V \rightarrow M$ constructs \bar{U} . By a theorem by Cummings-Woodin [CW], we may assume that $M = \{j(f)(\bar{U} \upharpoonright \xi) : \xi \text{ is novel}\}$. The point is that any measure sequence is equivalent, in the sense of the Radin forcing defined from it, to another one constructed by the embedding formed by taking the limit ultrapower along the novel points on the measure sequence.

By the simplification, we can find f and a novel $\xi < lh(\bar{U})$ such that $j(f)(\bar{U} \upharpoonright \xi) = E$, which is cofinal in $lh(\bar{U})$ and of size $< 2^\kappa$, as computed in M . Let $A \in U(\xi) - \bigcup \bar{U}(< \xi)$. Consider $B = \{\bar{u} : \text{if } \xi \text{ is the least such that } \bar{u} \upharpoonright \xi \in A, lh(\bar{u}) \in f(\bar{u} \upharpoonright \xi)\}$.

We claim for $\tau > \xi$, $B \in U(\tau)$ iff $\tau \in E$. If $\tau > \xi$, and $\bar{U} \upharpoonright \tau \in j(B)$, then by the definition, ξ is the least such that $\bar{U} \upharpoonright \xi \in j(A)$, then τ has to be in $j(f)(\bar{U} \upharpoonright \xi) = E$. Conversely, if $\bar{U} \upharpoonright \tau$ is not in $j(B)$, then it must be that $\tau \notin E$ by the same reasoning.

Definition 4.10. We say the *Strong Club Guessing* (SCG) holds on $S \subset \kappa$ if there exists a sequence $\langle t_\alpha \subset \alpha : \alpha \in S \rangle$ for any club $C \subset \kappa$, there exists another club D , such that for any $\alpha \in D \cap S$, t_α is unbounded in α and $t_\alpha \subset^* C$.

By taking closure, we could also assume $t_\alpha \subset \alpha$ is closed for each $\alpha < \kappa$.

Proposition 4.11. In $V^{R\bar{v}}$ where κ is regular, SCG holds on the set of singular cardinals below κ .

Lemma 4.12. Let \bar{U} be a measure sequence without repeat points and $cf(lh(\bar{U})) \leq \kappa$. Then in $V[G]$, there exists a cofinal subset c of κ such that for any tail measure one set $B \in V$, $c \subset^* O(B \cap \mathcal{MS}_G)$.

Proof. First assume that $cf(lh(\bar{U})) = \lambda < \kappa$. Since we assume \bar{U} does not have a repeat point, we can fix $\langle \nu_i : i < \lambda \rangle$ cofinal in $lh(\bar{U})$ such that for each $i < \lambda$, there is some $A_{\nu_i} \in U(\nu_i) - \bigcup \bar{U}(< \nu_i)$. In $V[G]$, define the following sequence recursively: $c = \langle c_i : i < \lambda \rangle$, where for each $i < \lambda$, c_i is the least element in $G \cap A_{\nu_i}$ above $\sup_{j < i} c_j$. It is well known that in $V[G]$, $cf(\kappa) = \lambda$ (see [Git10]). Hence c is well-defined. It is also easy to see that this sequence is cofinal in κ . Let us verify that for any tail measure one set B , it is the case that $c \subset^* O(B \cap \mathcal{MS}_G)$. For this, we use the density argument. Fix $p = p^\frown(\bar{U}, A) \in R_{\bar{U}}$ and B whose tail-measure-one-ness is witnessed by $\gamma < lh(\bar{U})$. Consider $A' = A \cap (B \cup \bigcap_{i < \lambda, \nu_i > \gamma} A_{\nu_i}^c)$. Then it is easy to verify that $p_0 \frown(\bar{U}, A')$ forces \dot{c} is almost contained in $O(B \cap \mathcal{MS}_G)$.

Now suppose $cf(lh(\bar{U})) = \kappa$. Let $\langle \nu_i : i < \kappa \rangle$ be cofinal in $lh(\bar{U})$ such that for each $i < \kappa$, there is some $A_i \subset V_\kappa \cap \mathcal{MS}$ such that $A_i \in U(\xi)$ iff $\xi > \nu_i$. By [Git10], we know in $V[G]$, $cf(\kappa) = \omega$, as witnessed by some $\langle \kappa_n : n \in \omega \rangle$. Recursively define κ'_n such that κ'_n the least element $\geq \kappa_n$ such that there is some $\bar{w} \in \mathcal{MS}_G \cap \bigcap_{i < \kappa, \nu_i < \kappa_n} A_i$ with $\kappa'_n = \kappa(\bar{w})$. Hence, the sequence $\langle \kappa'_n : n \in \omega \rangle$ has the property that it is almost contained in $O(\mathcal{MS}_G \cap A_i)$ for any $i < \kappa$. Suppose B is any other tail measure one set, then there exists some large enough $i < \kappa$ such that $A_i \cap B^c$ is \bar{U} -null. As a result, in $V[G]$, a tail of G avoids $A_i \cap B^c$. Therefore, $A_i \cap \mathcal{MS}_G \subset^* B \cap \mathcal{MS}_G$. Therefore, $\langle \kappa'_n : n \in \omega \rangle$ is almost contained in $O(B \cap \mathcal{MS}_G)$. \square

Proof of Proposition 4.11. Let $G \subset R_{\bar{U}}$ be generic over V . In $V[G]$, if $\bar{w} \in G$ such that $cf(\kappa(\bar{w})) < \kappa(\bar{w})$, then we let $c_{\kappa(\bar{w})}$ be the set as given by Lemma 4.12 applied to $R_{\bar{w}}$. We claim this sequence is strong club guessing on the set of singulars below κ . Given any club $D \subset \kappa$, by Theorem 2.10 we may find a tail measure one set B , witnessed by $\gamma < lh(\bar{U})$ such that $O(B \cap \mathcal{MS}_G) \subset D$. For any \bar{w} such that $B \cap V_{\kappa(\bar{w})}$ is tail measure one with respect to \bar{w} , Lemma 4.12 implies that $c_{\kappa(\bar{w})} \subset^* O(B \cap \mathcal{MS}_G)$. Since $\{\bar{w} : B \cap V_{\kappa(\bar{w})} \text{ is tail measure one with respect to } \bar{w}\}$ is tail measure one with respect to \bar{U} , we can apply Theorem 2.11 to get the conclusion as desired. \square

Even though there are many instances of club guessing principles valid in ZFC (see [She94]), SCG at singulars does not necessarily hold in ZFC. For example, in the forcing extension where we add λ^+ many Cohen subsets of λ , SCG on the singulars below λ fails. We describe another scenario of a different nature.

Proposition 4.13. *Suppose κ is a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, then SCG fails on the set $S = \{\alpha \in \lambda : cf(\alpha) < |\alpha|\}$.*

Proof. Suppose for the sake of contradiction that $\bar{t} = \langle t_\alpha : \alpha \in S \rangle$ is a strong club guessing sequence. We may assume t_α is a club in α for each $\alpha \in S$. Let $j : V \rightarrow M$ witness that κ is λ -compact. In particular, ${}^\kappa M \subset M$ and $\sup j''\lambda =_{def} \delta < j(\lambda)$. Let $t = j(\bar{t})_\delta$ and $A = j^{-1}acc(t \cap j''\lambda)$. Recall for a set Z , $acc(Z) = \{\delta \in Z : Z \cap \delta \text{ is unbounded in } \delta\}$. It is not hard to see the set A is $< \kappa$ -closed and unbounded. Consider \bar{A} , the closure of A . By the hypothesis, there exists a club $D \subset \lambda$, such that for any $\alpha \in D \cap S$, $t_\alpha \subset^* \bar{A}$. By elementarity of j , we know that $\delta \in j(D \cap S)$. As a result, $t \subset^* j(\bar{A})$. This in particular implies $t \cap j''\lambda \subset^* j(\bar{A}) \cap j''\lambda = j''\bar{A}$.

We check that for any $\nu \in t \cap j''\lambda$, there is some $\nu' \geq \nu$ in $t \cap j''\lambda$ that does not belong to $j''\bar{A}$. This clearly gives a contradiction as desired. Given such a ν , if it already does not belong to $j''\bar{A}$, then we are done. Suppose there is some $\xi \in \bar{A}$, $j(\xi) = \nu$. Let ν' be $\min((t \cap j''\lambda) - (\nu + 1))$. In particular, ν' is in $nacc(t \cap j''\lambda)$. Observe that $\nu' \notin j''\bar{A}$. To see this, suppose for contradiction that there is $\xi' \in \bar{A}$ such that $j(\xi') = \nu'$. Note that $\xi' > \xi$. By the definition of A , $\xi' \in \bar{A} - A$. We can then find $\eta \in (\xi, \xi') \cap A$. But then $\nu < j(\eta) < \nu'$ and $j(\eta) \in t \cap j''\lambda$, contradicting with the choice of ν' . \square

Remark 4.14. *Proposition 4.13 can be used to separate the \diamond^+ -principles from the SCG principles. For example, [Ish05, Question 1] asks if for a regular uncountable cardinal μ , $\diamond^+(\mu)$ implies SCG on μ . By [CFM01, Theorem 22], if $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \lambda^{++}$, then there is a λ^+ -directed closed and λ^{++} -c.c poset that forces $\diamond^+(\lambda^+)$. Therefore, relative to the existence of a supercompact cardinal κ , we can produce a model (using the technique by Laver [Lav78]) where κ remains supercompact and $\lambda > \kappa$ satisfies $\diamond^+(\lambda^+)$ holds. In this model, SCG on λ^+ fails by Proposition 4.13. There are two things worth noting:*

1. Large cardinals should not be needed to separate \diamond^+ from SCG. For example, [IL12] shows it is consistent relative to ZFC that $\diamond^+(\omega_1)$ holds and SCG on ω_1 fails.
2. If λ is regular with $\lambda^{<\lambda} = \lambda$, then [FK05] proves there is a λ -closed and λ^+ -c.c poset that adds a SCG-sequence on λ .

The truth of SCG on regulars is constrained by the compactness principle to be discussed in the next section.

5 A weak consequence of ineffability does not imply the diamond principle

To relate the compactness principle we will consider in this section with standard large cardinals, we first supply a characterization of ineffable cardinals in terms of club sequences. The following lemma is essentially due to Todorcevic [Tod07], who proved the corresponding version for weakly compact cardinals. Recall κ is *ineffable* iff for any $\langle d_\alpha \subset \alpha : \alpha < \kappa \rangle$, there exists $A \subset \kappa$ such that $\{\alpha < \kappa : A \cap \alpha = d_\alpha\}$ is a stationary subset of κ .

Definition 5.1. A C -sequence on κ is a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ where each C_α is a club subset of α for any $\alpha < \kappa$.

Lemma 5.2. For a strongly inaccessible κ , we have that: κ is ineffable iff for any C -sequence on κ $\langle C_\alpha \subset \alpha : \alpha < \kappa \rangle$, there exists a club $C \subset \kappa$, such that $\{\alpha < \kappa : C \cap \alpha = C_\alpha\}$ is stationary.

Proof. Let T be a tree of height κ and the ξ -th level of the tree is $T(\xi) = [\delta_\xi, \delta_{\xi+1})$, where $\langle \delta_\xi : \xi < \kappa \rangle$ is an increasing continuous sequence of ordinals below κ . We will also assume that T does not split at limit levels, in the sense that for any limit $\xi < \kappa$, and $\alpha_0, \alpha_1 \in T(\xi)$, if $\{\eta : \eta <_T \alpha_0\} = \{\eta : \eta <_T \alpha_1\}$, then $\alpha_0 = \alpha_1$.

We claim that there is a branch d through T such that $\{\xi < \kappa : \delta_\xi \in d\}$ is stationary in κ . Note that there is a club C consisting limit ξ such that $\delta_\xi = \xi$. Consider the following C -sequence \bar{C} : for any $\xi < \kappa$, if $\xi \in \lim C$, then we let C_ξ be the closure of $E_\xi =_{\text{def}} \{\eta < \xi : \eta <_T \xi\}$ and if $\xi \notin \lim C$, then $C_\xi = (\max C \cap \xi, \xi)$. Apply the hypothesis, we get a club $D \subset \kappa$ such that $S = \{\xi < \kappa : D \cap \xi = C_\xi\}$ is stationary in κ . We claim that $S \cap \lim C$ forms a branch as desired. Suppose $\alpha < \beta \in S \cap \lim C$, let $\alpha' \in T(\alpha)$ be the unique node such that $\alpha' <_T \beta$. We need to show that $\alpha = \alpha'$. By the fact that T does not split at limit levels, we only need to show that α and α' have the same set of T -predecessors. Because T is a tree, it suffices to show there are arbitrarily large $\eta < \alpha = \delta_\alpha$ with $\eta <_T \alpha, \alpha'$. Given $\xi < \alpha$, note that $D \cap T(\xi + 1)$ has a unique element. To see this, by the hypothesis we have $D \cap \alpha = C_\alpha$, as a result $D \cap T(\xi + 1)$ consists of the unique node that is T -below α , since $D \cap T(\xi + 1) = C_\alpha \cap T(\xi + 1) = E_\alpha \cap T(\xi + 1)$. Let $\eta \in D \cap T(\xi + 1)$. The same argument shows that $\eta <_T \alpha'$. We are done.

It is left to see the principle as above implies κ is ineffable. Given a list $\langle d_\alpha \in 2^\alpha : \alpha < \kappa \rangle$, consider the tree $T' = \{d_\beta \upharpoonright \alpha : \alpha \leq \beta\}$. Note T' does not split on limit levels. Since κ is strongly inaccessible, it is possible to find an increasing continuous sequence $\langle \delta_\xi : \xi < \kappa \rangle$ such that there is a bijection $b_\xi : [\delta_\xi, \delta_{\xi+1}) \leftrightarrow T'(\xi)$ with $b_\xi(\delta_\xi) = d_\xi$ for each $\xi < \kappa$. Let T be the pull-back of T' , namely, for any α_0, α_1 with $\xi_0, \xi_1 < \kappa$ be such that $\alpha_i \in [\delta_{\xi_i}, \delta_{\xi_i+1})$ for $i < 2$, $\alpha_0 <_T \alpha_1$ iff $b_{\xi_0}(\alpha_0) <_{T'} b_{\xi_1}(\alpha_1)$. It can be easily verified that T is of the form described above. Then there is a branch d through T such that $S = \{\xi < \kappa : \delta_\xi \in d\}$ is stationary. Consider $d' = \{b_\xi(\delta_\xi) =_{def} d_\xi : \xi \in S\}$. Then for any $\xi_0 < \xi_1 \in S$, $\delta_{\xi_0} <_T \delta_{\xi_1}$, which implies $b_{\xi_0}(\delta_{\xi_0}) = d_{\xi_0} <_{T'} b_{\xi_1}(\delta_{\xi_1}) = d_{\xi_1}$, hence $d_{\xi_0} = d_{\xi_1} \upharpoonright \xi_0$. This implies that κ is ineffable. \square

The following notion is related to constructions of distributive Aronszajn trees, which strenghtens the principle $\otimes_{\vec{C}}$ from [She94, p. 134].

Definition 5.3 ([BR19]). A C -sequence on κ $\langle C_\alpha : \alpha < \kappa \rangle$ is said to be *amenable* if there is no club $D \subset \kappa$, such that $\{\alpha < \kappa : D \cap \alpha \subset C_\alpha\}$ is stationary.

Therefore, the assertion that *there is no amenable C -sequence on κ* is a weakening of ineffability. To be more explicit, the compactness principle states:

for any C -sequence $\langle C_\alpha \subset \alpha : \alpha < \kappa \rangle$, there exists a club $D \subset \kappa$, such that $\{\alpha : D \cap \alpha \subset C_\alpha\}$ is a stationary subset of κ .

Remark 5.4. *That “there is no amenable C -sequence on κ ” is a non-trivial compactness principle. For example, it implies that the diagonal stationary reflection principle at κ , which means: for any κ -sequence of stationary subsets of κ $\langle S_i : i < \kappa \rangle$, there is some $\delta < \kappa$ such that for any $i < \delta$, $S_i \cap \delta$ is stationary. However, it is not a consequence of κ being weakly compact, while the diagonal stationary reflection principle at κ is. For instance, in L , there is no amenable C -sequence on κ iff κ is ineffable.*

Remark 5.5. *To relate to the Strong Club Guessing principle discussed in the previous section, notice that if there is no amenable C -sequence on κ , then the Strong Club Guessing principle on the regulars below κ fails.*

In [BN19], it was shown that relative to the existence of large cardinals, it is consistent that there is a strongly inaccessible cardinal κ where the diagonal stationary reflection principle holds yet $\diamond(\kappa)$ fails. We strengthen the compactness principle in this model to “there is no amenable C -sequence on κ ”.

Theorem 5.6. *If \bar{U} is a measure sequence on κ such that $cf(lh(\bar{U})) \geq \kappa^{++}$, then in $V^{R_{\bar{U}}}$, there is no amenable C -sequence on κ .*

Lemma 5.7. *Suppose $cf(lh(\bar{U})) \geq \kappa^+$. Let $E \subset \mathcal{MS}$ be \bar{U} -stationarily positive, namely $\{\tau < lh(\bar{U}) : E \in U(\tau)\}$ is a stationary subset of $lh(\bar{U})$. Given a tail measure function b , there exists $A \in \bigcap \bar{U} \upharpoonright \geq \beta$ for some $\beta < lh(\bar{U})$, such that in any generic $G \subset R_{\bar{U}}$, $\{\alpha < \kappa : O(A \cap \mathcal{MS}_G) \cap \alpha \subset^* O(b(\bar{w}) \cap \mathcal{MS}_G), \bar{w} \in \mathcal{MS}_G \cap E, \kappa(\bar{w}) = \alpha\}$ is stationary.*

Proof. Fix a tail measure function b . Then there is some $\tau = \tau_{\bar{w}} < lh(\bar{w})$ such that $b(\bar{w}) \in \bigcap \bar{w} \upharpoonright \geq \tau$. Let f be the function $\bar{w} \mapsto \tau_{\bar{w}}$. Let $S = \{\tau < lh(\bar{U}) : E \in U(\tau)\}$. Consider $g : S \rightarrow lh(\bar{U})$ such that $g(\gamma) = j(f)(\bar{U} \upharpoonright \gamma) < \gamma$. By Pressing Down, there is a stationary $S' \subset S$ and $\beta < lh(\bar{U})$ such that $g''S' \subset \beta$. We may assume β is novel since \bar{U} does not contain a repeat point. Pick $A_\beta \in \bigcap \bar{U} \upharpoonright \geq \beta$ with $A_\beta^c \in \bigcap \bar{U} \upharpoonright \beta$.

We check that $F = \{\bar{w} \in E : A_\beta \cap b(\bar{w})^c \text{ is } \bar{w}\text{-null}\}$ is \bar{U} -positive. Indeed, for any $\tau \in S'$, $\bar{U} \upharpoonright \tau \in j(F)$, since $A_\beta \cap j(b)(\bar{U} \upharpoonright \tau)^c \notin U(i)$ for any $i < \tau$. To see this, if $i \geq \beta$, $j(b)(\bar{U} \upharpoonright \tau) \in U(i)$ and if $i < \beta$, $A_\beta^c \in U(i)$.

Let $G \subset R_{\bar{U}}$ be generic over V , then $O(F \cap \mathcal{MS}_G)$ is a stationary subset of κ , by Theorem 2.10. For each $\bar{w} \in F \cap \mathcal{MS}_G$, we know a tail of \mathcal{MS}_G below \bar{w} avoids $A_\beta \cap b(\bar{w})^c$. As a result, $O(A_\beta \cap \mathcal{MS}_G) \cap \kappa(\bar{w}) \subset^* O(b(\bar{w}) \cap \mathcal{MS}_G)$. \square

Lemma 5.8. *Suppose $cf(lh(\bar{U})) \geq \kappa^+$. Let $p \Vdash \dot{c}$ is a name for a C -sequence. Then there exists $q \leq^* p$ and a tail measure function b such that for some $B \in \bigcap \bar{U}$, for any $\bar{w} \in B$, if $cf(lh(\bar{w})) \geq \kappa(\bar{w})^+$, it is the case that $q \restriction \bar{w} \Vdash O(b(\bar{w}) \cap G) \subset^* \dot{c}(\bar{w})$.*

Proof. Suppose $p = p_0 \restriction (\bar{U}, A)$. By Lemma 2.13, we can find \bar{U} -measure-one $A' \subset A$, and some \dot{d} such that $\dot{d}(\bar{w})$ is a $R_{\bar{w}}$ -name for a club subset of $\kappa(\bar{w})$ and letting $q = p_0 \restriction (\bar{U}, A')$ we have that $q \restriction \bar{w} \Vdash \dot{c}_{\kappa(\bar{w})} = \dot{d}(\bar{w})$ for any $\bar{w} \in A'$.

For each $\bar{w} \in A'$ with $cf(lh(\bar{w})) \geq \kappa(\bar{w})^+$, $t \in V_{\kappa(\bar{w})} \cap R_{<\kappa}$, we can find (by Theorem 2.10) $A_{\bar{w}}^t \in \bigcap \bar{w}$ and some $\Gamma_{\bar{w}}^t \in \bigcap \bar{w} \upharpoonright \geq \tau$ where $\tau < lh(\bar{w})$ such that $t \restriction (\bar{w}, A_{\bar{w}}^t) \Vdash_{R_{\bar{w}}} O(\Gamma_{\bar{w}}^t \cap \dot{G}_{\bar{w}}) \subset \dot{d}(\bar{w})$. Let $A_{\bar{w}} = \Delta_t A_{\bar{w}}^t \in \bigcap \bar{w}$ and $\Gamma_{\bar{w}} = \Delta_t \Gamma_{\bar{w}}^t \in \bigcap \bar{w} \upharpoonright \tau$ for some $\tau < lh(\bar{w})$.

Let b be a tail measure function satisfying that $b(\bar{w}) = \Gamma_{\bar{w}}$ for any $\bar{w} \in A'$ with $cf(lh(\bar{w})) \geq \kappa(\bar{w})^+$. We claim that $q \restriction \bar{w} \Vdash O(b(\bar{w}) \cap G) \subset^* \dot{c}(\bar{w})$ for all $\bar{w} \in A'$ with $cf(lh(\bar{w})) \geq \kappa(\bar{w})^+$. Fix such \bar{w} .

It is left to check that $p_0 \restriction (\bar{w}, A' \cap V_{\kappa(\bar{w})}) \Vdash_{R_{\bar{w}}} O(\Gamma_{\bar{w}} \cap G_{\bar{w}}) \subset^* \dot{d}(\bar{w})$. This follows immediately from the fact that for any $t \in V_{\kappa(\bar{w})}$ with $t \restriction (\bar{w}, A_{\bar{w}}) \in R_{\bar{w}}$, $t \restriction (\bar{w}, A_{\bar{w}}) \Vdash_{R_{\bar{w}}} O(\Gamma_{\bar{w}} \cap \dot{G}_{\bar{w}}) - (rank(t) + 1) \subset \dot{d}(\bar{w})$. To see this fact, suppose for the sake of contradiction that this is not the case for some t , then we have an extension $t^* = t \restriction (\bar{w}, e) \restriction (\vec{\eta}, \vec{A}_{\vec{\eta}}) \restriction (\bar{w}, A^*) \Vdash \kappa(\bar{u}) \notin \dot{d}(\bar{w})$ for some $\bar{u} \in \Gamma_{\bar{w}}$. Notice that $\bar{u} \in A_{\bar{w}}^t$ and $\vec{\eta} \subset A_{\bar{w}}^t$ by the definition of $A_{\bar{w}}$. Hence t^* is compatible with $t \restriction (\bar{w}, A_{\bar{w}}^t)$, which forces $O(\Gamma_{\bar{w}}^t \cap \dot{G}_{\bar{w}}) \subset \dot{d}(\bar{w})$. As $\bar{u} \in \Gamma_{\bar{w}}^t$, we know that $t \restriction (\bar{u}, A_{\bar{w}}^t \cap V_{\kappa(\bar{u})}) \restriction (\bar{w}, A_{\bar{w}}^t) \Vdash \kappa(\bar{u}) \in \dot{d}(\bar{w})$, and is compatible with t^* that forces $\kappa(\bar{u}) \notin \dot{d}(\bar{w})$, which is absurd. \square

Proof of Theorem 5.6. By Lemma 5.8, in $V[G]$ where $\langle c_\alpha : \alpha < \kappa \rangle$ is given, we aim to find a club $C \subset \kappa$ such that $\{\alpha : C \cap \alpha \subset^* C_\alpha\}$ is a stationary subset of κ . By Lemma 5.8, we can find tmf $b \in V$, such that on a tail of \mathcal{MS}_G , if $\bar{w} \in \mathcal{MS}_G$ satisfies $cf(lh(\bar{w})) \geq \kappa(\bar{w})^+$, then $O(b(\bar{w}) \cap \mathcal{MS}_G) \subset^* c_{\kappa(\bar{w})}$. By Lemma 5.7 with $E = \{\bar{w} : cf(lh(\bar{w})) \geq \kappa(\bar{w})^+\}$, we can find a club D in $V[G]$

such that for stationarily many regular $\alpha \in C_G$ with $\alpha = \kappa(\bar{w})$ for $\bar{w} \in \mathcal{MS}_G$, it is the case that $D \cap \alpha \subset^* O(b(\bar{w}) \cap \mathcal{MS}_G) \subset^* c_\alpha$. \square

If we prepare the ground model such that κ is $(\kappa + 2)$ -strong and $2^\kappa > \kappa^{++}$, then for some elementary embedding, we can derive a measure sequence \bar{U} of length κ^{++} . By Theorem 4.5 and Theorem 5.6, we have no amenable C-sequence on κ and $\neg \diamond(\kappa)$ in $V^{R_{\bar{U}}}$.

Corollary 5.9. *Relative to the existence of suitable large cardinals, it is consistent that there does not exist an amenable C-sequence at κ and $\diamond(\kappa)$ fails.*

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