PIMS Talk, Vancouver - 2013

- joint work with Tom Meyerovitch

All measures will be shift-invariant probability measures.

A Markov random field \( \mu \) (MRF) is a probability measure on \( \mathbb{Z}^d \) such that for all \( A, B, C, Z \subset \mathbb{Z}^d \) finite satisfying \( A \cap B \cap C = A \cap B \cap C = A \cap B \cap C = \emptyset \),

\[
\mu \left( E_{A} \cap E_{B} \right) = \mu \left( E_{A} \cap E_{B} \right)
\]

The collection of these objects is called a specification.

(need not correspond to a measure.)

Question: Given \( \text{supp}(\mu) \) can there exist a finite description of the specification for any MRF with support \( \mu \)?

A nearest neighbour interaction is \( V: \mathbb{Z}^d \rightarrow \mathbb{R} \).

A Gibbs state is an MRF with \( \mu \) such that

\[
\mu \left( E_{A} \cap E_{Z} \right) = \prod_{x \in \text{supp}(\mu)} V(x | A)
\]

for all \( Z \subset \mathbb{Z}^d \).

Topological support \( \mu \) is \( \text{supp}(\mu) \).

Hammersley-Clifford Theorem: If \( \text{supp}(\mu) \) has a safe symbol then \( \mu \) is an MRF if and only if \( \mu \) is a Gibbs state with some nearest neighbour interaction.
This does not hold without some symbol. There constructions by Muscari in the finite graphs.

\( \mu \) is a probability shift-invariant mean

Note: \( \text{supp}(\mu) \) is a shift

Note: \( \mu \) is a MRF \( \Rightarrow \text{supp}(\mu) \) is a shift

(say) Further is true. It is a topological Markov field.

\[ \Delta_x = \mathcal{E}(x,y) \text{ with } x \text{ and } y \text{ differ at finitely many sites.} \]

(suffice us to parametrise the specifications.)

**Markov Cocycles**

\[ X - \text{TMR} \]

\[ c: \Delta_x \to \mathbb{R} \]

such that

\[ c(x,y) = c(x,z) + c(z,y) \]

\[ c(x,y) \text{ is a function of } x/\mathcal{E}(x,y) \]

where \( \mathcal{E} = \{ i : x(i) \neq y(i) \} \).

Shift invariant

**Gibbs Cocycle:** Markov Cocycle + there exists

Schmidt & Petersen

interaction \( V \) such that

\[ c(x,y) = \sum_{c \in \mathcal{D}} v(x \uparrow c) - v(y \uparrow c) \]
1-1 correspondence between Markov cocycles and specifications. If \( \mu \) is an NRE
\[
c(x,y) = \log \frac{\mu(\{x,z\} \cap \{y,w\})}{\mu(\{y,z\} \cap \{x,w\})}
\]
where \( x,y \) different at \( \Gamma \) is a Markov cocycle.

Random NIko dynm cocycles.

Finite - \( G_x \) - Gibbs cocycles, \( \Gamma \) vector

Dimensional \( N \) - Markov cocycles.

Finite description corresponds to \( M_x \) finite.

Stronger version of Hammersley Clifford
\( x \) has a safe symbol. Then
\( G_x = M_x \).

Pivot property \( x \) has pivot property
if for all \( (x,y) \in A_x \) there exists
\( x = x_1, x_2, x_3 \) - \( x_0 \) such that
\( x_i, x_{i+1} \) differ at a single site.

Note: Then \( c(x,y) = \sum_{i=0}^{n-1} c(x_i, x_{i+1}) \) - \( M_x \) finite dim.
Main Examples: $x - 2$-coloured checkerboard

\[
\begin{array}{ccc}
2 & 1 & 2 \\
2 & 1 & 1 \\
3 & 3 & 3 \\
3 & 3 & 3 \\
\end{array}
\]

$\dim(M_x) = 3$
$\dim(\mathcal{G}_x) = 2$

But every Markov random field is Gibbs.

Is the pivot property important?

Close counting cousin of checkerboard model:

- Shift
- All $2 \times 2$ blocks
- Topologically transitive
- Square of various sizes
- Two distinct colours

- $\dim(M_x) = \infty$