A - finite set (discrete topology)
\( \mathbb{Z}^d \) - Cayley graph of \( \mathbb{Z}^d \) with standard generators.
\( d \geq 2 \)

Pattern an element of \( \mathcal{P}^A \) (for some \( A \subset \mathbb{Z}^d \) finite)

\[ X_f := \{ x \in A^{\mathbb{Z}^d} \mid \text{translates of patterns in } f \text{ do not occur in } x \} \]

Shift spaces.

If \( f \) can be chosen finite, \( X_f \) is called SFT (a shift of finite type (SFT))
The non-emptiness problem for SFTs is undecidable.
(say) The non-emptiness problem for SFTs is undecidable. So we study a more restricted class.

Hom-shift: $H$ - finite undirected graph (Connected always)
$G$ - maybe infinite
$A$ - graph
$G/H \Rightarrow$ adjacency in $G/H$ (say we drop subscript when $G = \mathbb{Z}^d$)

$f : G \rightarrow H$ homomorphism
\[ i \sim_g j \Rightarrow f(i) \sim f(j). \]

$\text{Hom}(\cdot G, \cdot H) = \text{space of all homomorphism from } G \text{ to } H$.

$K_n$ - complete graph with vertices $1, \ldots, n$
$\text{Hom}(G^\# , K_n) = \text{complete proper } n\text{-colours of } G$
$\text{Hom}(G, Co-I) = \text{hard core model}$
(no two 1s are adjacent)
\( x_H = \text{Hom}(\mathbb{Z}^d, H) \) 

shift space is called a hom-shift (say) why is this a shift space?

The language \( L(x) \) is the set of all patterns appearing in \( x \).

\( B_n \) is a box in \( \mathbb{Z}^d \) with side length \( n \).

\( L_n(x) = I_{n-1}(x) \cap A^n \) \( B_n, \) box of side length \( n \).

topological entropy

\[ h_{\text{top}}(x) = \lim_{n \to \infty} \frac{\log |L_{B_n}(x)|}{\log |B_n|} \]

Entropy minimality \( Y \subseteq X \)

then \( h_{\text{top}}(Y) \leq h_{\text{top}}(X) \)

\( X \) - entropy minimal if \( Y \subseteq X \)

\[ \Rightarrow h_{\text{top}}(Y) < h_{\text{top}}(X) \]

(say)

That is, if we forbid a pattern entropy drops.
Qn: When is $X^H$ entropy minimal?

Result: $C_n - n$-cycle with vertices $0, \ldots, n-1$

(C., Meyerowitz) '83

$X_{C_n}$ is entropy minimal.

(C. '14) $H$-four-cycle free if

$C_4$ is not a subgraph of $H$

$\Rightarrow X^H$ is entropy minimal.

Previous results:

Transitivity: st.

$X$ is strongly irreducible if

$\forall a \in A, b \in B$, st.

$a, b \in (\mathbb{R})$

$\exists x \in X$ st. $x \not\in A, x \not\in B$.

Schraudner '09: S.1 shift space is entropy minimal. (Say stronger here)

Qn: When is $X^H$ S.1?
Block-gluing $X$ is block-gluing $\forall x \in A \subset X \forall x \in B \subset X$ rectangular at

$E \times [0, 5] \times \mathbb{R} \times \mathbb{R}$

$\times |A| = a$, $\times |B| = b$

Boyle, Pavlov, Schrödmer 109 Block-gluing $\not\implies$ entropy minimal.

$c_3 - 1 \xrightarrow{2} x_c$ is not block-gluing (say) yet it is entropy minimal.
\( \mu \)-shift invariant probability measure.

Can associate entropy measure theoretic entropy denote by \( h \mu \cdot \text{supp}(\mu) \) topological support.

(Variational principle)
\[
\sup \, h \mu = h_{\text{top}}(x)
\]
\( \mu(x) = 1 \)

\( \mu \) \( \in \) which achieves this max. (NME)

for such \( \mu \)
\[
h \mu = h_{\text{top}}(x).
\]
\( \text{supp}(\mu) = \text{smallest closed set } \mathcal{Y} \text{ s.t } \mu(\mathcal{Y}) = 1 \).

\( \mu \) is an NME if and only if

\( \& \) Observe: \( \mu \) is a \( X \) is entropy minimal if

for all mme. \( \mu \cdot \text{supp}(\mu) = x \).

(\text{Can prove further X SFT, } \mu \text{ mme } \Rightarrow \mu \text{ adapted})

that is, \( \mu \cdot \text{mme. of } x \) \text{ for } x \in \text{supp}(\mu)

\( y \) \( \neq \) \( x \) drifts at finitely many \( y \) sites \( \neq \) from \( x \)
\[
\Rightarrow y \in \text{supp}(\mu).
\]

(\text{Give examples}).
Want to prove adapted to
\( M \) nice

\[ \mu \gg 0 \quad \mu \text{ nice adapted measure} \]

\[ \text{for } x \mapsto \frac{x}{||x||} \quad \text{modding map} \]

\[ \mathbb{Z} \rightarrow \mathbb{C}_3 \]

\[ x \equiv y \mod 3 \iff x = y \]

\[ \tilde{x}_t \text{ is unique given } \tilde{x}(0) \]

In fact, \( \mu \)-almost ergodic,

we can associate slopes for every direction \( \tilde{e}_1 \ldots \tilde{e}_n \)

\[ s \tilde{e}_i(x) = \frac{1}{n} \left( \tilde{x}(n\tilde{e}_i) - \tilde{x}(0) \right) \]

exists and is constant \( \mu \)-a.e.

Repro.
\text{Conjecture: } \# \text{ for all } H

\text{Hom}(\mathbb{Z}^2/H) \text{ is entropy minimal.}