Markov Random Fields and the 3-coloured Chessboard

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Outline

- Topological Markov fields
- Markov random fields and Gibbs measures with nearest neighbour interactions
- The pivot property
- Examples: 3-coloured chessboard and the Square Island shift.
Topological Markov Fields

A topological Markov field is a shift space $X \subset A^{\mathbb{Z}^d}$ with the ‘conditional independence’ property: for all finite subsets $F \subset \mathbb{Z}^d$, $x, y \in X$ satisfying $x = y$ on $\partial F$, $z \in A^{\mathbb{Z}^d}$ given by

$$z = \begin{cases} x & \text{on } F \\ y & \text{on } F^c \end{cases}$$

is also an element of $X$. 
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- $x$, $y$, and $z$ are depicted in the grid diagrams.
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Consider any one-dimensional shift space $X$. Make a two-dimensional shift space $Y$ where the horizontal constraints come from $X$. If $x$ and $y$ agree on $\partial F$, they must agree on $F$. Therefore $Y$ is a topological Markov field. There are uncountably many such shift spaces but there are only countably many nearest neighbour shift of finite type!!
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A Markov random field is a shift-invariant Borel probability measure \( \mu \) on \( \mathcal{A}^{\mathbb{Z}^d} \) with the property that for all finite \( A, B \subset \mathbb{Z}^d \) such that \( \partial A \subset B \subset A^c \) and \( a \in \mathcal{A}^A, b \in \mathcal{A}^B \) satisfying \( \mu([b]_B) > 0 \)

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The set of conditional measures \( \mu([\cdot]_A \mid [b]_{\partial A}) \) for all \( A \subset \mathbb{Z}^d \) finite and \( b \in \mathcal{A}^{\partial A} \) is called **specification** for the measure \( \mu \).
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The set of conditional measures $\mu([\cdot]_A \mid [b]_{\partial A})$ for all $A \subset \mathbb{Z}^d$ finite and $b \in \mathcal{A}^{\partial A}$ is called specification for the measure $\mu$. The specification might contain a huge lot of data!!!!
Given a shift space $X$ define a **nearest neighbour interaction** on $X$ as a shift-invariant function $V : \mathcal{B}(X) \rightarrow \mathbb{R}$ supported on configurations on edges and vertices.
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A Gibbs state with a nearest neighbor interaction $V$ is a Markov random field $\mu$ such that for all $x \in \text{supp}(\mu)$ and $A, B \subset \mathbb{Z}^d$ finite satisfying $\partial A \subset B \subset A^c$

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\mu([x]_A \mid [x]_B) = \frac{\prod_{C \subset A \cup \partial A} e^{V([x]_C)}}{Z_{A, x|\partial A}}
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The specification of a Gibbs measure with a nearest neighbour interaction has a finite description: all we need is the interaction $V$. 
**Question:** When is a Markov random field Gibbs with some nearest neighbour interaction?
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(*Hammersley-Clifford theorem*) Every Markov random field whose support has a *safe symbol* is Gibbs with some nearest neighbour interaction.
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Question: How can we weaken the hypothesis?
Pivot Property

A shift space $X$ is said to satisfy the pivot property if for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \ldots, x^n = y \in X$ such that $x^i, x^{i+1}$ differ on at most a single site.
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Examples:
- Any shift space with a safe symbol.
- $r$-coloured checkerboard for $r \in \{4, 5\}$.
- Domino tilings.
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The 3-coloured chessboard is a shift space with alphabet \{0, 1, 2\} such that adjacent colours are distinct.
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The tables below illustrate the 3-coloured chessboard:

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$$\frac{\mu([x]_F | [x]_{\partial F})}{\mu([y]_F | [x]_{\partial F})} = \prod_{i=1}^{n-1} \frac{\mu([x^i]_F | [x^i]_{\partial F})}{\mu([x^{i+1}]_F | [x^i]_{\partial F})} = \prod_{i=1}^{n-1} \frac{\mu([x^i]_{m_i} | [x^i]_{\partial m_i})}{\mu([x^{i+1}]_{m_i} | [x^i]_{\partial m_i})}. $$

Therefore the entire specification is determined by finitely many parameters viz. $\frac{\mu([x]_{0 \cup \partial 0})}{\mu([y]_{0 \cup \partial 0})}$ for configurations $x, y$ which differ only at 0, the origin.
Suppose $\mu$ is a Markov random field whose support has the pivot property. Then given $x, y \in \text{supp}(\mu)$ that differ exactly on $F$ there exists a chain $x = x^1, x^2, \ldots, x^n = y$ where $x^i, x^{i+1}$ differ exactly at a site $m_i \in \mathbb{Z}^2$ and consequently

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Therefore the entire specification is determined by finitely many parameters viz. $\frac{\mu([x]_{0\cup\partial 0})}{\mu([y]_{0\cup\partial 0})}$ for configurations $x, y$ which differ only at 0, the origin.

Thus the space of specifications on any topological Markov field with the pivot property can be parametrised by finitely many parameters.
**Question:** Suppose we are given a nearest neighbour shift of finite type with the pivot property. Is there an algorithm to determine the number of parameters which describes the specification?
A specification supported on the 3-coloured chessboard is determined the quantities 

\[ v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix})}, \]

If \( \mu \) is a Gibbs measure with nearest neighbour interaction \( V \) then 

\[ v_1 = \exp\left( V(0,1) + V(1,0) + V(0,2) + V(0,1) + V(1,2) + V(2,0) \right) \]

\[ v_2 = \exp\left( V(1,2) + V(2,1) + V(2,0) + V(0,2) + V(0,1) \right) \]

\[ v_3 = \exp\left( V(0,2) + V(2,0) + V(2,0) + V(0,2) + V(0,1) \right) \]

\( \mu \) is Gibbs if and only if 

\[ v_1 v_2 v_3 = 1. \]
A specification supported on the 3-coloured chessboard is determined the quantities \( v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix})} \), \( v_2 = \frac{\mu(\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix})}{\mu(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix})} \).
A specification supported on the 3-coloured chessboard is determined the quantities $v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix})}$, $v_2 = \frac{\mu(\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix})}$ and $v_3 = \frac{\mu(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix})}{\mu(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix})}$.
A specification supported on the 3-coloured chessboard is determined the quantities \( v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix})} \), \( v_2 = \frac{\mu(\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix})} \), and
\[ v_3 = \frac{\mu(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix})}{\mu(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix})}. \]

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A specification supported on the 3-coloured chessboard is determined the quantities
\[ v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix})}, \quad v_2 = \frac{\mu(\begin{bmatrix} 2 & 1 & 2 \end{bmatrix})}{\mu(\begin{bmatrix} 2 & 2 & 1 \end{bmatrix})} \]
and
\[ v_3 = \frac{\mu(\begin{bmatrix} 0 & 2 & 0 \end{bmatrix})}{\mu(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix})}. \]
If \( \mu \) is a Gibbs measure with nearest neighbour interaction \( V \) then

\[
\begin{align*}
v_1 &= \exp(V(01) + V(10) + V(0_1) + V(0_1) - V(21) - V(12) - V(2_1) - V(1_2)), \\
v_2 &= \exp(V(12) + V(21) + V(2_1) + V(1_2) - V(02) - V(20) - V(0_2) - V(2_0)), \\
v_3 &= \exp(V(02) + V(20) + V(2_0) + V(0_2) - V(01) - V(10) - V(0_1) - V(1_0)).
\end{align*}
\]

\( \mu \) is Gibbs if and only if \( v_1 v_2 v_3 = 1. \)
Therefore the Hammersley-Clifford type conclusion fails for specifications of the 3-coloured chessboard.
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Square Island Shift

Inspiration from checkerboard island shift by Quas and Şahin.
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There are two kinds of squares: ones with red dots and ones without red dots which float in a sea of blanks.
The Square Island shift does not have the generalised pivot property.
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There is no way to switch from a big square with red dots to a big square without red dots making single site changes (or even bigger regional changes).
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There exists a Markov random field supported on the shift space which is not Gibbs for any finite-range interaction.

Can more uniform mixing conditions help?
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There exists a Markov random field supported on the shift space which is not Gibbs for any finite-range interaction.

**Question:** Can more uniform mixing conditions help?
Thank You!