H - undirected graph.

G - undirected graph without self-loops.

\[ \text{Hom}(G, H) := \{ x : G \to H \mid \text{in}_G \Rightarrow x_i \sim_H x_j \} \]

H - \( \mathbb{Z}^3 \) - proper 3-coloring of \( G \).

H - \( \mathbb{C}^2 \) - no two 1's are adjacent (hard core model).

\[ H_{\text{walk}} = \left( \text{Hom}(\mathbb{Z}, H), E_{\text{walk}} \right) \]

\[ E_{\text{walk}} := \{ (x, y) : x_i \sim y_i \wedge i \in \mathbb{Z} \} \]

Q: When is \( \text{diam}(H_{\text{walk}}) < \infty \)?

Motivation: \( \text{Hom}(\mathbb{Z}^2, H) \) forms a dynamical system (translation of \( H \times \text{Hom}(\mathbb{Z}, H) \) is still a homomorphism).

Want to address: How do properties of \( H \) reflect in the dynamics of \( \text{Hom}(\mathbb{Z}^2, H) \)?
By properties $\rightarrow$ mixing properties.

What is a mixing property?

Let $A, B \subset \mathbb{Z}^2$-boxes.

\[ a \in \operatorname{Hom}(A, H), \quad b \in \operatorname{Hom}(B, H) \]

Then exists $x \in \operatorname{Hom}(\mathbb{Z}^2, H)$ s.t.

\[ x \cdot a = a, \quad x \cdot b = b \]

$n$ depends on $a, b, \exists n \rightarrow$ transitivity

($\forall x$) $\rightarrow$ holds for all $H$-connected

$n$ independent of $a, b \rightarrow$ block-gluing.

Qn: When is $\operatorname{Hom}(\mathbb{Z}^2, H)$ block-gluing?

Note if $H$ is bipartitive

$a, b$ are even patterns on $A, B$ singletons then $n$ depends on whether $a, b$ are of the same parity class.

$\text{H} \text{ - bipartitive. } \Rightarrow \text{Hom}(\mathbb{Z}^2, H) \text{ not block-gluing}$
\[ n \text{ independent of } A, B. \quad \text{phased block-glim} \]
\[ \text{(might depend on phase of) } q, b. \]

\[ \text{Diam} \left( \mathbb{Z}^2, H \right) \text{ is block phased block-glim} \]
\[ \text{iff} \]
\[ \text{Diam} \left( H \text{ walk} \right) < \infty \]

\[ \text{Really} \]

\[ \text{Examples:} \]

\[ 0 \quad 1 \]
\[ 1 \quad 2 \]

\[ \text{Anything at fixed distance looks like this to } (0, 2)^\infty \text{ looks like } (1, 3)^\infty \]

\[ \text{Distance } (1012)^\infty, (0132)^\infty = \infty \]

\[ \text{Diam} \left( (1 \rightarrow 2) \text{ walk} \right) = \infty \]
More generally, $H$-graph.

$\nu$ folds into $\omega$ if $N_+(v) \cap N_{\omega}(w) = \emptyset$.

Each $\nu \rightarrow H \nu \rightarrow \emptyset$.

$x \in \text{Hom}(Z, H)$

Shifting $x \sim \ast$"
A is called bipartite dismantlable if sequence of folds stabilizes with H and ending with ...

Then: \( A \) is a set \( H \) of \( H \) to be undirected graph without self loops and copies of \( E \)

Then: \( \text{Diam}(H \text{ walk}) < 2 \) if \( H \) is not bipartite dismantlable.

Remark: Converse is not true in general.

\[ \Rightarrow E \times H \Rightarrow \text{Diam}(H \text{ walk}) \leq 6 \]

=> Rennie Pantaleo

Conjecture: It is undecidable whether \( \text{Diam}(H \text{ walk}) \leq 6 \)
How to prove \( \text{diam} (H \text{ walk}) = \infty \)

for \( H = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \)

There is a natural map \( \mathbb{Z} \mod 3 : \mathbb{Z} \to \mathbb{Z} \mathbb{H} \)

This induces a covering map \( f \) from \( \mathbb{Z} \text{ walk} \) to \( H \text{ walk} \).

But distance \( \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \ast \end{pmatrix} \right\} = \infty \)

\( \to \) difference of height \( \leq 1 \)

\( \Rightarrow \) number of steps \( \geq \frac{n}{2} \) for all \( n \).

\( \Rightarrow \) \( \text{diam} (\mathbb{Z} \text{ walk}) = \infty \)

\( \Rightarrow \) \( \text{diam} (H \text{ walk}) = \infty \).

If \( H \) has no self-loops and four cycles use \( \pi : \text{universal cover of } H \to H \).
Thus, it is decidable whether $\text{diam}(H_{post}) \leq n$.

Conjecture: It is undecidable whether $\text{diam}(H_{walk}) < \omega$. 