Some universal Models

I will begin with two questions.

**Question 1:** $X$ - Standard Borel space
$T: Z^d \times X \rightarrow X$ action by Borel automorphism.
$(X, T) -$ Borel dynamical system.
$\mathcal{E}(X, T) -$ set of ergodic invariant probability measures.

$\mathcal{E}_0 \subset \mathcal{E}$ is "full" if $\mu(x_0) = 1$ for all $\mu \in \mathcal{E}(X, T)$.

**Definition:** $(X_0, T)$ and $(Y, S)$ are "almost Borel isomorphic" if
$x_0 \in C_X$, $y_0 \in C_Y$ full and invariant
and an isomorphism $\phi: (X, T) \rightarrow (Y, S)$.

**Question:** Suppose $\psi: \mathcal{E}(X, T) \rightarrow \mathcal{E}(Y, S)$ is an isomorphism
such that $(X, \rho, T) \cong (Y, \psi(\mu), S)$ for $\mu \in \mathcal{E}(X, T)$.

Are $(X, T)$ and $(Y, S)$ almost Borel isomorphic?

**Question 2:** Dominoes are rectangular parallelepipeds
with side length 2 and height 1.

$X_{\text{dom}}$ - space of tilings of $Z^d$ by dominoes.
$\sigma -$ $2d$ action on $X_{\text{dom}}$ by translations (shifts).

**Question:** Prove
$$\lim_{n \rightarrow \infty} \frac{1}{(2n)^d} \log \# \text{domino tilings of } [1, 2n]^d = h_{\text{top}}(X_{\text{dom}}).$$

For $d = 2$, follows from Veneklyan (1962).
\((x, t)\) - Borel dy. syst.

\[ h_{\text{gms}}(x, t) = \sup_{\mu \in \mathcal{F}(x, t)} h_{\mu} \]

\((x, t)\) is called "universal" if for all free, ergodic \((Y, \mu, S)\)

\[ h_{\mu} < h_{\text{gms}}(x, t) \]

invariant

\[ \exists Y_0 \subset Y, \mu(Y_0) = 1 \]

and an embedding \((Y_0, S)\) into \((x, t)\).

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What about non-ergodic measures?

\((x, t)\)

\[ h_{\text{gms}}(Y, S) < h_{\text{gms}}(x, t) \]

(Cauchy-Borel universal)

\[ \exists \text{ embedding of } (Y, S) \text{ into } (x, t) \]

\[ h_{\mu} < h_{\text{gms}}(x, t) \]

Hochman (2015) All mixing SFTs are Borel universal.

Motivation for question)

(What is the distinction between universal and almost Borel universal.)
Two topological dynamical systems.

$\mathbb{Z}^d$ - Cayley graph with standard generators.

Proper $k$-colouring of $\mathbb{Z}^d$.

$x \in \text{col.} \iff x \in S, \quad \forall j \in \mathbb{Z}^d : \text{adjacent to } j \text{ in } \mathbb{Z}^d$.

$\{x : x((i)^{r}) \neq x((j)^{s})\}$

Tiling shifts

- $F = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ are rectangular parallelepipeds in $\mathbb{Z}^d$.
- For all coordinate directions the g.c.d. of the corresponding side lengths is $1$. (mixing)
  
  e.g. dominoes. $X_{\text{dom}}$

$X_F = \text{set of tilings of } \mathbb{Z}^d$ by $\mathcal{F}_1, \ldots, \mathcal{F}_n$.

(general class of shifts and very little is known).
Prihodko (2002) proved. for all $(Y, \mu, s)$ a free $\mathcal{F}$ equivalent map

$$\varphi: \mathcal{Y} \to X$$
defined a.p.e.

(2nd Alperin's Lemma: equivalent tilting of orbits)

Galvin & Robinson (2001)
Are $X, \mathcal{Y}$ and $X, \text{dom.}$ universal in $d=2$?

Is there a Borel equivalent map into $X, \mathcal{Y}$ and $X, \text{dom.}$?

Thus: $X, \mathcal{Y}$ is $t$-almost Borel universal for some $t$.

This strengthens the 'almost' answers.

Thus, $(C \& H)$ $X, \mathcal{Y}$ is $t$-almost Borel universal for some.

$X, \mathcal{Y}$ and $X, \text{dom.}$ are almost Borel universal.
Completely answers \( \mathbb{C} \) [Stronger 2D - Alpern's Lemma]

"almost" answers \( \mathbb{C} \) [We don't know how to deal with part with which does not support probability measure]

("almost" equivalent fillings of an and colouring of andy Borel dynamical system)

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we prove further: \( H \) finite undirected graph

\( X \) vertex shift for \( H \)

\( \text{maps from } Z^d \text{ to } H \text{ which preserve adjacency} \)

\[ H = \begin{array}{c}
2 \\
\_ \_ \\
3
\end{array} \]

\( X_H \text{ is } X_{Z^3, \text{col}} \)

Thm: \( H \) is not bipartite \( \Rightarrow X_H \) is almost Borel universal

Question \( \mathbb{G} \) \( \Rightarrow X \text{ dom for } d > 2 \text{ is universal} \)

General Scheme

\[ \text{Want: } (X, T) \text{ universal} \]

- Prove \( (X, T) \) has specification-like condition
- Specification-like condition \( \Rightarrow \) universality
Let us discuss two such conditions:

- **Shift space** $X \subset \mathbb{Z}^d$

- **Strong irreducibility**
  \[ A, B, C \subset \mathbb{Z}^d \]
  \[ \forall x, y \in X \exists z \in \mathbb{Z} \ni x + z = y + z \]

- **Technical condition $\Rightarrow$ universality**
  \[ (\text{Salim} \& \text{Robinson}, 2001) \]
  \[ (\not\exists \text{ required is a result of our} \text{ theorem}) \]

- **Almost weak specification**
  \[ \exists f : \mathbb{N} \rightarrow \mathbb{N} \quad \forall n \in \mathbb{N} \quad f(n) = o(n) \]

  \[ |B_i| \geq f(|A_i|) \]

  Then for all $x_1, x_2, \ldots, x_n \in X$

  \[ \exists x \in X \ni x_{i+1} = x_i \quad \Rightarrow \text{universality} \]

  Answered *Lind & Thouvenot* 6

  *Queiroz & Soo* 2012

  Used this to prove
First Idea: Why not try to approximate our systems with these.

Then: $X_{3, col}$ does not contain any such system!

Second Idea: ($X_{3, col}$ and $X_{dom}$ do not contain any such have a non-trivial cocycle into $\pi$).

Second Idea: Why not use their method of proof?

They use Kuran - Rothstein Machinery.

First step: \( \mu \) on \((X, T)\) is \(\mu\) is not an n.m.e.

Find \(\mu'\) close to \(\mu\) (weak star sense).

\(\mu' > \mu\).

It is believed.

(Belief) \(\exists \mu \) on \(X_{H}\) (weaker shift).

\(\mu\) is not an n.m.e.

but \(\mu\) is a local n.m.e.
\( L(x, B) = 9 \times 1_B : x + x^2 \).

\( B_n \) - box of side length \( n \). Fix \( k \in \mathbb{N} \).

\( C_{nk} \cap L(x, B_nk) \) is called \( t \)-flexible if

\[
\liminf_{n \to \infty} \frac{1}{|B_nk|} \log |C_{nk}| = t.
\]

\[
\begin{align*}
\forall n, k & \quad a(nk) \in U \cup C_{nk} \\
\Rightarrow & \quad \exists a \in C_{nk} \text{ s.t. } a|_{B(nk)} = a(0).
\end{align*}
\]

Thus \( t \)-flexible \( \Rightarrow \) \( t \)-Borel universal.
$H$ is an undirected graph (connected).

$(v, w)$ edge

$C_n$: graph homomorphisms from $B_n$ to $H$

such that on $B_n \setminus B_{n-1}$ only $v$, $w$ appear

$\mathcal{F}$ - set of rectangular tiles. $k$: product of side lengths of tiles in $\mathcal{F}$

$\mathcal{C}_n(k)$: complete tilings of $B_n \times k$

Question: Is $h_{\mathcal{F}}(X \mathcal{F})$ always computable?