Modelling processes on the $\mathbb{Z}^d$-lattice

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Toy questions:

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*How can we record the tossing of a coin?*
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By the symbols $H$ and $T$. 
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By the symbols 1, 2, 3, 4, 5 and 6.
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By the symbols 1, 2, 3, 4, 5 and 6.

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*Can we record the throw of a dice using two symbols?*
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*How can we record the throw of a dice?*

By the symbols 1, 2, 3, 4, 5 and 6.

Question

*Can we record the throw of a dice using two symbols?*

No! We are constrained by the size of the sample space.
(Infinite) toy questions:

Question

*How can we record infinitely many dice throws?*
(Infinite) toy questions:

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*How can we record infinitely many dice throws?*

Answer.

We can use functions \( f : \mathbb{Z} \rightarrow \{1, 2, 3, 4, 5, 6\} \). \( f(i) \) is used to record the result of the \( i^{th} \) dice throw.
(Infinite) toy questions:

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*Can we record the sequence of dice throws using two symbols at each time point instead of six?*
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Question

_Can we record the sequence of dice throws using two symbols at each time point instead of six?_

Let us first define ‘recording’ rigorously.
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Let $A$ be a finite set. An element $\omega \in A^\mathbb{Z}$ can be thought both as a function

$$\omega : \mathbb{Z} \rightarrow A$$

and as a binfinite sequence $(\omega_i)_{i \in \mathbb{Z}}$ of the elements of $A$. 
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Given $j \in \mathbb{N}$, the sequence $(\omega_{i-j})_{i \in \mathbb{Z}}$ represents the function whose values have been shifted $j$ entries to the left.
Stationary stochastic processes

A stationary stochastic process is a sequence of finite-valued random variables \( \Omega = (\Omega_i)_{i \in \mathbb{Z}} \) for which \( \Omega_0, \Omega_1, \ldots, \Omega_i \) has the same distribution as \( \Omega_j, \Omega_{j+1}, \ldots, \Omega_{j+i} \) for all \( i \in \mathbb{N} \) and \( j \in \mathbb{Z} \).
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Stationary stochastic processes
Example: Bernoulli process

Let $\Omega$ be a fixed finite-valued random variable. Let $(\Omega_i)_{i \in \mathbb{Z}}$ be a sequence of independent copies of $\Omega$. $(\Omega_i)_{i \in \mathbb{Z}}$ is called a Bernoulli process.
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A sequence of dice throws forms a Bernoulli process where $\Omega$ takes values 1, 2, $\ldots$, 6 with equal probability.
A slightly more complicated stochastic process

Consider the stochastic process \( \Omega := (\Omega_i)_{i \in \mathbb{Z}} \) where

\[
\begin{align*}
\text{Prob}(\Omega_0 = 0) := \text{Prob}(\Omega_0 = 1) := \text{Prob}(\Omega_0 = 2) := \frac{1}{3}
\end{align*}
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and for all $i \in \mathbb{Z}$, given $\Omega_i = 0$

$\text{Prob}(\Omega_{i+1} = 0) := \text{Prob}(\Omega_{i+1} = 1) := \frac{1}{2},$

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It can be verified that this defines a stochastic process.
Recording of the stochastic process
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(\phi(\omega))(i) := \Phi(\omega(i)).
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Though $\phi$ is not injective on $\{0, 1, 2\}^\mathbb{Z}$, with probability one it is injective on the values taken by the stochastic process $(\Omega_i)_{i \in \mathbb{Z}}$: 
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Though \( \phi \) is not injective on \( \{0, 1, 2\}^\mathbb{Z} \), with probability one it is injective on the values taken by the stochastic process \((\Omega_i)_{i \in \mathbb{Z}}\): \( \omega \) can be recovered from \( \phi(\omega) \) by replacing the 1’s by alternating 1’s and 2’s with probability one.
Recording of the stochastic process

In other words, $\Omega$ has been recorded by $\{0, 1\}^\mathbb{Z}$. 
What is an embedding (recording)?

Suppose $\Omega = (\Omega_i)_{i \in \mathbb{Z}}$ is a stationary stochastic process where the $\Omega_i$’s take value in a finite set $A$. 
What is an embedding (recording)?

Suppose $\Omega = (\Omega_i)_{i \in \mathbb{Z}}$ is a stationary stochastic process where the $\Omega_i$’s take value in a finite set $\mathbb{A}$. We say that $\Omega$ is embedded into $\{1, 2, \ldots, k\}^\mathbb{Z}$ if there exists a measurable map

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for which the map $\phi : \mathcal{A}^\mathbb{Z} \to \{1, 2, \ldots, k\}^\mathbb{Z}$ given by

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For instance if \( \omega \in \mathcal{A}^\mathbb{Z} \),

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\phi(\omega)(0) = \Phi(\ldots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2 \ldots)
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For instance if $\omega \in A^\mathbb{Z}$,

$$\phi(\omega)(0) = \Phi(\ldots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2 \ldots)$$

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$$\phi(\omega)(1) = \Phi(\ldots \omega_{-1}\omega_0\omega_1\omega_2\omega_3 \ldots).$$
Embedding captures the idea of recording that we have spoken about until now!
Back to the (infinite) toy question

Question

*Can we embed infinite dice throws into \( \{1, 2\}^\mathbb{Z} \)?*
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Answer.
No!
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To understand why, we need to introduce *entropy* in our setting.
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To understand why, we need to introduce **entropy** in our setting. This replaces the size of the sample space of random variables that we were using before.
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Answer.

No!

To understand why, we need to introduce entropy in our setting. This replaces the size of the sample space of random variables that we were using before.

Suppose \( \Omega \) is a random variable which takes values 1, 2, \ldots, \( k \) with probabilities \( p_1, p_2, \ldots, p_k \). Then the Shannon entropy of \( \Omega \) is given by

\[
H(\Omega) := - \sum_{i=1}^{k} p_i \log(p_i).
\]
Some calculations of Shannon entropy

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If \( \Omega \) is a coin toss then

\[ H(\Omega) = - \log\left(\frac{1}{2}\right) = \log(2). \]
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If \( \Omega \) is a coin toss then

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In general if \( \Omega \) is the uniform random variable taking \( k \) values then

\[ H(\Omega) = \log(k). \]
Some properties of Shannon entropy

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Note that \( \log \) is a strictly concave function. Thus if \( \Omega \) takes only \( k \) values with positive probability then by Jensen’s inequality

\[ H(\Omega) = \sum_{i=1}^{k} p_i \log \frac{1}{p_i} \leq \log k, \]

where equality is attained if and only if \( \Omega \) is a uniform random variable (\( p_i = \frac{1}{k} \) for all \( 1 \leq i \leq k \)).
Some properties of Shannon entropy (contd.)

Suppose $\Omega$ is a random variable taking $k$ distinct values with probabilities $p_1, p_2, \ldots, p_k$. If $\Omega_1$ and $\Omega_2$ are independent copies of $\Omega$, then $(\Omega_1, \Omega_2)$ is a random variable taking $k^2$ distinct values with probabilities $(p_i p_j)_{1 \leq i, j \leq k}$ and

$$H(\Omega_1, \Omega_2) = -\sum_{i,j} p_i p_j \log(p_i) - \sum_{i,j} p_i p_j \log(p_j) = 2H(\Omega).$$

If $(\Omega_1, \Omega_2, \ldots, \Omega_n)$ are independent copies of $\Omega$, then

$$H(\Omega_1, \Omega_2, \ldots, \Omega_n) = nH(\Omega).$$
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If $(\Omega_1, \Omega_2, \ldots, \Omega_n)$ are independent copies of $\Omega$ then

$$H(\Omega_1, \Omega_2, \ldots, \Omega_n) = nH(\Omega).$$
Some more properties of Shannon entropy

If $\Omega$ is a finite-valued random variable taking $k$ values then $H(\Omega) \leq \log k$.
Some more properties of Shannon entropy

If $\Omega$ is a finite-valued random variable taking $k$ values then

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The converse is false. $H(\Omega) \leq \log(k)$ does not imply that $\Omega$ takes only $k$ values:
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The converse is false. $H(\Omega) \leq \log(k)$ does not imply that $\Omega$ takes only $k$ values:

For example, if
$$\Omega := \begin{cases} 1 & \text{with probability } \frac{19}{20} \\ 2, 3 & \text{with probability } \frac{1}{40} \text{ each} \end{cases}$$

then $H(\Omega) = .101 < \log 2$ but takes three different values.
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All this analysis is for a single random variable only, as opposed to a stochastic process.
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The converse is false. $H(\Omega) \leq \log(k)$ does not imply that $\Omega$ takes only $k$ values:

For example, if

$$\Omega := \begin{cases} 1 & \text{with probability } \frac{19}{20} \\ 2, 3 & \text{with probability } \frac{1}{40} \text{ each} \end{cases}$$

then $H(\Omega) = .101 < \log 2$ but takes three different values.

All this analysis is for a single random variable only, as opposed to a stochastic process.

For stochastic processes, we consider ‘entropy-per-site’ instead.
Kolmogorov-Sinai entropy (1958-1959)

Given a stationary stochastic process $\overline{\Omega} = (\Omega_i)_{i \in \mathbb{Z}}$ we define its entropy by

$$h(\overline{\Omega}) := \lim_{n \to \infty} \frac{1}{n} H(\Omega_1, \Omega_2, \ldots, \Omega_n).$$
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If $\Omega = (\Omega_i)_{i \in \mathbb{Z}}$ is a Bernoulli process (independent copies of a random variable $\Omega$) then

$$h(\Omega) := \lim_{n \to \infty} \frac{1}{n} H(\Omega_1, \Omega_2, \ldots, \Omega_n) = \lim_{n \to \infty} \frac{1}{n} nH(\Omega) = H(\Omega).$$
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In general, $h(\Omega) \leq H(\Omega_1)$. 

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Kolmogorov-Sinai entropy (1958-1959)

Theorem (Kolmogorov and Sinai, 1958-1959)

If \( \Omega \) can be embedded in \( \{1, 2, \ldots, k\}^\mathbb{Z} \) then \( h(\Omega) \leq \log k \).

Conversely, Theorem (Krieger's generator theorem (1972))

If \( h(\Omega) < \log k \) then \( \Omega \) can be embedded in \( \{1, 2, \ldots, k\}^\mathbb{Z} \).

The results are sharp.
Kolmogorov-Sinai entropy (1958-1959)

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If $\Omega$ can be embedded in $\{1, 2, \ldots, k\}^\mathbb{Z}$ then $h(\Omega) \leq \log k$.

If $\Omega$ is an infinite sequence of dice throws then

$$h(\Omega) = H(\Omega_1) = \log 6;$$

thus dice throws cannot be embedded in $\{1, 2\}^\mathbb{Z}$.
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Theorem (Krieger’s generator theorem (1972))

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$\mathbb{Z}^d$-stochastic processes
$\mathbb{Z}^d$-stochastic processes

A shift-invariant $\mathbb{Z}^d$-stochastic processes $\Omega = (\Omega_{\vec{i}})_{\vec{i} \in \mathbb{Z}^d}$ is a collection of random variables indexed by $\mathbb{Z}^d$, such that for all $\vec{j} \in \mathbb{Z}^d$,

$$(\Omega_{\vec{i}})_{\vec{i} \in \mathbb{Z}^d} \text{ has the same distribution as } (\Omega_{\vec{i} + \vec{j}})_{\vec{i} \in \mathbb{Z}^d}.$$
$\mathbb{Z}^d$-stochastic processes

A shift-invariant $\mathbb{Z}^d$-stochastic processes $\bar{\Omega} = (\Omega_i^j)_{i \in \mathbb{Z}^d}$ is a collection of random variables indexed by $\mathbb{Z}^d$, such that for all $\vec{j} \in \mathbb{Z}^d$,

$$(\Omega_i^j)_{i \in \mathbb{Z}^d} \text{ has the same distribution as } (\Omega_{i+j}^j)_{i \in \mathbb{Z}^d}.$$

For $d = 1$, this is the same as a stationary stochastic processes.
Entropy for $\mathbb{Z}^d$-stochastic processes

Let $B_n$ be a cube in $\mathbb{Z}^d$ of side length $n$. The entropy is defined by

$$h(\Omega):=\lim_{n\to\infty} \frac{1}{n} dH(\Omega_\vec{i}; \Omega_\vec{i} \in B_n).$$

Recall for $d=1$, we had

$$h(\Omega):=\lim_{n\to\infty} \frac{1}{n} H(\Omega_1, \Omega_2, \ldots, \Omega_n).$$

Again, if $\Omega= (\Omega_\vec{i})_{\vec{i} \in \mathbb{Z}^d}$ are independent copies of the same random variable $\Omega$ then

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Embedding of $\mathbb{Z}^d$-stochastic processes

Suppose $\Omega = (\Omega_i)_{i \in \mathbb{Z}^d}$ is a stationary stochastic process where the $\Omega_i$’s take values in a finite set $A$. 
Embedding of $\mathbb{Z}^d$-stochastic processes

Suppose $\Omega = (\Omega_i)_{i \in \mathbb{Z}^d}$ is a stationary stochastic process where the $\Omega_i$’s take values in a finite set $\mathcal{A}$. We say that $\Omega$ can be embedded in $\{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ if there exists a measurable map

$$\Phi : \mathcal{A}^{\mathbb{Z}^d} \to \{1, 2, \ldots, k\}$$

for which the map $\phi : \mathcal{A}^{\mathbb{Z}^d} \to \{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ given by

$$\phi(\omega)(j) := \Phi((\omega_{i-j})_{i \in \mathbb{Z}^d})$$

is injective with probability one.
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is injective with probability one.

We say that $\Omega$ can be embedded in a set $X \subset \{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ if in addition there exists $\phi$ as above for which $\phi(\omega) \in X$ with probability one.
Embedding of $\mathbb{Z}^d$-stochastic processes

Again, we have,

**Theorem (Robinson and Ruelle, 1967)**

If $\overline{\Omega}$ can embedded in $\{1, 2, \ldots k\}^{\mathbb{Z}^d}$ then $h(\overline{\Omega}) \leq \log k$.

and

**Theorem (Rosenthal, 1988 ($d = 2$) and Kammeyer, 1990 ($d > 2$))**

If $h(\overline{\Omega}) < \log k$ then $\overline{\Omega}$ can be embedded in $\{1, 2, \ldots, k\}^{\mathbb{Z}^d}$.

The results are sharp.
Embedding of $\mathbb{Z}^d$-stochastic processes

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*If* $h(\Omega) < \log k$ *then* $\Omega$ *can be embedded in* $\{1, 2, \ldots, k\}^\mathbb{Z}^d$.

The results are sharp.

But what if we want to embed in some $X \subset \{1, 2, 3, \ldots, k\}^\mathbb{Z}^d$?
Embedding under constraints

Let $X \subset \{1, 2, \ldots, k\}^\mathbb{Z}^d$ be closed and invariant under translations of the $\mathbb{Z}^d$-lattice. We define the topological entropy of $X$ as

$$h_{top}(X) := \lim_{n \to \infty} \frac{1}{n^d} \log(\#\{x|_{B_n} : x \in X\}).$$

The limit exists.
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Let \( X \subset \{1, 2, \ldots, k\}^{\mathbb{Z}^d} \) be closed and invariant under translations of the \( \mathbb{Z}^d \)-lattice. We define the topological entropy of \( X \) as

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$$h_{top}(\{1, 2, \ldots, k\}^{\mathbb{Z}^d})$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \log(\#\{x |_{B_n} : x \in \{1, 2, \ldots, k\}^{\mathbb{Z}^d}\})$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \log |\{1, 2, \ldots, k\}|^{n^d} = \log k.$$
Universality

$X$ is said to be **universal** if all stochastic process $\overline{\Omega}$ for which

$$h(\overline{\Omega}) < h_{\text{top}}(X)$$

$\overline{\Omega}$ can be embedded in $X$.

By the aforementioned results of Krieger, Rosenthal and Kammeyer, $\{1, 2, \ldots, k\}^\mathbb{Z}_d$ are universal.
Motivating Question

When is $X$ universal?
Example: Hom-shifts

We are going to think of $\mathbb{Z}^d$ as both the group and the Cayley graph with respect to standard generators. For instance, $\mathbb{Z}^2$ is the infinite grid.
Example: Hom-shifts

Given graphs $G, H$ a graph homomorphism from $G$ to $H$ is an edge preserving map from the vertex set of $G$ to the vertex set of $H$. 
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Figure: If $f(v)$ is green
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Figure: If $f(v)$ is green then $f(w)$ is either blue.
Example: Hom-shifts

Given graphs \( G, \mathcal{H} \) a graph homomorphism from \( G \) to \( \mathcal{H} \) is an edge preserving map from the vertex set of \( G \) to the vertex set of \( \mathcal{H} \). In other words, \( f : G \to \mathcal{H} \) is a graph homomorphism if \( v \) being adjacent to \( w \) in \( G \) implies that \( f(v) \) is adjacent to \( f(w) \) in \( \mathcal{H} \).

![Graphs G and H with vertices v and w and f(v) adjacent to f(w) in H](image)

Figure: If \( f(v) \) is green then \( f(w) \) is either blue or red.
Example: Hom-shifts

Hom-shifts $X_H$ are the space of graph homomorphisms from $\mathbb{Z}^d$ to $H$. 
Example: Hom-shifts

Hom-shifts $X_{\mathcal{H}}$ are the space of graph homomorphisms from $\mathbb{Z}^d$ to $\mathcal{H}$.

Examples: (Hard core model)
Example: Hom-shifts

Hom-shifts $X_\mathcal{H}$ are the space of graph homomorphisms from $\mathbb{Z}^d$ to $\mathcal{H}$.

Examples: (Proper 3-colourings)
Example: Domino tilings

The space of domino tilings $X_{dom}$ are all possible partitions of $\mathbb{Z}^d$ by rectangular parallelepipeds one of whose side lengths is 2 and rest are 1.
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The space of domino tilings $X_{dom}$ are all possible partitions of $\mathbb{Z}^d$ by rectangular parallelepipeds one of whose side lengths is 2 and rest are 1.

Figure: A domino tiling in $d = 2$. 
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In 2001, Şahin and Robinson initiated the study of the question: When is $X \subset \{1, 2, \ldots, k\}^\mathbb{Z}^d$ universal?
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In particular, they asked whether for $d = 2$, domino tilings and the space of proper 3-colourings are universal.
In 2001, Şahin and Robinson initiated the study of the question: When is $X \subset \{1, 2, \ldots, k\}^Z$ universal?

In particular, they asked whether for $d = 2$, domino tilings and the space of proper 3-colourings are universal.

In 2015, Jackson and Gao reiterated the question (in a stronger form).
Main result (Contd.)
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We answered the question by Şahin and Robinson.
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Theorem (Chandgotia and Meyerovitch, 2018)

1. All hom-shifts (for all $d$), $X_\mathcal{H}$ where $\mathcal{H}$ is not bipartite and
2. the space of domino tilings (for $d = 2$)

are universal.
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We can in fact also ‘embed’ arbitrary measurable $\mathbb{Z}^d$-actions on standard Borel spaces up to a universally null set provided the entropy constraint is satisfied.
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We can in fact also ‘embed’ arbitrary measurable $\mathbb{Z}^d$-actions on standard Borel spaces up to a universally null set provided the entropy constraint is satisfied.

A similar (but more technical) result holds for graphs $\mathcal{H}$ which are bipartite; we will skip it.
What is at stake? (Hom-shifts)
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Fix a connected graph $\mathcal{H}$ and vertices $v, w \in \mathcal{H}$ which form an edge.
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Let $L_n$ be the set of graph homomorphisms from $B_n$ to $\mathcal{H}$ with alternating $v$'s and $w$'s on the boundary and $G_n$ be the set of graph homomorphisms from $B_n$ to $\mathcal{H}$. 
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Figure: The graph $H$

Figure: This is an element of $L^4$ (only blue and green appear on the boundary)

Figure: This is an element of $G^4 \setminus L^4$ (all the three colours appear on the boundary)
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We prove that

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\lim_{n \to \infty} \frac{\log |L_n|}{n^d} = \lim_{n \to \infty} \frac{\log |G_n|}{n^d};
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this is sufficient to prove the universality of hom-shifts when \( \mathcal{H} \) is not bipartite.
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this is sufficient to prove the universality of hom-shifts when \( \mathcal{H} \) is not bipartite.

Note that when \( X_\mathcal{H} := \{1, 2, \ldots, k\}^{\mathbb{Z}^d} \) we have that

\[
|L_n| = 2k^{(n-1)^d} \quad \text{while} \quad |G_n| = k^{(n)^d}
\]

So the equation mentioned above follows automatically.
What is at stake? (Hom-shifts)

In fact, we prove that there is a $c_H > 0$ then given the uniform distribution on $G_n$

$$\text{Prob}(L_n) \geq e^{-c_H n^{d-1}}.$$
What is at stake? (Domino tilings)

Question (Open)

*Are domino tilings universal in all dimensions $d$?*

Recall that $B_n$ is the box of side length $n$ in $\mathbb{Z}^d$. Let $L_{2n}$ be the set of tilings of $B_{2n}$ by dominos and $G_{2n}$ be the set of tilings of $\mathbb{Z}^d$ by dominos restricted to $B_{2n}$. It follows that $L_{2n} \subset G_{2n}$.

![Figure: An element of $L_6$ (on the left) and of $G_6 \setminus L_6$ (on the right)]
What is at stake? (Domino tilings)

If the equation

\[
\lim_{n \to \infty} \frac{\log |L_{2n}|}{(2n)^d} = \lim_{n \to \infty} \frac{\log |G_{2n}|}{(2n)^d}
\]

holds then domino tilings are universal for all dimensions \(d\).

**Fact:** The number on the right is the topological entropy of the space of domino tilings.
What is at stake? (Domino tilings)

If the equation

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holds then domino tilings are universal for all dimensions $d$.

**Fact:** The number on the right is the topological entropy of the space of domino tilings.

For $d = 2$, the equation follows from some deep ideas from Kastelyn (1961) and also from the work of Cohn, Kenyon and Propp (2001). These ideas fail to extend to higher dimensions.
Thank You!