Workshop on Fractals, June 8-12 2014
The Hebrew University of Jerusalem

Sunday June 8 (morning)

8:30 Registration

9:00 Mark Pollicott, *Spiralling geodesics and logarithm laws*

There are classical connections between Diophantine approximation and the time spent by geodesics on negatively curved surfaces in a cusp. We consider related questions on the time spent by geodesics close to a given closed geodesic. There are analogues of Khintchine's theorem on Diophantine approximation and estimates on the Hausdorff Dimension of exceptional sets.

10:00 Coffee

10:30 Barak Weiss, *Badly approximable vectors on fractals*

In joint work with David Simmons, we prove that certain natural fractal measures arising from iterated function systems of contracting similarities, give zero measure to badly approximable vectors. Previous results in this direction involved a reduction to the measure rigidity results of Lindenstrauss. Our approach involved a reduction to the analysis of stationary measures for certain semigroups acting on homogeneous spaces, extending work of Benoist and Quint.

11:30 Lingmin Liao, *Uniform Diophantine approximation related to b-ary and β-expansions*

Let $b \geq 2$ be an integer and v a real number. Among other results, we compute the Hausdorff dimension of the set of real numbers ξ with the property that, for every sufficiently large integer N, there exists an integer n such that $1 \leq n \leq N$ and the distance between $b^n\xi$ and its nearest integer is at most equal to b^{-vN}. We further solve the same question when replacing $b^n\xi$ by $T_\beta^n\xi$, where T_β denotes the classical β-transformation. This is a joint work with Yann Bugeaud.

12:30 Lunch

Sunday June 8 (afternoon)

14:00 Tuomas Sahlsten, *Fourier transforms of fractal measures*

A measure is called a Rajchman measure if its Fourier transform decays to 0 at infinity. It is well-known that the middle third Cantor set cannot support Rajchman measures, but for other suitable classes this can be made to happen. For example, R. Kaufman proved in 80s that the self-conformal Cantor sets given by real numbers with bounded continued fraction expansions in fact support a specific Rajchman measure. Kaufman's original approach is actually quite flexible and we were able to adapt it to prove that a wide class of measures arising from thermodynamical formalism are in fact Rajchman measures with power decay on these Cantor sets. In particular, this applies to the conformal measure of the system with the natural dimension and thus to the Hausdorff measure. Joint work with Thomas Jordan (Bristol).

14:30 Lior Fishman, *Intrinsic, extrinsic and ambient Diophantine approximation on fractals*

Given a point in a subset G of Euclidean space, one may ask about the quality of approximation to this point by rational points that:

1. lie in G (intrinsic approximation)
2. lie outside of G (extrinsic approximation)
3. lie either outside or in G (ambient approximation)

We shall present results and open questions with respect to those three notions of approximation given G is a fractal.

15:00 Coffee

15:30 Alby Fisher, *Renewal Flows: an example of fractal return-time structure in infinite measure ergodic theory*

With some luck, the collection of times of return to a set of finite measure in an infinite measure transformation may exhibit a fractal-like structure. We study the example of a renewal process with regularly varying gap distribution of parameter alpha between 0 and 1, showing its return sets have as a limiting scenery flow the set of points of increase of a Mittag-Leffler process of parameter alpha (which also equals the Hausdorff dimension of the sets). Precisely, this limit approximation is given by a log-average almost-sure invariance principle, i.e. a joining of the two processes such that paths are forward asymptotic under scaling up to a set of times of density zero.

We show the magnification (geodesic) scenery flow of the Mittag-Leffler sets is a Bernoulli flow of infinite entropy, while their translation (horocycle) scenery flow satisfies an order-two ergodic theorem with respect to its natural infinite invariant measure. This result then passes over to the renewal processes themselves, providing as a corollary a new proof of a theorem with Aaronson and Denker.

(Joint with Marina Talet of Aix-Marseille University.)"

16:30 Antti Käenmäki, *Dynamics of the scenery flow and geometry of measures*

We present applications of the recently developed ergodic theoretic machinery on scenery flows to classical geometric measure theoretic problems in Euclidean spaces. We also review the enhancements to the theory required in our work. Our main results include a sharp version of the conical density theorem, which we reduce to a question on rectifiability. Moreover, we show that dimension theory of measure theoretical lower porosity can be reduced back to the set theoretical analogue.

Monday June 9 (morning)

9:00 Jonathan Fraser, *The Assouad dimension of self-similar sets with overlaps*

Historically, the Assouad dimension has been important in the study of quasi-conformal mappings and embedding problems, but recently it has also been gaining some notoriety in the realm of fractal geometry. In general it is an upper bound for the Hausdorff dimension, but it is easily seen that these dimensions coincide for self-similar sets in Euclidean space which satisfy the open set condition. In 2011 Lars Olsen asked if this remained true even if the defining iterated function system had non-trivial overlaps. In my PhD thesis I answered this question in the negative by constructing a self-similar subset of the unit interval which had Assouad dimension one, but whose Hausdorff dimension could be made arbitrarily small. In more recent work in collaboration with Henderson (Reno, Nevada), Olson (Reno, Nevada) and Robinson (Warwick), we have been able to provide the following precise dichotomy. Consider a self-similar set in the line. Either the weak separation property is satisfied, in which case the Hausdorff and Assouad dimensions coincide, or the weak separation property is not satisfied, in which case the Assouad dimension is one, independent of the Hausdorff dimension. In this talk I will discuss this result and its proof and also some examples in higher dimensions which exhibit different types of behaviour.

10:00 Coffee

10:30 Yuval Peres, Brownian motion with variable drift and self-affine graphs

Let B be a Brownian motion in d dimensional space. For any Borel function f from the unit interval to $\mathbf{R}^d$, we express the Hausdorff dimension of the image and the graph of $B + f$ in terms of f. The expression involves an adaptation of the parabolic dimension previously used by Taylor and Watson to characterize polarity for the heat equation. In the case when the graph of f is a self-affine McMullen-Bedford carpet, we obtain an explicit formula for the dimension of the graph of $B + f$ in terms of the generating pattern. In particular, we show that it can be strictly bigger than the maximum of the Hausdorff dimensions of the graphs of f and B. Despite the random perturbation, the Minkowski and Hausdorff dimension of the graph of $B + f$ can disagree. (Joint work with Perla Sousi, Cambridge).

11:30 Karoly Simon, *The large deviation multifractal analysis of a process modeling TCP traffic in the Internet*

I will talk about the large deviation multifractal analysis of certain random real functions which can be presented as infinite sums of certain piecewise continuous random functions. *Our research was motivated by a recent paper:

M. Rams and L. Vehel Large Deviation Multifractal Analysis of a Class of Additive Processes with Correlated Non-Stationary Increments, IEEE/ACM Transactions on Networking 21 (2013).

In this paper, the authors described the large deviation multifractal analysis of the TCP Reno model. However, in the last decades TCP Reno has been replaced by the so-called TCP CUBIC.

Using the methods of Rams and Vehel, we describe the large deviation multifractal analysis for the TCP CUBIC model.

(joint result with Sandor Molnar, Julia Komjathy and Peter Mora)

12:30 Lunch

Monday June 9 (afternoon)

14:00 Julia Romanowska, *On the dimension of the graph of the classical Weierstrass function*

In my talk I will examine dimension of the graph of the famous Weierstrass non- differentiable function

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$$

for an integer $b \geq 2$ and $1/b < \lambda < 1$. In our recent paper, together with Balázs Bárány and Krzysztof Baranski, we prove that for every b there exists (explicitly given) $\lambda_b \in (1/b, 1)$ such that the Hausdorff dimension of the graph of $W_{\lambda,b}$ is equal to $D = 2 + \frac{\log \lambda}{\log b}$ for every $\lambda \in (\lambda_b, 1)$. We also show that the dimension is equal to D for almost every λ on some larger interval. This partially solves a well-known thirty-year-old conjecture. Furthermore, we prove that the Hausdorff dimension of the graph of the function

$$f(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x)$$

for an integer $b \geq 2$ and $1/b < \lambda < 1$ is equal to D for a typical $\mathbb{Z}$-periodic C^3 function ϕ.

In my talk I will talk about these results as well as I will introduce Ledrappier-Young theory and results of Tsujii, which were used in the proofs.

14:30 Tom Kempton, *Slicing Fractals and Equidistribution for sets of beta expansions*

We discuss natural measures on slices through self similar sets, with a particular focus on measures on sets of beta expansions. This allows us to apply work of Lindenstrauss, Peres and Schlag to the question of equidistribution of beta expansions, which in turn is relevant to the study of Bernoulli convolutions.

15:00 Coffee

15:30 Thomas Jordan, *Local dimension of Bernoulli convolutions and related self-affine measures*

We look at the self-affine iterated function (IFS) with maps $T_0(x, y) = (\lambda x, y/2)$ and $T_1(x, y) = (\lambda x + (1 - \lambda), y/2 + 1/2)$ for $\lambda > 1/2$. This IFS was studied in the 1980's by Przytycki and Urbański where they were able to calculate the Hausdorff dimension of the attractor under the condition that the associated Bernoulli convolution has dimension 1. We look at a self-affine measure with weights p and $1 - p$ and study the local dimension spectra for this measure. We show how the local dimension spectra can be related to the local dimension spectra of a biased Bernoulli convolution by adapting Przytycki and Urbański's method. We then look at what can be said about the local dimension spectra for biased Bernoulli convolutions and in turn what this implies for the associated self-affine measures. Joint work with Pablo Shmerkin and Boris Solomyak.

16:30 Balázs Bárány, *Dimension of slices of Sierpinski-like carpets*

We investigate the dimension of intersections of the Sierpisnski-like carpets with lines. We show a sufficient condition that for a fixed rational slope the dimension of almost every intersection w.r.t the natural measure is strictly greater than s-1, and almost every intersection w.r.t the Lebesgue measure is strictly less than s-1, where s is the Hausdorff dimension of the carpet. Moreover, we give partial multifractal spectra for the Hausdorff and packing dimension of slices. This is a joint work with Michal Rams

Tuesday June 10 (morning)

9:00 Joerg Schmeling, On Fourier dimension and its modifications

This talk is joint work with Magnus Aspenberg, Fredrik Ekström and Tomas Persson. We first give an example that the Fourier dimension is not stable under countable unions of Borel sets (due to Fredrik Ekström). Then we propose a way of modifying the Fourier dimension, still having many important properties that are used in metric number theory, fractal geometry and harmonic analysis, but behaving much more regular from the point of view of dimension theory. Finally we show that the measures exceeding a given value in their modified Fourier dimension are characterized by their joint zero sets.

10:00 Coffee

10:30 Elon Lindenstrauss, *Spectral gap and self-similar measures*

In contrast to the two dimensional case, in dimension $d \geq 3$ averaging operators on the $d-1$-sphere using finitely many rotations, i.e. averaging operators of the form $Af(x) = |S|^{-1} \sum_{\theta \in S} f(sx)$ where S is a finite subset of $SO(d)$, can have a spectral gap on L^2 of the $d-1$-sphere.

We prove a uniform spectral gap result for averaging operators corresponding to finite subsets of the isometry group of R^d, which is a semi-direct product of $SO(d)$ and R^d, provided the averaging operator corresponding to the rotation part of these elements has a spectral gap. This spectral gap result has several applications, in particular to the study of self similar measures in $d \geq 3$ dimensions, and can be used to sharpen a previous result of Varju regarding random walks on R^d using elements of the isometry group.

Joint work with Peter Varju.

11:30 Nicolas de Saxce, *On Hausdorff dimension of sum-sets and product-sets*

Given a subset A of the real line, one may study the sets $A+A$ (resp. AA) of elements that can be obtained as a sum (resp. as a product) of two elements of A. Typically, one wants to show that if A has Hausdorff dimension strictly between 0 and 1, then either $A+A$ or AA has larger Hausdorff dimension than A. I will present some results of Bourgain in this context, and then discuss some analogs in the Lie group setting.

12:30 Lunch

14:00 Excursion (bus leaves from conference site)

No talks on the afternoon of Tuesday June 10 due to Excursion

Wednesday June 11 (morning)

9:00 Esa Järvenpää, *Falconer-Sloan condition and random affine code tree fractals*

For self-affine iterated function systems, we study the validity of the Falconer-Sloan condition $C(s)$ introduced by Falconer and Sloan. We verify that, in general, systems having a small number of maps do not satisfy condition $C(s)$. However, there exists a natural number n such that for any typical system the family of all iterates up to level n satisfies condition $C(s)$. Generalising the earlier planar results, we utilise this result to calculate the almost sure dimension for a general class of random affine code tree fractals in $\mathbb{R}^d$.

10:00 Coffee

10:30 Pablo Shmerkin, *Projections of self-similar sets and measures, and their products*

It follows from Marstrand's projection theorem that for any set S of Hausdorff dimension larger than 1, the projection of S onto a line making angle θ with the x-axis has positive Lebesgue measure, outside of a (possible) Lebesgue null set of exceptional angles θ.

When S is a planar self-similar set, or the product of two linear self-similar sets, we improve this by showing the exceptional set of directions has zero Hausdorff dimension. This follows from a corresponding result for measures, that I will also discuss.

This is joint work with B. Solomyak.

11:30 Michał Rams, *Random Palis problem*

I will present results (joint with Karoly Simon) on the Hausdorff dimensions of algebraic sums of randomly generated Cantor sets (fractal percolations). We prove that if $E_1, \ldots, E_k$ are fractal percolations with the sum of expected Hausdorff dimensions $s \leq 1$ then $\dim_H E_1 + \ldots + E_k = s$ almost surely, and when $s > 1$ then $E_1 + \ldots + E_k$ almost surely contains an interval.

12:30 Lunch

Wednesday June 11 (afternoon)

14:00 Yaar Solomon, *The Danzer problem*

Is there a discrete set Y in $\mathbb{R}^d$, with growth rate $O(T^d)$, that intersects every convex set of volume 1? This question is due to Danzer, and it is open since the sixties. I'll present our progress regarding this question, which includes both negative and positive results. On one hand we are able to rule out the candidacy of certain constructions of uniformly discrete sets that arise in natural mathematical constructions, such as sets that correspond to substitution tilings, and cut-and-project sets. On the other hand we construct a Danzer set of growth rate $O(T^d(logT))$ for every d, improving the previous result from 1971 that gave a growth rate of $O(T^d(logT)^{d-1})$. This is a joint work with Barak Weiss.

14:30 Xiong Jin, *Projections and slices of self-similar measures*

We use the local entropy average method introduced by Hochman & Shmerkin and the classical compact group extension theorem to calculate the Hausdorff dimension of projections and slices of self-similar measures on self-similar sets without any separation condition. As a consequence we show that if the rotation group of a self-similar set is finite or connected, then every projection is dimension conserving for the self-similar measures on the self-similar set. This is joint work with Kenneth Falconer.

15:00 Coffee

15:30 Julien Barral, *Inverse problems in multifractal analysis of measures*

Multifractal formalism is designed to describe the distribution at small scales of the elements of $\mathcal{M}_c^+(\mathbb{R}^d)$, the set of positive and compactly supported Borel measures on $\mathbb{R}^d$. It is valid for such a measure μ when its Hausdorff spectrum is the upper semi-continuous function given by the concave Legendre-Fenchel transform of the free energy function τ_μ associated with μ; this is the case for fundamental classes of exact dimensional measures. For any function τ candidate to be the free energy function of some $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, we build such a measure, exact dimensional, and obeying the multifractal formalism. Also, for any upper semi-continuous function candidate to be the lower Hausdorff spectrum of some exact dimensional $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, we build such a measure.

16:30 Márton Elekes *The dimension of the graph and level sets of "almost every" continuous function*

Let $C([0,1]^n, \mathbb{R}^m)$ denote the space of continuous functions from $[0,1]^n$ into $\mathbb{R}^m$ equipped with the sup norm. We examine the various fractal dimensions (Hausdorff, packing, topological) of the graph and fibers (i.e. level sets if $m = 1$) of "almost every" function $f \in C([0,1]^n, \mathbb{R}^m)$, where almost every either means generic in the Baire category sense, or refers to Christensen's notion of a Haar null set in non-locally compact groups.

Thursday June 12 (morning)

9:00 Tamás Keleti, *Are lines bigger than line segments?*

"We pose the following conjecture:

(*) *For any positive integer n, if A is the union of line segments in $\mathbb{R}^n$, and B is the union of the corresponding full lines then the Hausdorff dimensions of A and B agree.*

We prove that this conjecture would imply that every Besicovitch set (compact set that contains unit line segment in every direction) in $\mathbb{R}^n$ has Hausdorff dimension at least $n-1$ and Minkowski dimension n.

We also prove that conjecture (*) holds if the Hausdorff dimension of B is at most 2, so in particular it holds in the plane."

10:00 Coffee

10:30 De-Jun Feng, *Affine embeddings and intersections of Cantor sets*

In this talk, we consider the problem that for given two Cantor sets, whether one of them can be affinely embedded into another one, and whether the intersection of them has a drop in dimension. The talk is based on joint work with Wen Huang and Hui Rao.

11:30 András Máthé, *Disintegrating measures onto Lipschitz curves and surfaces*

Is it true that every set of n^3 points in the three dimensional space contains n^2 points on a Lipschitz surface (with fixed Lipschitz constant)? Is it true that every set of positive measure can be mapped to a ball by a Lipschitz map? These questions are unsolved and related to the differentiability of Lipschitz maps. I will briefly survey this area and talk about a related recent construction of a particular measure in $\mathbb{R}^3$. Let us say that a measure in a Euclidean space has a k-dimenisonal Alberti representation if we can disintegrate it onto measures on k-dimensional Lipschitz surfaces which are absolutely continuous with respect to the k-dimensional Hausdorff measure. We present a measure in $\mathbb{R}^3$ which has two independent 1-dimensional Alberti representations (where independent means that the two families of Lipschitz curves go in separate directions) yet it is supported by a 2-purely unrectifiable set (so it does not have a 2-dimensional representation).

12:30 Lunch

Thursday June 12 (afternoon)

13:55 (Math hall 2) Ville Suomala, *Random measures, intersections, and applications*

In joint work with Pablo Shmerkin, we consider a large class of fractal measures (μ) in $\mathbb{R}^d$ including many 'natural random constructions' such as Poissonian cut-out measures, fractal percolation, and more general multiplicative cascades. We pair these random measures with parametrized families of deterministic measures $(\eta_t)_t$ and study the 'intersection measures' $\mu \cap \eta_t$. In our main result, we show that under some natural conditions on both the random measures and the deterministic family, the intersections $\mu \cap \eta_t$ are defined for all t and behave in a H¨older continuous manner with respect to t. Applying these results for various specific models, we obtain a number of corollaries. For instance, we find classes of random measures μ of given dimension 0<s<d such that a.s.,

1. There are no exceptional directions in the Marstrand-Mattila type projection and intersection theorems.
2. If $s > k$, all orthogonal projections of μ onto k-planes have a H¨older continuous density (If $d = 2$ and $s > 1$, this holds for all polynomial projections, but the projections are only piecewise H¨older).
3. Almost surely, the support of μ intersects all OSC self-similar sets E in a set of dimension $\leq \max(0, s + \dim E - d)$
4. If μ_1 and μ_2 are independent realizations of the random measure and if $s > d/2$, then the convolution $\mu_1 * S(\mu_2)$ is absolutely continuous with a H¨older density, for all $S \in GL_d(\mathbb{R})$. In particular, if A_1 and A_2 denote the supports of μ_1, μ_2, then $A_1 + G(A_2)$ has nonempty interior for all S.
5. If $s > d/2$, for all $S \in GL_d(\mathbb{R})$, the self-convolution $\mu * S(\mu)$ is absolutely continuous with Hölder density and $\text{supp}\mu + S(\text{supp}\mu)$ has nonempty interior..
6. If A is the support of μ and E is a given Borel set such that $\dim E > d - \min(1, s)$, then $A + S(E)$ has nonempty interior for all $S \in GL_d(\mathbb{R})$.
7. We also find the sharp dimension threshold for the support of μ to containing given patterns (all progressions, all angles etc.)

14:30 (Math hall 2) Colloquium (S. Sodin)

15:00 (Math lounge) Coffee

15:30 (Math hall 2) David Simmons, *Geometry and dynamics of groups acting on Gromov hyperbolic metric spaces*

In this talk, I will discuss two theorems about groups acting by isometries on Gromov hyperbolic metric spaces. The first theorem is a generalization of a theorem of Bishop and Jones ('97) and Paulin ('97) to this setting. The second is a construction of Patterson-Sullivan measures in a setting where compactness is not assumed. Both theorems are part of an ongoing collaboration with Tushar Das (University of Wisconsin - La Crosse) and Mariusz Urba\'nski (University of North Texas).

16:30 (Math hall 2) Kenneth Falconer, *Intersections of random isometric images of subsets of Cantor sets*

Let C be the regular m-ary Cantor set and let E and F be compact subsets of C. Let G be the group of isometries of C (corresponding to the automorphism group of the rooted m-ary tree) endowed with the natural invariant probability measure. We will discuss a codimension formula for the dimension of $E \cap \sigma(F)$ where $\sigma \in G$ is a random isometry.