RECURRENCE AND TOPOLOGICAL ENTROPY

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Here are a few interesting problems relating recurrence and topological entropy.

The first I got from Benjy Weiss, and appears in his paper "Multiple recurrence and doubly minimal systems", in *Topological dynamics and applications*, Contemporary Mathematics 1998, volume 215,pages 189–196.

Problem. Let $T : X \to X$ be a homeomorphism of a compact metric space. Suppose that every $(x, y) \in X \times X$ is two-sided recurrent, i.e. there is a sequence $n_k \neq 0$ such that $(T^{n_k}x, T^{n_k}y) \to (x, y)$. Does this imply $h(X, T) = 0$?

Weiss showed that under the additional assumption that $n_k > 0$ – i.e. that every point in the product is forward recurrent – then the answer is positive. But nothing seems to be known about the two-sided case above.

Forward recurrence of every point in $X \times X$ implies that the system is topologically deterministic in the sense of Kamiński, Siemaszko, and Szymański (Bulletin of the Polish Academy of Sciences. Mathematics 2003, Vol. 51, no 4, pp. 401–417). This can be defined in a number of equivalent ways, among them:

(1) Every continuous $S : Y \to Y$ which is a factor of (X, T) must be invertible.
(2) For every continuous function $f \in C(X)$,

$$f \in \langle 1, T^n f \,:\, n = 1, 2, 3, \ldots \rangle$$

where $\langle \mathcal{F} \rangle$ is the closed algebra in $C(X)$ generated by $\mathcal{F}$.

Note that (2) is a generalization of the condition that every finite partition is measurable with respect to its past (or future), which for measure preserving systems implies 0-entropy. For measure preserving systems 0-entropy also implies that the map is invertible on a set of full measure, and since 0-entropy implies the same for factors we see that 0-entropy for measure-preserving systems means that each factor is essentially invertible (and positive entropy systems have factors which are not).

Kamiński et. al. proved that a topolgically deterministic system has zero topological entropy. In my paper *On notions of determinism in topological dynamics* I generalized this to $\mathbb{Z}^d$-actions, in the sense that if $f \in C(X)$ and

$$f \in \left\langle 1, T^u f \,:\, 0 < u \in \mathbb{Z}^d \right\rangle$$

then $h(T) = 0$. Here $<$ is the lexicographical order on $\mathbb{Z}^d$.

Both the original proof for $\mathbb{Z}$-actions and my proof for $\mathbb{Z}^d$ use in an essential way the fact that $\{u < 0 \,:\, u \in \mathbb{Z}^d\}$ is the "past" of the process. For general groups there is no good notion of past. However one can still ask the following question, which makes sense for any amenable group:

Problem. Suppose an infinite discrete amenable group G acts by homeomorphisms on X. Let $S \subseteq G$ be a semigroup that does not contain 1_G. Suppose that for every $f \in C(X)$ we have $f \in \langle sf \,:\, s \in S \rangle$. Does this imply that $h(X, G) = 0$?

A positive answer would be interesting even under stronger assumptions on S.