Are the periodic points dense in multidimensional strongly irreducible SFTs?

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This is one of the most frustrating problems I know. Let Σ be a finite alphabet and recall that

1. A subshift $X \subseteq \Sigma^{\mathbb{Z}^d}$ is a *shift of finite type* (SFT) if there is a finite set $E \subseteq \mathbb{Z}^d$ and a family $L \subseteq \Sigma^E$ such that

$$X = \{x \in \Sigma^{\mathbb{Z}^d} \text{ : no pattern } a \in L \text{ appears in } x\}$$

2. A subshift $X \subseteq \Sigma^{\mathbb{Z}^d}$ is *strongly irreducible* if there is a constant $R > 0$ such that, for any two patterns $a \in \Sigma^A$, $b \in \Sigma^B$, if $\|a - b\| > R$ for all $a \in A$ and $b \in B$, then there is an $x \in X$ with $x|_A = a$ and $x|_B = B$.

3. A periodic point for the shift action on $\Sigma^{\mathbb{Z}^d}$ is a point whose stabilizer has finite index in $\mathbb{Z}^d$.

Problem If X is a strongly irreducible shift of finite type, are the periodic points dense?

This is known to be true in dimensions $d = 1$ and $d = 2$. For $d = 1$ it is classical and very easy once one knows how to represent an SFT as the space of infinite paths in a directed graph. For $f = 2$ there is a clever argument which can be found e.g. in Sam Lightwood's paper "Morphisms from non-periodic $\mathbb{Z}^2$-subshifts. I. Constructing embeddings from homomorphisms", Ergodic Theory Dynam. Systems 23 (2003), no. 2, 587–609. (I do not know if this is where the argument originates).

For $d \geq 3$ the problem is open. I have very little to contribute to this question, except one observation. An old argument by Wang shows that if the periodic points in an SFT are dense then the extension problem is solvable, that is, there is an algorithm that, for any finite pattern $a \in \Sigma^F$, decides whether there is an $x \in X$ with $x|_F = a$. The extension problem for general SFTs is not decidable in any dimension $d \geq 2$, but in my paper with Tom Meyerovitch we've shown that for strongly irreducible SFTs in any dimension it is solvable. This may be viewed as very week support of a positive answer to the problem.