NOTE ON ONE-SIDED AND TWO-SIDED ERGODIC THEOREMS

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Abstract. We show that, for non-integrable f in a measure preserving system, the limits of the forward, backward and two-sided ergodic averages coincide if any of them exist, improving results of Yves Derriennic whos proved that they converge or diverge together. Similar results were recently obtained by Benjamin Weiss.

1. Notation and statement

Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic probability-preserving system and let $f : X \to \mathbb{R}$ be a measurable function. Write

$$A_n f = \frac{1}{2n+1} \sum_{i=-n}^{n} T^i f$$

for the two-sided ergodic average, and denote the one-sided ergodic averages by

$$\begin{aligned} A_n^+ f &= \frac{1}{n+1} \sum_{i=0}^{n} T^i f \\ A_n^- f &= \frac{1}{n+1} \sum_{i=0}^{n} T^{-i} f \end{aligned}$$

When convenient we omit f and write A_n, A_n^+, etc. We also write $A_n(f)(x)$ or $A_n(x)$ for evaluation at $x \in X$. More generally if $I \subseteq \mathbb{Z}$ is finite (generally an interval), we write

$$A_I f = \frac{1}{|I|} \sum_{i \in I} T^i f$$

and omit f when convenient.

If $f \in L^1$ then the ergodic theorem tells us that all three converge μ-a.e. to the same constant function of value $\int f d\mu$. When f is not integrable, however, the relation between the limits, if they exist, is less clear. The following is known:

Theorem. *If $A_n^+ f$ converges a.e. then so does $A_n^- f$.*

Yves Derriennic has a proof based on representing f as the sum of an L^1 function and a special type of co-boundary.

During the special semester on ergodic theory and additive combinatorics at MSRI in the fall of 2008, Benjamin Weiss raised the question of the relation between convergence of one sided and two-sided averages for non-integrable functions. We give below a direct proof below that ll three must converge to the same limit. We remark that Benjamin Weiss has independently answered this question using similar methods.

Theorem 1. *Each of the limits of* A_n, A_n^+ *and* A_n^- *exists a.e. if and only if the others do, and in this case they are all equal.*

We do not know what happens for the analogous questions in $\mathbb{Z}^2$.

2. Proof

Lemma 2. *If any one of the averages* $A_n^\pm$ *converges then its limit is invariant. Similarly, the limsup and liminf of these sequences are invariant.*

Proof. Note that A_n^+ and

$$\widehat{A}_n^+ f = \frac{1}{n}\sum_{i=1}^{n} T^i f$$

have the same limits whenever they converge (after replacing $1/n$ with $1/(n+1)$ in $\widehat{A}_n^+$ they differ by the term $f/(n+1)$), but $\widehat{A}_n^+ f = A_{n-1}^+(Tf)$. Hence the limit is invariant. □

Lemma 3. *If* $A_n f$ *converges a.e. to a limit* g *and the limit is finite with positive probability, then* g *is constant.*

Proof. Choose a k such that there is a positive measure set B of points at which both $A_n f$ and $A_n(T^k f)$ converge to a finite limit. For every point in B the difference $A_n f - A_{n-k}(T^k f)$ converges to a finite limit, and this difference and the quantity

$$\Delta_n^k = \frac{1}{n+1}(T^{-n} f + T^{-n+1} f + \ldots + T^{-n+2k} f)$$

differ by a term tending to 0.

The set where $\lim \Delta_n^k$ exists is clearly invariant and it contains B, hence Δ_n converges a.e. The limit function $h = \lim_n \Delta_n^k$ is also seen to be invariant, and thus equal to some constant c; since the limit $\lim_n \Delta_n^k$ is finite on B, we see that c is finite.

It follows that $T^k g = g + c$, with the understanding $\infty + c = \infty$, so the set where $g = \infty$ is invariant under T^k, and similarly the set where $g = -\infty$.

Denote by $X_0, \ldots, X_{k-1}$ the ergodic components of T^k, so these sets are disjoint and $T^i X_0 = X_{i \bmod k}$. Suppose without loss of generality that g is finite on X_0 (we know it is finite on at least one of them by assumption that g is finite with positive measure, and since $\{g = \infty\}$ and $\{g = -\infty\}$ are T^k-invariant). Suppose $g = \infty$ on some X_m for $0 < m < k$. Define Δ_n^m as above; it follows that $\lim_n \Delta_n^m = \infty$ a.e. But then we also have that

$$\lim_n (A_{n+km} - A_n) = \infty$$

a.e., which is impossible, since it must be finite on X_0. This shows that $g < \infty$ a.e.; one similarly shows that g>-∞ a.e.

The same proof as above shows that $Tg = g + c/k$. It remains to show that $c = 0$. If $c \geq 0$ then for any M the set $g \geq M$ is forward invariant and the set $g \leq M$ is backward invariant (if $c \leq 0$ the directions are reversed). This implies that $|g| = \infty$ a.e. unless $c = 0$. Hence $c = 0$ and $Tg = g$. □

Proposition 4. *If A_n converges a.e. then so does A_n^+.*

Proof. Suppose $A_n \to c$ a.e. for a constant c. Let $\varepsilon > 0$ be such that on some set B with $\mu(B) > 0$ we have $\limsup A_n^+ f > \varepsilon$ and $\liminf A_n^+ f < -\varepsilon$ (we may ensure this by subtracting an appropriate constant from f). Assume without loss of generality that $c \leq 0$; otherwise we can apply the argument to $-f$.

Choose $\delta > 0$ and n_0 so that

$$\mu(|A_n(f) - c| > \frac{1}{10}\varepsilon \text{ for some } n > n_0) < \delta$$

Let B denote the event above. By the mean ergodic theorem, for almost every point x, if n is large enough then

$$\frac{1}{n}\#\{1 \leq i \leq n \,:\, T^i x \in B\} < \delta$$

and for a suitable constant L and every large enough n we have

$$\frac{1}{n}\#\{1 \leq i \leq n \,:\, |f(T^i x)| > L\} < \delta$$

Choose a typical $x \in X$ and choose $n \geq n_0$ and large enough so that the two conditions above hold and such that $A_n^+ f > \varepsilon$.

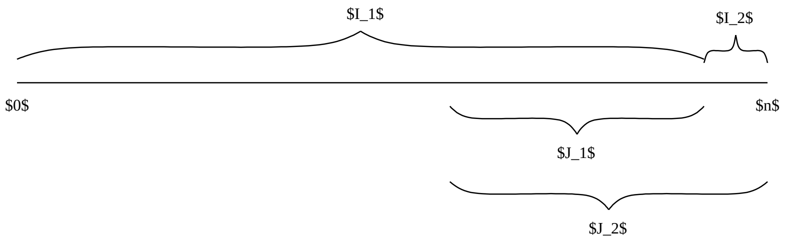
We now claim that we can write $[0, n]$ as a union $[0, n] = I_1 \cup I_2$ of intervals with centers i_1, i_2 and lengths $2r_1 + 1, 2r_2 + 1$ respectively, and intersecting at most at one point, and such that

$$T^{i_1} x \in B \text{ and } |f(T^{i_1 + r_1} x)| \leq C$$

and furthermore

$$r_2 \leq \delta n$$

It is clear that what we must find is only an index $i_i \in [\frac{1}{2}n - 2\delta n, \frac{1}{2}n]$ that satisfies the above for the choice $r_1 = i_1$, and this can be done because the interval $[\frac{1}{2}n - 2\delta n, \frac{1}{2}n]$ has length $2\delta n + 1$, while each of the properties of i_1 that we wish to avoid each has density $\leq \delta n$.



It follows that

$$\varepsilon < A_n^+(x) = \frac{|I_1|}{n} A_{I_1}(T^{i_i}x) + \frac{|I_2|}{n} A_{I_2}(T^{i_2}x) + \tau \cdot \frac{f(T^{i_1+r_1}x)}{n}$$

where $\tau = 0$ or 1, depending on whether $I_1 \cap I_2$ is empty or not (this depends on the parity of n, which we do not control). Since $|A_{I_1}(T^{i_1}x) - c| \leq \frac{1}{10}\varepsilon$ we have

$$A_{I_2}(T^{i_2}x) \geq \frac{4\varepsilon}{5} \cdot \frac{n}{|I_2|}$$

Next, using similar reasoning we can find segments $J_1 \subseteq J_2 \subseteq [0,n]$ of lengths that are odd and at equal to at least $2n_0 + 1$, so that their centers j_1, j_2 satisfy $T^{j_!}x, T^{j_2}x \in B$ and such that $J_2 = J_1 \cup I_2$ and $J_1 \cap I_2$ contains at most one point. It follows that

$$A_{J_2}(T^{j_2}x) = \frac{|J_1|}{|J_2|} A_{J_2}(T^{j_2}x) + \frac{|I_2|}{|J_2|} A_{I_2}(T^{i_2}x) + \tau \cdot \frac{f(T^{i_1+r_1}x)}{|J_2|}$$

with τ as before. But the first two averages above are within $\frac{1}{10}\varepsilon$ of c and hence $\leq \frac{\varepsilon}{10}$, and this contradicts the previous estimate for $A_{I_2}(T^{i_2}x)$. □

Proposition 5. *If A_n^+ converges to a finite limit then so does A_n^-, and the limits are the same.*

Proof. Similar to the above. Briefly, assume $\lim A_n^+ \leq 0$ and $\lim A_n^- \geq \varepsilon > 0$. For typical x and large n we have $A_n^+ \leq 0$ and can find $i \in [n-\delta n, n]$ so that $A_n^-(T^i x) > \varepsilon$. Hence $\sum_{j\in[i,n]} T^j x \geq \frac{n}{2\varepsilon}$. But we can find $i' < i$ so that $A^+_{[i',i]}, A^+_{[i',n]} \leq 0$ and this is a contradiction. □