Irreducibility and periodicity in $\mathbb{Z}^2$ symbolic systems

Michael Hochman*

Abstract

We show that there exist $\mathbb{Z}^2$ symbolic systems that are aperiodic and strongly irreducible.

1 Introduction

1.1 Statement of the main result

Many mixing conditions have been introduced in the theory of $\mathbb{Z}^d$ symbolic dynamics, with the goal of finding a class of systems which bahaves analogously to one-dimensional case. This line of investigations begins with Ruelle's notion of specification [14], which was introduced in the context of the theormodynamic formalism. Other aspects of the theory have lead to other notions: block gluing, corner gluing, uniform filling and square mixing (e.g. [6, 13, 10, 12, 3, 7].

The strongest and perhaps most natural notion is that of strong irreducibility (e.g. [4]): Given $E, F \subseteq \mathbb{Z}^d$, one says that a $\mathbb{Z}^d$-subshift $X \subseteq A^{\mathbb{Z}^d}$ admits **gluing** along E, F if for every $x, y \in X$ there exists a point $z \in X$ such that $z|_E = x|_E$ and $z|_F = y|_F$. In this case z is called a gluing of $x|_E, y|_F$. A subshift X is **strongly irreducible (SI) with gap** $g \geq 0$ if it admits gluing along every pair of sets E, F satisfying $d(E, F) > g$.

Strongly irreducible systems behave in many respects like one-dimensional mixing shifts of finite type: They factor onto any full shift of lower entropy [6, 9], and if a subshift factors into a SI subshift then it also embeds bijectively into it, giving a version of Krieger's embedding theorem [10, 11, 2]. They admit fully supported measures of maximal entropy, although Burton and Steif proved that these need not be unique, nor Bernoulli [4].

The intersection of the class of strongly irreducible systems with the class of the shifts of finite type is even better-behaved. These systems, unlike general SFTs, have a decidable language and computable entropy [8]. In dimension $d = 2$, Lightwood showed that the periodic points are dense, so SI $\mathbb{Z}^2$-SFTs satisfy specification in the sense of Ruelle, and these systems contain isomorphic copies of every minimal subshift of lower entropy, fully generalizing Krieger's embedding theorem. It is not known whether the last two properties hold in $\mathbb{Z}^d$ for $d \geq 3$ [10, 11].

*Research supported by ISF research grant 3056/21.

In the present paper we consider the question, raised in [5], of existence of periodic points in SI systems (without assuming the SFT propery). Here, a **periodic point** means a point with finite orbit under the shift action. For $\mathbb{Z}$-subshifts, an elegant argument by Bertrand [1, Theorem 1] and, independently, Weiss [5, Theorem 5.1] shows that strongly irreducible $\mathbb{Z}$-subshifts admit dense periodic points (see [5]), and as we have noted, the same is true in $\mathbb{Z}^2$ if we add an assumption that the subshift is an SFT.

Joshua Frisch has pointed out to us that the question of existence of periodic points in SI $\mathbb{Z}^2$ subshifts is heuristically related to the question of their existence in SI SFTs in $\mathbb{Z}^3$. Indeed, if X is a SI $\mathbb{Z}^2$-subshift, then for every k, the $\mathbb{Z}$-subshift consisting of restrictions of its configurations to $\mathbb{Z} \times \{1, \ldots, k\}$ (with $\mathbb{Z}$ acting by horizontal shift) is again SI, so by Weiss's result it contains $\mathbb{Z}$-periodic points. Similarly, if Y is a SI $\mathbb{Z}^3$ SFT, then for every k one can consider locally admissible configurations on $\mathbb{Z}^2 \times \{1, \ldots, k\}$, and this is a SI $\mathbb{Z}^2$ SFT, so by Lightwood's result it contains $\mathbb{Z}^2$-periodic points. In both cases, there is no obvious way to extend this lower-dimensional periodicity to actual periodic points, but one might hope that there was a way to do this.

Our main result is

Theorem 1.1. *For every $d \geq 2$, there exist strongly irreducible $\mathbb{Z}^d$ subshifts without periodic points.*

We prove the theorem for $d = 2$, but this implies it for all $d > 2$ by considering the subshifts whose 2-dimensional slices belong to our example. Our construction is non-canonical and far from optimal in every sense. We obtain a result for gap 10 and a very large alphabet, and have made no attempt to minimize these parameters. But we see no obstruction to there existing examples with gap 1 and alphabet $\{0, 1\}$.

We also note there do not exist so many "non-trivial" examples of SI subshifts – previous examples either had "free symbols" or "free subsets of symbols", or were factors of systems with such symbols. Our example does not rely on this mechanism.

.

1.2 Discussion

A point $x \in A^{\mathbb{Z}^2}$ is aperiodic if and only if, for each $n \in \mathbb{N}$, there exists a pair $(u_n, v_n) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ with $u_n = v_n \bmod n$, and $x_{u_n} \neq x_{v_n}$. Such a pair will be called an ***n*-aperiodic pair for** x. The set of configurations that posses an n-aperiodic pair for a given n, or for all n, is closed under shifts, but not under taking limits. To ensure that the orbit closure of x contains no periodic points, one must require that there exists a sequence of sets $\mathcal{W}_n \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$ and a sequence $R_n > 0$ such that

- **Aperiodicity:** Every $(u, v) \in \mathcal{W}_n$ is an n-aperiodic pair for x.
- **Syndeticity:** Every closed R_n-ball contains u, v for some $(u, v) \in \mathcal{W}_n$.

If we are given the sequence $R_n \to \infty$, the set $X = X_{(R_n)}$ of all configurations that admit sets $\mathcal{W}_n$ as above is an aperiodic subshift, and if $R_n \to \infty$ quickly enough then $X \neq \emptyset$.

This construction never yields a SI subshift in dimension $d = 1$, for, by Weiss's result, if $X_{(R_n)}$ were SI it would contain periodic points, which it does not. Our main theorem shows that this argument cannot be applied in higher dimensions, but the following construction shows that for at least some sequences R_n, the subshift $X_{(R_n)}$ is not SI.

Example 1.2. For $A = \{0, 1\}$ we shall construct a sequence $R_n \to \infty$ such that $X_{(R'_n)} \subseteq \{0, 1\}^{\mathbb{Z}^2}$ is not SI. More precisely we construct an integer sequence $n_k \to \infty$ such that each integer ℓ divides some n_k, and a sequence $r_k \to \infty$. We then set $R_n = \min\{r_k \mid k \in \mathbb{N}\ ,\ n|n_k\}$, and take $X_{(R_n)}$ to be our example.

Fix a one-dimensional aperiodic sequence $z \in \{0, 1\}^{\mathbb{N}}$ and an auxiliary sequence $s_n \to \infty$ with the property that every subword of length s_n of z contains an n-aperiodic pair. A square pattern will be said to be z-lined if the pattern along its sides (excluding the corners) are an initial segment of z. Every square $N \times N$ pattern can be extended to a z-lined pattern of dimensions $(N + 2) \times (N + 2)$. Observe that

> If we tile a large square with z-lined $(2s_n + 2) \times (2s_n + 2)$ patterns, then every $2s_n \times 2s_n$ sub-pattern b will contain the initial s_n-segment of z along some row or column, a therefore b contains an n-aperiodic pair.

Given a sequence $r_1 < \ldots, < r_k$, we say that an $r_k \times r_k$ pattern is acceptable (with respect to $r_1, \ldots, r_k$ and $n_1, \ldots, n_k$) if for every $m \leq k$, every $r_m \times r_m$ subpattern of a admits an n_m-aperiodic pair..

We shall inductively define integers $n_1, n_2, \ldots$ and $r_1 < r_2 < \ldots$ with $r_m \geq 2s_{n_m}$, and acceptable patterns $x^{(n)}$ over $\{0, 1\}$, each $x^{(n)}$ being a square pattern centered at z and extending $x^{(n+1)}$, as follows.

For $k = 1$ let $n_1 = 1$ and let $x^{(1)}$ be any non-1-periodic pattern of dimension $2s_1 \times 2s_1$ centered at 0 and with $x^{(1)}_0 = 0$. Set $r_1 = 2s_1$. Evidently $x^{(1)}$ is acceptable, since the only $r_1 \times r_1$ sub-pattern of $x^{(1)}$ is $x^{(1)}$ itself.

Now suppose that $n_!, \ldots, n_k$ and $r_1 < \ldots < r_k$ have been defined with $r_m \geq 2s_m$, and let $x^{(k)}$ be an acceptable pattern with respect to them.

Choose the minimal $g \in \mathbb{N}$ and the lexicographically least pattern $b \in \{0, 1\}^{[-g,g]^2}$ with the property that

$$b_0 = 1$$

and such that the $r_k \times r_k$ pattern $y^{(k)}$ given by

$$y^{(k)} = \begin{cases} y^{(k)}_u = x^{(k)}_u & u \in [r_k \times r_k]^2 \setminus [-g, g]^2 \\ b_u & u \in [-g, g]^2 \end{cases}$$

is also acceptable. Let

$$n_{k+1} = k(r_k + 2)$$

and let

$$r_{k+1} \geq 2s_{n_k}$$

be a multiple of n_k.

To define $x^{(k+1)}$, first extend $x^{(k)}, y^{(k)}$ to z-lined patterns $\widehat{x}^{(k)}, \widehat{y}^{(k)}$ of dimensions $(r_k+2) \times (r_k+2)$. Then extend $\widehat{x}^{(k)}$ to an $r_{k+1} \times r_{k+1}$ pattern by surrounding $\widehat{x}^{(k)}$ with translates of $\widehat{y}^{(k)}$. We require that the original $\widehat{x}^{(k)}$ not be in one of the corners of $x^{(k+1)}$.

We claim that $x^{(k+1)}$ is acceptable with respect to $r_1, \ldots, r_{k+1}$ and $n_1, \ldots, n_k$. Indeed,

- The only $r_{k+1} \times r_{k+1}$ sub-pattern is $x^{(k+1)}$ itself, and $\widehat{x}_0^{(k)} = 0 \neq 1 = \widehat{x}_{(0,r_{k+1})}^{(k+1)}$ (because $(0, r_{k+1})$ is the center of a copy of $\widehat{y}^{(k)}$).
- For $1 \leq m \leq k$, if b is an $r_m \times r_m$ sub-pattern of $\widehat{x}^{(k+1)}$. then either it is a sub-pattern of $x^{(k)}$ or a translate of $y^{(k)}$, in which case b contains an n_m-aperiodic pair because $x^{(k)}, y^{(k)}$ are acceptable, or else b intersects the boundary of a translate of $\widehat{x}^{(k)}$ or $\widehat{y}^{(k)}$ in a segment of length at least $r_m/2$, and there one sees an initial segment of z of length $\geq r_m/2 - 1 = s_m$, so there is an n_m-aperiodic pair.

Now, the sequence $x^{(k)}$ increases to $x \in \{0,1\}^{\mathbb{Z}^2}$ and clearly $x \in X_{(R_n)}$ for R_n defined as above. On the other hand, form the construction it is clear that for every gap g, if we set flip $x_0 = 1$, there is no way to redefine $x|_{[-g,g]^2 \setminus \{0\}}$ in a manner that will keep x in $X_{(R_n)}$. For we have constructed the $x^{(k)}$ precisely so that however we fill in the symbols in $[-g,g]^2$, this pattern will have been tried at some step int he construction, and inserting it results in a periodic $r_{k+1} \times r_{k+1}$ rectangle tiled by copies of $\widehat{y}^{(k)}$ and this pattern is $n_k = 2r_k + 1$ periodic.

This construction demonstrates why controlling the effects of gluing is so difficult: when a finite region is cut out of a configuration and another pattern glued in its place, the change potentially can create periodicity at any one of infinitely many "scales". Thus, if one wants to glue in a way that prevents periodicity, one potentially must satisfy infinitely many constraints, arising from symbols arbitrarily far away. At the same time, we are free to re-define only finitely many symbols near the boundary where the gluing occurred. So there are not always enough degrees of freedom to ensure aperiodicity.

We note that the example above produced very special sequences R_n. We do not know if some sequences (R_n) can give rise to SI subshifts $S_{(R_n)}$. Even if the answer is negative, it seems possible that subshifts defined as above can lead to SI aperiodic systems in the followins sense.. Given a rule S for gluing configurations together, every subshift X defines a strongly irreducible closure $\langle X \rangle_S = \bigcup_{n=1}^{\infty} X_n$, where $X_1 = X$ and X_{n+1} is obtained by gluing patterns from X_n in all possible ways using S. One can then ask if there is a sequence $R_n > 1$ and a gluing rule S such that $\left\langle X_{(R_n)} \right\rangle_s$ is aperiodic?

1.3 Ideas from the construction

In order to obtain an aperiodic and SI subshift we will have to strengthen the aperiodicity and syndeticity assumptions above, thus making the aperiodicity more "robust".

Our problem is as follows. When z is a gluing of $x|_E$ and $y|_F$, it may happen that some pair $(u, v) \in E \times F$ is an n-aperiodic pair for x or for y (i.e. $u = v \bmod n$, and $x_u \neq x_v$ or $y_u \neq y_v$) but not for z (i.e. $z_u = x_u = y_v = z_v$).

Since we are free to re-define the symbols in $(E \cup F)^c$, we can try to resolve this by creating a new aperidic pair, e.g. by "moving" u to the set $(E \cup F)^c$, whose size is at least proportional to the boundary of E. But this strategy faces two problems:

1. There may not be enough space in $(E \cup F)^c$ to create new n-aperiodic pairs for every pair that was destroyed. Indeed, we can only assume that $(E \cup F)^c$ is proportional to the length of the boundary of E, whereas, potentially, the number of sites belonging to n-aperiodic pairs in E may be proportional to the size of E itself. Actually the situation is not quite this bad, since we need not worry about pairs (u, v) for which both u and v in the same set E or F, so, assuming we bound the distance between u, v by some R_n, we need "only" worry about n-aperiodic pairs that lie within R_n of the boundary of E. Nevertheless, since R_n grows with n, when E is finite the number of pairs that need fixing easily exceeds the number of sites we can use to fix them.

   Consequently, in addition to the syndeticity condition on n-aperiodic pairs, we will impose a complementary "sparseness" condition to ensure that the number of n-aperiodic pairs $(u, v) \in E \times F$ is at most proportional to the size of the boundary.

2. When one member $u \in E$ of an aperiodic pair $(uv) \in E \times F$ is "moved" from E to a site $u' \in (E \cup F)^c$, we may not be able to ensure that $u = u' \bmod n$, so, even though we may be able to ensure different symbols at u' and v, the pair (u', v) is no an n-aperiodic pair. Instead, we could try to pair u' with a new site v' to get an n-aperiodic pair, but it may happen that all points that agree with u' modulo n are already part of an aperiodic pair, or that this cannot be done without violating the sparsity condition discussed in (1).

   To avoid this problem, we will not work with aperiodic pairs (u, v) as above. Instead, introduce larger "aperiodic sets" $W = \{u_1, \ldots, u_{N_n}\} \subseteq \mathbb{Z}^2$ which will be required to always contain an aperiodic pair among its members. The point of working with a larger collection of sites is that if W is large enough then there are guaranteed to be pairs in it lying in the same residue class mod n. Thus, if some element of W is replaced by another, we can hope to choose the symbol at the new site so as to make sure that W still contains an n-aperiodic pair.

We have arrived at the following strategy. For a given gap $g > 0$, we will define a subshift consisting of configurations x that admit a sequence $\mathcal{W}_n$, with each $\mathcal{W}_n$ consisting of subsets $W \subseteq \mathbb{Z}^2$ of size N_n, with the following properties:

- **Aperiodicity:** Every $W \in \mathcal{W}_n$ contains a pair $u \neq v$ with $u = v \bmod n$ and $x_n \neq x_v$ Furthermore, the pattern $x|_W$ will be "robust" in the sense that its properties can be preserved despite local changes.

- **Syndeticity:** For every closed R_n-ball B there exists $W \in \mathcal{W}_n$ with $W \subseteq B$.
- **Sparsity:** For every finite region E, the set $\{u \in \mathbb{Z}^2 \setminus E \mid d(u,E) > g\}$ is larger than $\sum_n |E \cap (\cup \mathcal{W}_n)|$.

These conditions are good approximations of the ones we will actually use.

It is important to keep in mind that **the structure ($\mathcal{W}_n$) and other auxiliary structure supporting it will not be encoded in the configurations of the subshift**. This makes the structure flexible, and when gluing configurations these structure may change also far from the gluing boundary. We shall call this auxiliary structure a **certificate**. Thus, in order to belong to our subshift, a point must admit compatible certificate, and our task will be to show that, when two configurations x, y are glued together to give z, we can simultaneously "glue" their certificates to give an admissible certificate for z.

1.4 Standing notation

Throughout the paper we write $a \gg b$ for "a sufficiently large relative to b". Thus, a statement of the form "If $a \gg b$, then..." means "For every b, if a is sufficiently large in a manner depending on b, then...". A statement such as "Since $a \gg b$ we have..." means "Since we are free to choose a arbitrarily large relative ot b, we can choose it large enough that...".

1.5 Organization of the paper

In Section 2 we explain how to address the robustness alluded to in paragraph (2) above.

In Section 3.1 we explain the sparsity requirement we impose in order to satisfy paragraph (1) above.

In Section 4 we describe the construction in detail.

In Section 5 we show that the resulting subshift is strongly irreducible.

Section 6 contains further remarks and open problems.

1.6 Acknowledgment

I would like to thank Joshua Frisch for bringing this problem to my attention.

2 Coins and buckets (a combinatorial lemma)

In this section we analyze a combinatorial game which arises in our context when x is a configuration, $W \in \mathcal{W}_n$ are as described at the end of the introduction, and a site from W is moved to a new location which we do not control, but we are able to choose the symbol of x at the new location. We want to ensure that after the move the set W still contains two sites that are equal modulo n but carry different symbols.

This leads us to the following model, in which the buckets represent residue classes, coins represent sites $u \in W$, and the value x_u is represented by the orientations of the coin (Heads or Tails; we assume a binary symbol). We note that in our application later on, the buckets will represent other data in addition to the residue classes.

- A **coin-and-bucket configuration** is an arrangement of finitely many coins in finitely many buckets.

- Each coin has an **orientation** of "heads" or "tails".

- Configurations are modified by applying one or more **moves** in sequence. A move consists of choosing a bucket B and orientation σ, removing from B a coin of orientation σ, and placing it in a bucket B' with orientation σ', where σ' must be equal to the less common orientation among the coins that were in the destination bucket before the transfer, unless there was a tie, in which case we can specify the orientation of the moved coin.

  A move is legal for a given configuration if the source bucket contains a coin of the desired orientation.

- A bucket is **oriented** if all the coins in it have the same orientation. A configuration is oriented if all its buckets are oriented (the orientation can depend on the bucket).

- A configuration is **orientable** if there exists a legal sequence of moves that, when applied to the configuration, leads to an oriented configuration. If no such sequence exists, the configuration is **unorientable**.

Note that when coins are moved from one bucket to another we often do not control the orientation they will receive. In particular, if a coin is added to a non-empty bucket, that bucket cannot be oriented after the move. Observe also that that legal moves do not change the total number of coins in a configuration, and that in each bucket the coins of a given orientation are interchangeable. in the sense that if one of them is to be moved, it does not matter which it is.

One can interpret this as a game. The first player sets up the initial configuration, and the second player applies a sequence of moves (choosing orientations as he pleases when relevant). The second player wins if he manages to reach an oriented configuration.

Let $c_{k,n}$ denote the configuration consisting of k buckets, all of which are empty except for the first, which contains $n_1 = n$ coins, out of which $n_2 = \lfloor n_1/2 \rfloor$ are heads and $\lceil n_1/2 \rceil$ are tails. Consider the following sequence of moves starting from $c_{k,n}$. First, move n_2 heads from the first bucket to the second bucket, one by one, orienting them as "tails" whenever there is a choice. Thus, when the first coin lands in an empty bucket its orientation is tails, the second coin is forced to be heads, the next is again tails, and so on. When all n_2 coins have been transferred, we will have $n_3 = \lfloor n_2/2 \rfloor$ heads and $\lceil n_2/2 \rceil$ tails in the second bucket. Now move all heads from the second bucket to the third in the same manner, and continue, ending in the k-th bucket. At the end. $\lceil n_i/2 \rceil$ tails remain in the buckets $i = 1, \ldots, k-1$, and n_k coins remain in the last bucket. The final configuration is be oriented if and only if $n_k \leq 1$, and, tallying how many coins are left in each bucket have shown that $c_{k,n}$ is orientable if $n \leq 2^{k+1} - 1$. Conversely,

Proposition 2.1. *If $n \geq 2^k$ then $c_{k,n}$ is unorientable.*

We begin the proof. Assume that k, n are given and that $m_1 \ldots m_N$ is a sequence of legal moves taking $c = c_{k,n}$ to an oriented configuration c'. Our goal is to deduce that $n \leq 2^k - 1$. Our strategy will be to reduce the given moves to a different sequence, in which each bucket is dealt with separately; then we will be able to directly analyze the situation siilarly to the exmaple above.

A modified game

To simplify analysis, we first modify the rules of the game. Introduce a special bucket called "the pile", which serves as a temporary holding place for coins. While in the pile, coins do not have a specific orientation. We only allow moves that transfer a coin from a bucket to the pile, or from the piile to a bucket. The game begins with all coins in the piule, and must end in an empy pile and oriented buckets.

We translate the original list of moves to this new game as follows:

1. Append n new moves $m_{-n}, \ldots, m_{-1}$ to the start of the sequence, each one taking a coin from the pile to the first bucket, requiring it to be set to "tails" if relevant.

2. An original move that transfers a coin from bucket B to bucket B' is replaced by two moves:L one taking a coin from B to the pile, and the other taking a coin from the pile to B'.

Evidently, starting from n coins on the pile and k empty buckets, the first n moves in the updated sequence resut in the configuration c, and the ramining move bring us to c'.

We keep the same notation, using $m_1, \ldots, m_N$ for the updated sequence of moves. Our goal is still to show that $n \leq 2^k - 1$.

Final states and how to reach them

For each bucket B, let

$$\ell_B = \# \text{ of coins in } B \text{ in the final configuration } c'$$

and let σ_B denote their common orientation (if B is empty in c' set σ_B =tails). Let $\sigma'_B \neq \sigma_B$ denote the other orientation, and set

$$\ell'_B = \max\{\ell_B - 1, 0\}$$

Define the seqeunce of moves $\underline{m}_B = m_{B,1}, \ldots, m_{B,\ell_B + 2\ell'_B}$ as follows:

- The first $\ell_B + \ell'_B$ moves insert coins into B, with th provision that if relevant, the orientation is σ_B.
- The last ℓ'_B moves remove σ'_B-oriented coins from B.

Then, if we apply $\underline{m}_B$ starting from a configuration in which B is empty and there are at least $\ell_B + \ell'_B$ coins in the pile, then the net result is that ℓ_B coins are transfered to B with orientation σ_B. Note that no shorter sequence of moves can achieve this result.

Peak states

We say that B is in its **peak state** if the number p_B of σ_B-coins in it is maximal relative to all states it passes through in the course of the game, and, subject to this, the number p'_B of σ'_B-coins in it is maximal. Note that $p_B \geq \ell_B$.

Let t_B denote the maximal integer $1 \leq t_B \leq N$ such that, after performing the move m_{t_B}, the bucket is in its peak state, but was not before. Observe that the move at time t_B, if it exists, must move a coin to B, so the times t_B are distinct for different buckets, provided that $p_B > 0$, while if $p_B = 0$, then the bucket remains throughout, and we can delete all such buckets and reduce k. Thus, we may assume all t_B to be distinct.

Separating moves of different buckets

We now perform a sequence of changes on the list of moves. In order not to confuse matters by frequent re-indexing, we think of the index i of the move m_i as part of the move, rather than its position in the list; so after deleting other moves and/or inserting moves, the move will still carry the index i (new moves do not receive an index at this point). Thus, if we began with $m_1m_2m_3m_3$, insert two new moves m', m'' in the middle, and delete m_2, we are left with the sequence $m_1m'm''m_3m_4$, so m_3 refers to the fourth element of the sequence.

We now modify the list of moves as follows: for each bucket B in turn, replace m_{t_B} with the sequence of moves $\underline{m}_B$ above, and delete all other moves involving B.

Then the new sequence is a list of legal moves taking the initial configuration to c'. Indeed, It is clear that if the sequence is legal, then it results in c'. To see that it is indeed legal, one should note that we are moving fewer than $p_B + p'_B$ moves into B and doing so no early than t_B, so at times before t_B, the pile is larger than it previously was; thus, there will be no shortage of coins for moves involving other buckets, or for the moves that put coins into B. Similarly, we return the maximal number of coins to the pile after time t_B, so there will be no shortage afterwards.

Completing the proof

The situation now is this: starting from the configuration c of empty buckets and n coins in the pile, we have k sequences of moves, $m^{(1)}, \ldots, m^{(k)}$, such that

1. $m^{(i)}$ moves $\ell_{B_i} + \ell'_{B_i}$ coins from the pile to B_i and then returning ℓ'_{B_i} coins of orientation σ_{B_i} to the pile,
2. The concatenated sequence $m = m^{(1)} m^{(2)} \ldots m^{(k)}$ takes c to the oriented configuration c'.

Let p_i, $i = 1, \ldots, k$, denote the number of coins in the pile after completing the i-th sequence $m^{(i)}$ and $p_0 = n$. Then

- The block of moves $m^{(i)}$ is legal and transfers $\ell_{B_i} + \ell'_{B_i}$ coins from the pile to B_i, so
$$p_i \geq \ell_{B_i} + \ell'_{B_i}$$
- $\ell_{B_i} \leq \ell'_{B_i} + 1$ so
$$p_i \geq 2\ell_{B_i} - 1$$
- $p_{i+1} = p_i - \ell_{B_i}$, so
$$\begin{aligned} 2p_{i+1} &= 2p_i - 2\ell_{B_i} \\ &\geq 2p_i - (p_i + 1) \\ &= p_i + 1 \end{aligned}$$
- m$\ell_k \leq 1$ and $p_k = 0$, so $p_{k-1} \leq 1$. Using the relation $p_i \leq 2p_{i+1} + 1$ from the previous step, it follows inductively that
$$p_{k-i} \leq 2^{i-1} + 2^i + \ldots + 1 = 2^i - 1$$
and in particular
$$n = p_0 \leq 2^k - 1$$

This concludes the proof.

3 Paths avoiding sparse obstacles (more combinatorics)

In this section we give a condition that, when satisfied by a family of obstacles in the plane, ensures that every point sufficiently far from the obstacles lies on an infinite, relatively flat polygonal path that avoids the obstacles entirely, and such that this property persists under changes to obstacles sufficiently far from the point.

3.1 Definitions and main statement

We introduce the following definitions:

- A **(rectilinear) rectangle** of dimensions $w \times h$ is a translate of $[0, w] \times [0, h]$.
- A **(rectilinear) diamond** of dimensions $w \times h$ is a translate of the closed convex hull of the points $(\pm w/2, 0), (0, \pm h/2)$. The points in a diamond with maximal and minimal y-coordinate are called the north/south poles, and the horizontal line segment passing through its center of mas is called its equator.
- Given a bounded measurable set $E \subseteq \mathbb{R}^2$ of positive Lebesgue measure and $c > 0$, denote by cE the convex set that is a translate of $\{cx \mid x \in E\}$ and has the same center of mass as E. .
- Give a collection $\mathcal{E}$ of bounded convex sets and $c > 0$, we say that $\mathcal{E}$ is **c-sparse** if for every $E, E' \in \mathcal{E}$ with $E \neq E'$, we have $E \cap cE' = cE \cap E' = \emptyset$.
- A **polygonal graph** $\gamma \subseteq \mathbb{R}$ is the graph of a piecewise linear function $f : \mathbb{R} \to \mathbb{R}$. If the absolute values of slopes of all line segments in γ is $\leq \alpha$ we say that γ has **slope** $\leq \alpha$ (equivalently, f is α-Lipschitz).
- A **polygonal path** will mean a polygonal path in some (not necessarily standard) coordinate system

The remainder of this section is devoted to the proof of the following theorem.

Proposition 3.1. *Let $w_1, \ldots, w_N > 0$ and $h_1, \ldots, h_N \geq 0$. If*

$$\begin{aligned} h_n &\gg w_{n-1}, h_{n-1} \quad \text{for } n = 1, \ldots, N-1 \\ w_n &\gg h_n, w_{n-1} \quad \text{for } n = 1, \ldots, N \end{aligned}$$

then the following holds: To every sequence $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_N)$, where $\mathcal{R}_n$ is a 80-sparse collection of rectilinear rectangles of dimensions $w_n \times h_n$, we can associate a set $\text{safe}(\mathcal{R}) \subseteq \mathbb{R}^2$*, the set of **safe points** relative to $\mathcal{R}$, such that*

1. $\mathbb{R}^2 \setminus \bigcup_{n=1}^{N} \bigcup_{R \in \mathcal{R}_n} 20R \subseteq \text{safe}(\mathcal{R}_1, \ldots, \mathcal{R}_N) \subseteq \mathbb{R}^2 \setminus \bigcup_{n=1}^{N} \bigcup_{R \in \mathcal{R}_n} 2R$
2. *Every vertical line segment of length $8h_N$ intersects* $\text{safe}(\mathcal{R})$.
3. *Every $p \in \text{safe}(\mathcal{R})$ lies on a polygonal graph $\gamma \subseteq \text{safe}(\mathcal{R})$ of slope $\leq h_1/w_1$.*
4. *For every $p \in \text{safe}(\mathcal{R})$, if $\mathcal{R}' = (\mathcal{R}'_1, \ldots, \mathcal{R}'_N)$ with $\mathcal{R}'_n$ a 40-sparse family of rectilinear $w_n \times h_n$ rectangles, and if $\mathcal{R}'$ agrees with $\mathcal{R}$ near p in the sense that*

   $$\{R \in \mathcal{R}_n \mid p \in 40R\} = \{R' \in \mathcal{R}'_n \mid p \in 40R'\}$$

   then $p \in \text{safe}(\mathcal{R}')$.

When applying the theorem, we often identify $(\mathcal{R}_1, \dots, \mathcal{R}_N)$ with $\mathcal{R} = \bigcup_{n=1}^N \mathcal{R}_n$. There is no ambiguity in this, provided that all rectangles in $\mathcal{R}_n$ have dimensions $w_n \times h_n$, and that these dimensions are distinct for different n, since then $\mathcal{R}_1, \dots, \mathcal{R}_N$ can be reconstructed from $\bigcup_{n=1}^N \mathcal{R}_n$.

We shall apply the proposition later to collections of rectangles in arbitrary orthogonal coordinate system. Since this section is devoted to the proof of the proposition above, we work only with rectilinear rectangles (and diamonds), and therefore omit the qualifier "rectilinear" for convenience.

The remainder of this section is devoted to the proof of Proposition 3.1.

3.2 Jigsaws

Definition 3.2. A **jigsaw** is a non-negative piecewise linear function function supported on a bounded interval. An α-jigsaw is a jigsaw of slope $\leq \alpha$ (equivalently, α-Lipschitz).

If $p = (x_0, y_0) \in \mathbb{R}^2$ with $y_0 \geq 0$, define the function

$$\Delta_{p,\alpha}(x) = \max\{0, y_0 - \alpha|x - x_0|\}$$

This is the α-jigsaw supported on $[x_0 - y_0/\alpha, x_0 + y_0/\alpha]$. Its graph on this interval is an isosceles triangle whose base is the interval above, whose sides of slope $\pm\alpha$, and whose its peak at p.

Lemma 3.3. *Let f be a jigsaw and $p = (x_0, y_0) \in \mathbb{R}^2$ with $y_0 \geq f(x_0)$. Suppose that $\beta > \alpha$ and set*

$$r = \frac{y_0 - f(x_0)}{\beta - \alpha}$$

Suppose that f has slope at most α on $I = [x_0 - r, x_0 + r]$. Then

$$g = \max\{f, \Delta_{p,\beta}\}$$

is a jigsaw such that

- $f \leq g \leq f + \Delta_{(x_0, y_0 - f(x_0)), \beta - \alpha}$.
- $g|_{\mathbb{R}\setminus I} = f|_{\mathbb{R}\setminus I}$.
- *The slope of g on I is bounded by β* .

We omit the elementary proof.

3.3 Double jigsaws

We identify a jigsaw with the closed set that it bounds with the x-axis. More precisely,

Definition 3.4. Given a jigsaw g whose minimal closed supporting interval is I, let

$$A(g) = \{(x, y) \mid x \in I \ , \ 0 \leq y \leq g(x)\}$$

If h is a function such that $-h$ is a jigsaw, define $A(h)$ similarly:

$$A(h) = \{(x, -y) \mid (x, y) \in A(-h)\}$$

By joining regions of the types $A(g), A(h)$ above one can form more complicated shapes (e.g. dimaonds):

Definition 3.5. A **double jigsaw** $G \subseteq \mathbb{R}^2$ **with equator** J is a set of the form $G = (A(g^+) \cup A(-g^-)) + p$ and an interval $J = I + p$, where

- $p \in \mathbb{R}^2$.
- g^+, g^- are jigsaws.
- I is the minimal closed interval supporting g^+, g^-.

If g^+, g^- are α-jigsaws then G is called a double α-jigsaw.

Note that G and J and p determine I, g^+, g^- uniquely, and vice versa.

3.4 The hull of collections of diamonds

Let $\vee$ denote the maximum operator between numbers and functions, and observe that $f \vee g$ is jigsaw whenever f, g are.

Definition 3.6 (Adjoining a diamond to a double jigsaw)**.** Let $I = [-r, r]$ be a symmetric interval and g^+, g^- jigsaws supported on I, and let $G = A(g^+) \cup A(-g^-)$ denote the corresponding double jigsaw.

Let D be an α-diamond (we do not require the equator to be on the x-axis). Let q^+, q^- be the north and south poles of D respectively. Then the double jigsaw obtained by **adjoining** D to G, denoted $G \ltimes D$, is the set $A(h^+) \cup A(-h^-)$, where

- $h^+ = g^+ \vee \Delta_{q^+,\alpha}$ if q^+ is in the upper half plane, and $h^+ = g^+$ otherwise.
- $h^- = g^- \vee \Delta_{-q^-,\alpha}$ if q^- is in the lower half plane, and $h^- = g^-$ otherwise.

When the equator $I \subseteq \mathbb{R}^2$ of G is not contained in the x-axis, let p be the center of I, and set

$$G \ltimes D = ((G - p) \ltimes (D - p)) + p$$

Remark 3.7. Note that

1. $G, D \subseteq G \ltimes D$.
2. The equator of $G \ltimes D$ contains the equator of G.

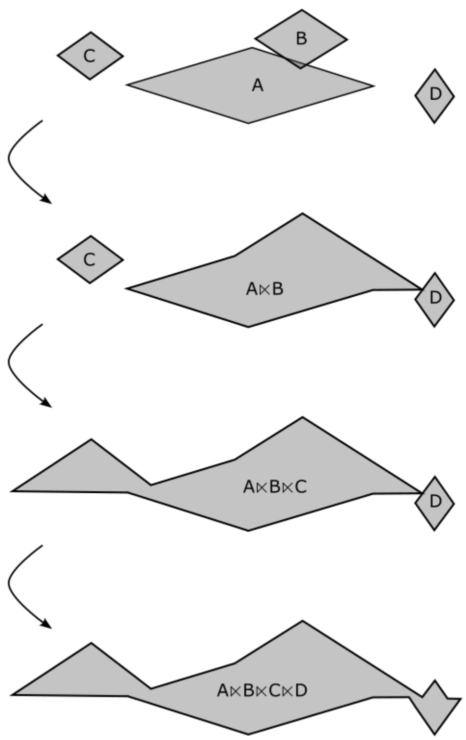

Figure 1: Merging diamonds to create their hull.

3. The operation $\ltimes$ is asymmetric, because the right-hand term must be a diamond, while the left can be a jigsaw. But for pairs of diamonds it is commutative, because the maximum operation is. This allows the operation to be extanded to pairs of jigsaws generated by diamonds, and it is comutative on this class.

Definition 3.8 (The hull of a family of diamonds)**.** Let $\mathcal{D}_1, \ldots, \mathcal{D}_N$ be families of diamonds, with all elements of $\mathcal{D}_n$ having dimensions $w_n \times h_n$ with $w_1 < w_2 < \ldots < w_N$.

The **hull** $\mathcal{H} = \mathcal{H}(\mathcal{D}_1, \ldots, \mathcal{D}_N)$ of the sequence is the collection $\mathcal{H}$ of double jigsaws constructed as follows:

Initialize $\mathcal{H} = \emptyset$.

For each n from N down to 1,

For each $D \in \mathcal{D}_n$,

(a) If $d(D, G) > h_n$ for all $G \in \mathcal{H}$, add D to $\mathcal{H}_n$.

(b) Otherwise, choose a set $G \in \mathcal{H}_n$ with $d(D, G) \leq h_n$, delete G from $\mathcal{H}$ and replace it with $G \ltimes D$.

There are many possible orders with which to choose the diamonds in round n of the construction, and there may be more than one choice for G in step (b). We assume for now that these choices have all been made according to some rule that is fixed in advance, so as to make the hull well defined. The next proposition shows that, under some separation assumptions, these choices do not affect the outcome.

Proposition 3.9. *Let $\mathcal{D}_1, \ldots, \mathcal{D}_N, w_n, h_n$ be as in Definition 3.8. Let $\alpha_n = h_n/w_n$, so $\mathcal{D}_n$ consists of α_n-diamonds, write $h_{\leq n} = \sum_{i \leq n} h_i$ and $w_{\leq n} = \sum_{i \leq n} w_i$. Suppose that*

(a) $\mathcal{D}_n$ is 20-sparse.

(b) $h_n > 10h_{\leq n-1}$ and $w_n > 10w_{\leq n-1}$ for all $1 < n \leq N$.

(c) $\alpha_n < \frac{1}{10}\alpha_{n-1}$ for $n > 1$.

Then

1. *If $D \in \mathcal{D}_n$ is merged with a double jigsaw $J \in \mathcal{H}$ is step (b) of Definition 3.8, then $(J \ltimes D) \setminus J$ is properly contained in a rectangle of dimensions $5w_n \times 5h_n$ that contains D.*

   In particular, since $\mathcal{D}_n$ is 20-separated, the order with which diamonds are chosen in round n of the construction does not affect whether part (a) or (b) are invoked for a given diamond.

2. *If $D \in \mathcal{D}_n$ was added in step (a) of round n of Definition 3.8, then in the final hull, the double jigsaw J containing D is properly contained in a rectangle of dimensions $(w_n + 10w_{\leq n-1}) \times (h_n + 10h_{n-1})$, and hence in a rectangle of dimensions $2w_n \times 2h_n$.*

3. *The double jigsaws in $\mathcal{H}$ are pairwise disjoint, and every vertical line segment of length $2h_N$ contains points in the complement of $\mathcal{H}$.*

4. *If $p \in \mathbb{R}^2 \setminus \cup\mathcal{H}$, then there exists a polygonal graph γ passing through p, disjoint from $\mathcal{H}$, and with all line segments in the path having slope at most α_1 in absolute value.*

5. *Let $p \in \mathbb{R}^2 \setminus \cup\mathcal{H}$. Suppose that $\mathcal{D}'_1, \ldots, \mathcal{D}'_N$ are collections of diamonds satisfying the same conditions as $\mathcal{D}_1, \ldots, \mathcal{D}_N$ and agreeing near p with the original collection, in the sense that for each $1 \leq n \leq N$,*

   $$\{D \in \mathcal{D}_n \mid p \in 10D\} = \{D' \in \mathcal{D}'_n \mid p \in 10D'\}$$

   Then $p \in \mathbb{R}^2 \setminus \mathcal{H}'$, where $\mathcal{H}'$ is the hull of $\mathcal{D}'_1, \ldots, \mathcal{D}'_N$.

Proof. (1) First, By induction, when the construction enters stage n, every $J \in \mathcal{H}$ has slopes at most α_{n+1} (or 0 when $n = N$).

For each n, we induct on the sequence (D_i) of elements of $\mathcal{D}_n$ processed in stage n of the construction. Consider what happens when we reach D_i, and its distance from some double jigsaw $J \in \mathcal{H}$ is less than h_n. Let (x_0, y_0) denote the center of D_i. Observe that J arose by adjoining zero or more of the diamonds $D_1, \dots, D_{i-1}$ to a double jigsaw J' that was in $\mathcal{H}$ at the beginning of stage n, and, by our induction hypothesis, each time one of these D_j was adjoined, the change to J' was limited to a box of dimensions $5w_n \times 5h_n$; since $\mathcal{D}_n$ is 20-sparse, these changes are disjoint from the box of the same dimensions with center (x_0, y_0). Thus, the jigsaws defining J agree with those defining J' on the interval $[x_0 - 5w_n, x_0 + 5w_n]$, and in particular its slopes on this interval are at most α_{n+1}. This implies, in particular, that the closest point in D_i to J is its top or bottom vertex (or that one of these is actually contained in J), so the distance of the other vertex from the jigsaw must be at most $2h_n$. Now we invoke Lemma 3.3 to see that adjoining D_i affects J only in the interval $[x_0 - r, x_0 + r]$ for $r = 2h_n/(\alpha_n - \alpha_{n+1})$, and since all changes happen in a region whose boundary has slopes α_n, we have confined the change to a box of dimension $2r \times (\alpha_n \cdot 2r)$. Using the definition of r, and the assumptions $h_n = \alpha_n w_n$ and $\alpha_{n+1} < \alpha_n/10$, we see that the changes are confined to a box of dimensions $5w_n \times 5h_n$ containing D_i.

The second part of (1) follows from the first part and the sparsity assumption, because together they imply that the region added to the jigsaw when $D_i \in \mathcal{D}_n$ is added does not come within h_n of any other diamond in $\mathcal{D}_n$, and so does not affect the choice of option (a) or (b) in future stages of the ocnstruction.

(2) follows because (1) shows that the additions in stage n do not interact, and so if J is a double jigsaw in $\mathcal{H}$ before entering stage n, then it is extended by at most $5h_n$ upwards and downwards, and $5w_n$ left and right, in the course of stage n. Each double jigsaw in the hull started from a diamond $D \in \mathcal{D}_n$ for some n, whose dimensions are $w_n \times h_n$. Thus the cumulative vertical height of a double jigsaw is $< h_n + 10h_{\leq n-1} < 2h_n$ and the cumulative width is $< w_n + 10w_{\leq n-1} < 2w_n$.

(3) The first part follows from (2), using 20-sparsity. The second part follows from (2) and the first part of (3), since they imply that any vertical line intersects elements of $\mathcal{H}$ in disjoint intervals of length $\leq 2h_N$.

(4) If p is in the complement of $\cup\mathcal{H}$, we can form a polygonal path through p by traveling horizontally in either direction; if we hit a double jigsaw in $\mathcal{H}$, we follow its boundary until it is possible to continue horizontally in the original direction, at which point we do so. This path is not quite disjoint from $\cup\mathcal{H}$, since part of it may lie in the boundary of $\cup\mathcal{H}$, but we can perturb the path by slightly raising or lowering each line segment that follows a boundary of an element of $\mathcal{H}$ in order to avoid the boundary entirely.

(5) This again follows from (1), because in the course of the construction of the hull, any diamond $D \in \mathcal{D}'_n \setminus \mathcal{D}_n$ is far enough away that adjoining cannot capture p nor cause p to be closer than h_n to an element of $\mathcal{H}$ if it was not this close already. □

3.5 Application to rectangles

Proof of proposition 3.1. If R is a rectangle of dimensions $w \times h$, let $\diamondsuit(R)$ denote the closed diamond of dimensions $2w \times 2h$ with the same center as R, whose diagonals are parallel to the sides of R, and that contains R.

Let $\mathcal{R}_1, \ldots, \mathcal{R}_N$ be as in the statement and define

$$\mathcal{D}_n = \{\Diamond(2R) \mid R \in \mathcal{R}_n\}$$

Then $D \in \mathcal{D}_n$ are 20-sparse sets of diamonds of dimensions $4w_n \times 4h_n$. Define $\text{safe}(\mathcal{R}_1, \ldots, \mathcal{R}_N) = \text{safe}(\mathcal{D}_1, \ldots, \mathcal{D}_N)$. Statements (2)–(4) are now immediate consequences of the corresponding statements in Proposition 3.9.

For (1), note that since $\text{safe}(\mathcal{R}_1, \ldots, \mathcal{R}_N)$ is disjoint from all $D \in \mathcal{D}_n$ it is disjoint from $2R$ for all $R \in \mathcal{R}_n$, leading to the right inclusion in (1). For the left inclusion, note that by part (1) of Proposition 3.9, for each $D \in \mathcal{D}_n$ there is a $5w_n \times 5h_n$ rectangle $\widetilde{R}(D)$ containing D such that

$$\mathcal{H}(\mathcal{D}_1, \ldots, \mathcal{D}_N) \subseteq \bigcup_{n=1}^{N} \bigcup_{D \in \mathcal{D}_N} \widetilde{R}(D)$$

Since $\widetilde{R}(D) \subseteq 10D$, and since there is a rectangle $R \in \mathcal{R}_n$ such that $D = 2R$, and hence $\widetilde{R}(D) \subseteq 20R$, we conclude that

$$\mathcal{H}(\mathcal{D}_1, \ldots, \mathcal{D}_N) \subseteq \bigcup_{n=1}^{N} \bigcup_{R \in \mathcal{R}_n} 20R$$

This implies the left inclusion in (1). □

4 Main construction

4.1 Notation and definitions

- Elements of $\mathbb{R}^2$ are called **points**.

  Elements of $\mathbb{Z}^2$ are called **sites**.

- A **disk** $D \subseteq \mathbb{R}^2$ is a closed Euclidean ball. If the radius of a disk is r, we call it an r-disk.

- A set $E \subseteq \mathbb{R}^2$ is r**-separated** if $\|x - y\| \geq r$ for all distinct $x, y \in E$.

- A set $E \subseteq \mathbb{R}^2$ is R**-dense** if it intersects every R-disk.

The following definition is not standard:

- Let $u \in \mathbb{R}^2$ be a unit vector and D an r-disk. An **almost radial rectangle** R with **orientation** u (relative to the disk D) is a rectangle satisfying

  1. $R \subseteq D$.
  2. $R \cap \partial D \neq \emptyset$.

3. The long sides of R are parallel to u, and when the long sides are extended in direction u, the resulting ray passes within $\frac{1}{100}r$ of the center of D.

The orientation of R is

$$\theta(R) = u$$

4.2 Parameters

For each $n \in \mathbb{N}$, define the parameter

$$N_n = 2^{1,000,000 \cdot n^2}$$

By Theorem **??**, there exist unorientable coin-and-bucket configurations with $1000^2 n^2$ buckets. We introduce two sequences of real parameters

$$h_1 \leq h_2 \leq \dots$$
$$w_1 \leq w_2 \leq \dots$$

representing the height and width of rectangles that will appear in the construction, and

$$r_n = \frac{10}{9} w_n$$

representing radii of disks. We assume further that

$$\frac{h_1}{w_1} \ll 1$$

and that

$$h_{n+1} \gg w_n, h_n, n$$
$$w_{n+1} \gg h_{n+1}, w_n, n$$

4.3 Certificates and their parts

Recall from the introduction that our goal is to define a structure called a certificate, consisting of

- A sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$, where
- Each $\mathcal{W}_n$ is a ("syndetic" and "separated") collection of sets $W \subseteq \mathbb{R}^2$, and
- Each W is a finite set whose elements are called **witnesses**.

  The coordinates of $w \in W$ are real, but from w a site in $\mathbb{Z}^2$ by taking the coordinate-wise integer part.

To help foordinate this data we shall introduce additional geometric structures:

- Each $W \in \mathcal{W}_n$ will be contained in a Euclidean disk called a **frame**.
- Each witness $w \in W$ will be contained in an associates **box**, which is an almost radial rectangle relative to the frame that w belongs to (it is long and thin and approximates a radius of the frame).
- Each box will be further subdivided into rectangular **sections**.

Here, finally, is the full definition. A **certificate** consists of

- Countably many **levels**, numbered $n = 1, 2, 3, \ldots$ (corresponding to the sets $\mathcal{W}_n$).
- The n-th level consists of countably many r_n-discs, called n-**frames**.
  - The **center** of an n-frame F is denoted $c(F)$.
  - The set of centers of n-frames in a certificate is $\frac{1}{2}r_n$-separated.

  If the set of centers of n-frames is $10r_n$-dense, we say that the n-th level is **dense**. The certificate is dense if all its levels are dense. We allow non-dense certificates, which arrise naturally in the intermediate stages of constructing a certificate.
- Each n-frame F contains N_n rectangles $B_1, \ldots, B_{N_n}$called n-**boxes**, satisfying
  - The dimensions of n-boxes is $w_n \times h_n$.
  - The B_i are almost-radial relative to F.
  - Every two distinct boxes $B_i \neq B_j$ satisfy $d(B, B') \geq 100h_n$.
  - All the B_i in a frame have a common orientation, and furthermore the orientations lies in the set
  $$\Theta = \{\frac{2\pi k}{1000} \mid 0 \leq k < 1000\}$$
  - Each n-box B associated to a frame F is divided into 1000 rectangular **sections** $B^{(1)}, B^{(2)}, \ldots, B^{(1000)}$ of dimensions $(w_n/1000) \times h_n$, numbered by increasing distance from the center of F.
  - The set of indices of sections is denoted
  $$\Sigma = \{1, \ldots, 1000\}$$
  and the section of a point $u \in B$ is denoted
  $$\sigma(u) = \sigma_B(u) \in \Sigma$$
  so for $u \in B$ we always have $u \in B^{(\sigma(u))}$. Note that $u \in \mathbb{R}^2$ may belong to several boxes (from different frames and levels) so $\sigma(u)$ is not well defined, hence the notation $\sigma_B(u)$. However, when the box is clear from the context, we omit the subscript and write $\sigma(u)$.

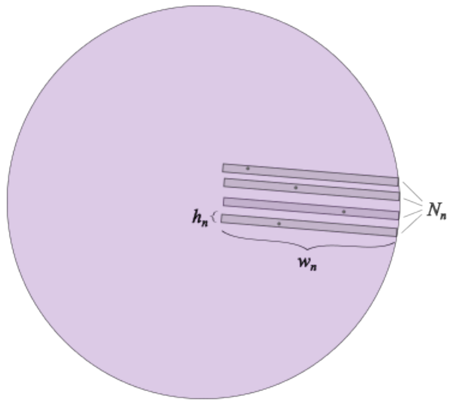

Figure 2: A frame and its boxes. Each box has a witness in it indicated by a dot. We have not depicted the sections.

- Each box B is associated with one **witness** $w = w(B) \in B$ (Note: this is a point in $\mathbb{R}^2$, not $\mathbb{Z}^2$).
  - The witness $w(B)$ must be safe (in the sense of Proposition 3.1) with respect to the collection of rectangles

    $$\mathcal{R}_C(B) = \{B', B''\} \cup \left\{ R^{(i)} \;\middle|\; \begin{array}{c} R \text{ is a } k\text{-box for } k < n \text{ such that } \theta(R) = \theta(B) \\ \text{and } 2R^{(i)} \cap B^{(i)} \neq \emptyset \text{ for some } i \in \{1, \ldots, 1000\} \end{array} \right\}$$

    Here, B', B'' denote the rectangles of dimensions $w_n \times \frac{1}{100} h_n$ that share one long side with B and whose interiors are disjoint from B. We note that this collection is 80-sparse, so it satisfies the hypothesis of Proposition 3.1.
  - The orientation of the box B associated to w is denoted $\theta(w) = \theta(B)$.
  - The section $\sigma(w) = \sigma_B(w)$ is also defined, as above.

- The set of all witnesses associated to a frame F denoted

  $$W(F) = \{w \mid w \text{ is a witness in } F\}$$

  and the sets $\mathcal{W}_1, \mathcal{W}_2, \ldots$ associated to the certificate C are

  $$\mathcal{W}_n(C) = \{W(F) \mid F \text{ is an } n\text{-frame in } C\}$$

4.4 Comments and first corrolaries of the definition

Notation

We will be flexible in our notation: A witness may well belong, as a point in $\mathbb{R}^2$, to many boxes, including more than one on each level, but we assume that witnesses "remember"

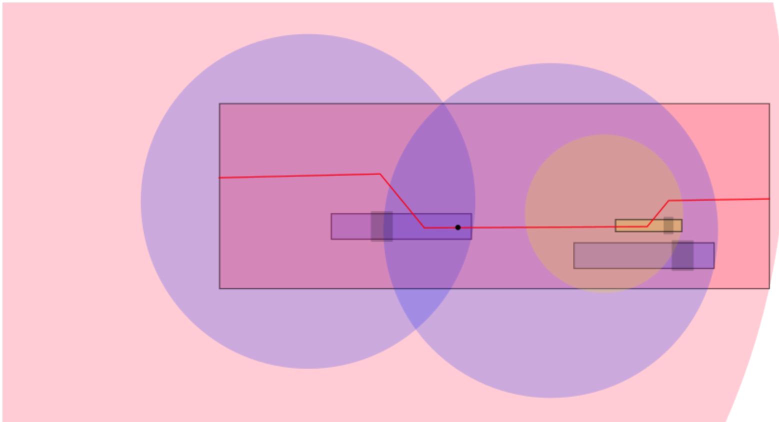

Figure 3: Part of a frame F and a box B in F (indicated in red), and two lower level frames with boxres in the same orientation (indicated in blue and yellow). The gray rectangles indicate sections of the lower-level boxes which would be included in $\mathcal{R}_C(B)$. The witness w of B is indicated by a black dot, and the red path passing through w consists entirely of safe points relative to $\mathcal{R}_C(B)$.

which box they came from. Similarly, by convention, each box "remembers" which frame it is associated to. We shall write $w \in B$ and $B \in F$ for these associations, and $w \in F$ to indicate that w is a witness associated to some box in F.

Separation of witnesses

Ideally, we would like different witnesses to occupy different sites, but it is convenient to aim for something slightly weaker: that each site can host at most a bounded number of witnesses.

We implement this a follows. Each witness belongs to some section S, which has an associated position in its box ($\sigma(S) \in \Sigma$), and the box has an orientation ($\theta(S) \in \Theta$). This partitions witnesses into $|\Theta| \cdot |\Sigma| = 1000^2$ **classes** parameterized by $\Theta \times \Sigma$. One should think of these classes as occupying different "layers" of the plane, avoiding each other. We must still ensure that witnesses from each class do not come close to each other. This is done in the following lemmas:

Lemma 4.1. *If F, F' are n-frames whose centers are at least $r_n/2$ separated, and if $B \in F$, $B' \in F'$ are boxes with $\theta(B) = \theta(B')$, then $d(B^{(i)}, (B')^{(i)}) > \frac{1}{4} r_n$ for all $i \in \{1, \dots, 1000\}$.*

Proof. Let p, p' denote the points in B, B' closest to the centers of F, F' respectively. Since the rectangles are almost radial, $d(p, c(F)) < r_n/100$ and $d(p', c(F')) < r_n/100$. Since $d(c(F), c(F')) > r_n/2$, we have $d(p, p') > r_n/3$. Each section is of dimensions $(w_n/1000) \times h_n$ and $h_n < w_n = \frac{9}{10} r_n$, so the diamater d of a section is at most $d < \sqrt{2} \cdot w_n < r_n/100$. Finally, $\theta(B) = \theta(B')$, so for each $i \in \Sigma$ there are points $p_i \in B^{(i)}$

and $p'_i \in (B')^{(i)}$ such that $d(p_i, p'_i) = d(p, p') > r_n/3$. It follows that

$$d(B^{(i)}, (B')^{(i)}) > d(p, p') - 2d > r_n/4$$

as claimed. □

Lemma 4.2. *Let C be a certificate and $w, w' \in C$ distinct witnesses belonging to levels $n \neq n'$, respectively. If $\sigma(w) = \sigma(w')$ and $\theta(w) = \theta(w')$, theSn $d(w, w') \geq h_{\min\{n,n'\}}$.*

Proof. Without loss of generality assume $n' > n$. Write $i = \sigma(w) = \sigma(w')$. Let B, B' denote the boxes and $S = B^{(i)}, S' = (B')^{(i)}$ the sections containing w, w' respectively. Since w' is safe relative to $\mathcal{R}_C(B')$, there are two possibilities:

- $S \notin \mathcal{R}_C(B')$, meaning that $2S \cap S' = \emptyset$, and in particular $w' \notin 2S$.
- $S \in \mathcal{R}_C(B')$ Then, since w' is safe with respect to $\mathcal{R}_C(B')$, by part (1) of Proposition 3.1, we know that $w' \notin 2S$.

In both cases we have $w' \notin 2S$, hence (since S is a box of dimensions $\frac{w_n}{1000} \times h_n$ and using $w_n \gg h_n$) we have $d(w', w) \geq h_n$. □

Corollary 4.3. *For every pair $w \neq w'$ of distinct witnesses in a certificate, at least one of the following hold:*

- $\theta(w) \neq \theta(w')$ *.*
- $\sigma(w) \neq \sigma(w')$ *.*
- $d(w, w') \geq h_1$.

Proof. If w, w' come from the same frame, and $\theta(w) = \theta(w')$, then their boxes are at least $100h_n$ apart, so their distance is certainly at least h_1.

If w, w' come from different frames in the same level, then Lemma 4.1 says that either $\theta(w) \neq \theta(w')$ or $\sigma(w) \neq \sigma(w')$.

Finally, if w, w' come from different levels $n \neq n'$, and Lemma 4.2 says that if their directions and sections agree then $d(w, w') \geq 4h_{\min\{w,w'\}} > h_1$. □

Inserting new frames into a certificate

If C is a certificate, and if u is a point that is $\frac{1}{2}r_n$-far from the centers of all n-frames in C, then it may be possible to add an n-frame F with center u to C, but for this we must also define new boxes and witnesses in F. Observe that

- In order to define a new box $B \in F$, it (or, rather, certain of its sections) must not come too close to witnesses from higher levels.
- In order to define a new witness $w \in B$, we need to stay away from (certain sections of) boxes from lower levels.

Thus, the task of defining a frame is split into two essentially independent parts.

The following lemma will be useful when we need to define boxes in a new frame, and want them to intersect a set E (the set where we want to place witnesses, because we have reedom to define the symbols there).

Lemma 4.4. *For every n-frame F and every path-connected set $E \subseteq F \setminus \frac{1}{5}F$ of diameter $> \frac{1}{100}r_n$, there are $N_n + 1$ rectangles $R_1, \ldots, R_{N_n+1}$ such that*

1. *The R_i are almost radial relative to F.*
2. *$d(R_i, R_j) > 100h_n$ for $i \neq j$.*
3. *All $\theta(R_i)$ are equal and are in Θ.*
4. *For each i the set $E \cap R_i$ contains a connected component that intersects both long edges of R_i.*

Proof. Without loss of generality we can assume that E is a simple path. Choose $u, v \in \Theta$ that differ by an angle of $2\pi/1000$ and such that the projection of E to $u^{\perp}$ and $v^{\perp}$ both lie within $r_n/50$ of the corresponding projection of $c(F)$. It is elementary to see that there is a universal constant $c > 0$ such that one of these projections of E is an interval of length at least cr_n. Suppose u is this direction. Since $r_n \gg h_n, n$ we may assume the projection of E in direction $u^{\perp}$ has length is at least $1000(N_n + 1)h_n$. This allows us to construct $N_n + 1$ almost radial rectangles in direction u satisfying (1)-(3) and such that their projection to $u^{\perp}$ lies inside the projection of E, and this ensures also (4). ☐

4.5 The alphabet

Recall that Θ is the set of angles that are multiples of $2\pi/1000$ and $\Sigma = \{1, \ldots, 1000\}$ is the index set of the sections of boxes. Our alphabet will be

$$A = \{H, T\}^{\Theta \times \Sigma}$$

where H, T represent "heads", "tails" respectively. For $(\theta, \sigma) \in \Theta \times \Sigma$ we write

$$\pi_{(\theta,\sigma)} : A \to \{H, T\}$$

for the corresponding coordinate projection, and extend $\pi_{(\theta,i)}$ to a projection $A^{\mathbb{Z}^2} \to \{H, T\}^{\mathbb{Z}^2}$.

One should think of a configuration $x \in A^{\mathbb{Z}^2}$ as consisting of 1000^2 "layers" $\pi_{\theta,\sigma}x \in \{H, T\}^{\mathbb{Z}^2}$, each layer indexed by $\Theta \times \Sigma$ and corresponding classes of sections determined by orientation ($\theta \in \Theta$) and section number ($\sigma \in \Sigma$).

4.6 Compatibility

Recall that we have defined witnesses to be points in $\mathbb{R}^2$, but would like to use them to index configurations in To this end, for $w = (w_1, w_2) \in \mathbb{R}^2$ write $[w] = ([w_1], [w_2])$, where $[\cdot]$ denotes the integer part. Then we define

$$x_w = x_{[w]}$$

If $x \in A^{\mathbb{Z}^2}$ is a configuration and F is an n-frame, we define a **coin and bucket configuration** $CBC(x, F)$ as follows:

- The number of buckets is $n^2 \cdot |\Theta| \cdot |\Sigma|$, indexed by $(\mathbb{Z}/n\mathbb{Z})^2 \times \Theta \times \Sigma$.
- For every witness $w \in F$ there is associated a coin with orientation $\pi_{(\theta(w),\sigma(w))}(x_w) \in \{H, T\}$ in the bucket indexed by $([w] \bmod n, \theta(w), \sigma(w))$.

  Since each n-frame contains N_n witnesses, there are altogether N_n coins in $CBC(x, F)$.

We say that $x \in A^{\mathbb{Z}^2}$ is:

- **Compatible with a frame** F if $CBC(x, F)$ is unorientable in the sense of Section 2.
- **Compatible with a certificate** C if it is compatible with all frames (from all levels) of C.

4.7 Definition of the subshift

Let

$$X = \left\{ x \in A^{\mathbb{Z}^2} \mid \text{There exists a dense certificate compatible with } x \right\}$$

Lemma 4.5. *If x is compatible with a certificate C then x is aperiodic:*

Proof. Fix $n \in \mathbb{N}$; it suffices to show that x is n-aperiodic. Let F be an n-frame in C. Then x is compatible with F, so $CBC(x, F)$ is unorientable and, in particular, there is a bucket (n, θ, σ) in $CBC(x, F)$ contains coins of different orientations. If $w, w' \in F$ are witnesses in corresponding to these coins, then the fact that they are in the same bucket means that $w = w' \bmod n$, $\theta(w) = \theta(w') = \theta$ and $\sigma(w) = \sigma(w') = \sigma$. The fact that the coins have different orientations means that $\pi_{\theta,\sigma}(x_w) \neq \pi_{\theta,\sigma}(x_{w'})$. Thus $x_w \neq x_{w'}$, and x not n-periodic. □

Lemma 4.6. *X is shift invariant.*

Proof. If x is compatible with a certificate C, then every shift of x is compatible with the corresponding shift of C, defined in the obvious way, so the shifts of x are also in X. □

Next, there is an obvious local way to define the distance between n-boxes, n-frames and certificates, which makes the space of certificates compact and metrizable. We would like the following to be true:

Lemma 4.7. *X is closed.*

The proof should go as follows: Suppose $x_n \to x$ in $A^{\mathbb{Z}^2}$ and x_n in compatible with a certificate C_n. By compactness of the space of certificates, we can pass to a subsequence and assume that $C_n \to C$. Then x is compatible with C, so $x \in X$.

This argument is flawed because pointwise convergence of witnesses, as points in $\mathbb{R}^2$, does not imply convergence of their integer parts. For this reason, compatibility does not pass to the limit.

One can correct this flaw as follows: rather than defining witnesses to be points in $\mathbb{R}^2$, define them as pairs (w, s) where $w \in \mathbb{R}^2$ and $s \in \mathbb{Z}^2$ with $\|w - s\|_\infty \leq 1/2$, and use s in place of $[w]$ in the definition of compatibilty. With this change, the space of certificates is still compact, and convergence of witnesses includes convergence of the "integer parts" s, so compatibility does pass to the limit. and the proof above goes through.

We continue to write $[w]$ instead of s but this should not cause any confusion.

Lemma 4.8. *$X \neq \emptyset$.*

Proof. Fix a certificate C (it is not hard to see that, taking all h_n, w_n large relative to their predecessors, certificates exist).

By Corollary 4.3, witnesses $w, w' \in C$ with the same orientation and section must satisfy $d(w, w') > w_1 > 2$ and hence the integer points derived from w, w' satisfy $[w] \neq [w']$.

It follows that if for every frame $F \in C$ we fix a coin and bucket configuration c_F on $n^2 \cdot |\Theta| \cdot |\Sigma|$ buckets, then there is a configuration $x \in A^{\mathbb{Z}^2}$ with $CBC(x, F) = c_F$; for, by the above, the symbols $(x_w)_{(\theta(w),\sigma(w))} \in \{H, T\}$ can be defined independently as w ranges over all witnesses in C.

By choice of N_n we can choose c_F to be unorientable for each frame $F \in C$. Then the configuration x above is compatible with C, and hence $x \in X$. This shows that the subshift X is non-empty. □

Combining the lemmas above we get:

Corollary 4.9. *X is a non-empty subshift.*

5 Strong irreducibility

In this section we show that the subshift X defined above is strongly irreducible.

5.1 Reduction to bounded, simply connected regions

Throughout this section and sometimes later, we view $\mathbb{Z}^2$ as the vertex set of a graph with edges between vertices at distance one (so $(u,v) \in \mathbb{Z}^2$ is connected to the vertices $(u \pm 1, v)$, $(u, v \pm 1)$), and identify a subset of $\mathbb{Z}^2$ with the induced subgraph. Thus we will speak of connected or disconnected subsets of $\mathbb{Z}^2$, of connected components in the set, etc.

Strong irreducibility with gap g is the property

> (A) X admits gluing along all pairs $E, F \subseteq \mathbb{Z}^2$ satisfying $d(E,F) > g$.

We now perform a sequence of reductions which show that in order to establish strong irreducibility with gap g, it suffices to show that the subshift admits gluing for $(\frac{1}{2}g - 2)$-separated sets that satisfy some additional properties.

We can assume F is finite and E has finitely many connected components

Indeed, we first claim that (A) follows from

> (B) X admits gluing along all pairs $E, F \subseteq \mathbb{Z}^2$ with $d(E,F) > g$ and F finite.

For, given arbitrary E, F with $d(E,F) \geq g$, and given $x, y \in X$, for each n we can use (B) to find a gluing $z_n \in X$ of $x|_E$ and $y|_{F \cap [-n,n]^2}$. Any accumulation point of (z_n) is then a gluing of $x|_E$ and $y|_F$.

Next, we claim that (B) follows from

> (C) X admits gluing along all all pairs $E, F \subseteq \mathbb{Z}^2$ such that
>
> C1. $d(E,F) > g$.
>
> C2. F is finite and E has finitely many connected components.

Indeed, if E, F satisfy the assumptions of (B), and if $x, y \in X$, replace E by $E' = \{u \in \mathbb{Z}^2 \mid d(u,F) > g\}$. Since F is finite, E' has finitely many connected components, so by (C) there is a gluing z of $x|_{E'}$ and $y|_F$. Since $E \subseteq E'$, the configuration z is also a gluing of $x|_E$ and $y|_F$.

We can assume the connected components are $\frac{1}{2}g - 2$ separated

Specifically, we claim that (C) follows from

> (D) X admits gluing along pairs E, F that satisfy
>
> D1. $d(E,F) > \frac{1}{2}g - 2$.
>
> D2. F is finite and E has finitely many connected components.
>
> D3. Every pair of connected components in E or F is $(\frac{1}{2}g - 2)$-separated,

Indeed, suppose that E, F satisfy the conditions (C1) and (C2). We will enlarge and merge pairs of components to obtain sets E', F' containing E, F, respectively, in a way that (D1)-(D3) are satisfied; then (D) ensures gluing along E', F', which implies gluing along E, F.

To this end, view E, F as subsets of $\mathbb{R}^2$, and enlarge E by adding to it all line segments between pairs $u, v \in E$ that are neighbors in $\mathbb{Z}^2$. Do the same for F. Note that subsets of $E \cap \mathbb{Z}^2$ that were previously connected components in the induced graph from $\mathbb{Z}^2$ are now connected components in the topological sense, and that E, F are still g-separated after adding these line segments.

Given any $u, v \in E \cap \mathbb{Z}^2$ that lie in different connected components of E and satisfy $d(u, v) < \frac{1}{2}g$, introduce a line segment $\ell = \ell_{u,v}$ connecting u, v. Observe that if $w \in \mathbb{Z}^2$ and $d(w, \ell) \leq \frac{1}{2}g$, then the point $w' \in \ell$ closest to w is within $g/2$ of one of the points u, v, and therefore either $d(w, u) < g$ or $d(w, v) < g$.

Let $\widehat{E}$ denote the union of E and all line segments $\ell_{u,v}$ introduced in the previous paragraph to $u, v \in E$, and similarly $\widehat{F}$. If $I \subseteq \widehat{E}$ is a connected component, then for any $u \in I$ we have $d(u, w) > \frac{1}{2}g$ for $w \in \widehat{F}$ and $w \in \widehat{E} \setminus I$: Indeed, we showed this above for $w \in (F \cup (E \setminus I)) \cap \mathbb{Z}^2$, but it holds also for $w \in \widehat{F} \cup (\widehat{E} \setminus I)$ because the distance between a pair of line segments is attained at their endpoints, which in our case belong to E or F).

We now return to $\mathbb{Z}^2$: For each line segment ℓ in $\widehat{E}$ add to E the integer points in a 1-neighborhood of ℓ. In the resulting set E', each connected component (in the graph $\mathbb{Z}^2$) is obtained from a connected component of $\widehat{E}$, and the former lies in the 1-neighborhood of the latter. Fo the same to obtain F' from $\widehat{F}$. Now $E \subseteq E'$, $F \subseteq F'$, and E', F' satisfy (D1) and (D3); they also satisfy (D2) because E, F satisfy (C2).

We can assume that E, F are connected

This is because condition (D) above follows from

(E) X admits gluing along pairs E, F provided

E1. $d(E, F) > \frac{1}{2}g - 2$.

E2. F is finite and E, F are connected.

E3. Every pair of connected components in E or F is $(\frac{1}{2}g - 2)$-separated,

For suppose that E, F satisfy the conditions (D1)-(D3) and assume that (E) holds. We show that every $x, y \in X$ admit a gluing along E, F. We do so by induction on the total number of connected components in E, F. For, assuming as we may that one of the sets E, F has more than one component, let J be one of these components. If $J \subseteq E$, apply the induction hypothesis to obtain a gluing z of x, y along $E \setminus J$ and F (which now have one component less), and then glue x and z along J and $J' = \{u \in \mathbb{Z}^2 \mid d(u, J) > \frac{1}{2}g - 2\}$. The resulting configuration w is also a gluing of x, y along E, F. In the case $J \subseteq F$, first glue x, y along $E, F \setminus J$, and glue the result and y along J' and J

We can assume that F is simply connected and $E = \{u \in \mathbb{Z}^2 \mid d(u, F) > \frac{1}{2}g - 2\}$

This reduction is not strictly necessary, but it gives something more concrete to think about. In fact, we can achieve this simply by "filling in the holes" in F, and then setting E as above.

Summary

We have reduced the problem of proving SI with gap g to proving that condition (E) above holds. This is what we do: we begin with $g = 10$ and must prove condition (E), noting that $\frac{1}{2}g - 2 = 3$.

5.2 Proof setup

Let $x, y \in X$ with compatible with dense certificates C_x, C_y respectively.

Let $E_x, E_y \subseteq \mathbb{Z}^2$ be connected sets with E_y finite and $E_x = \{u \in \mathbb{Z}^2 \mid d(u, E_y) > 3\}$, so $d(E_x, E_y) > 3$. We introduce the "filled in" versions of the sets

$$\widehat{E}_x = E_x + (-1, 1)^2$$
$$\widehat{E}_y = E_y + (-1, 1)^2$$

and

$$\widehat{E} = \mathbb{R}^2 \setminus (\widehat{E}_x \cup \widehat{E}_y)$$

By our assumptions, $\widehat{E}$ separates E_x from E_y and is at least 1-far from each of them. We call $\widehat{E}$ the **gap**. We call $\widehat{E}_x$ the **zone** of C_x and $\widehat{E}_y$ the **zone** of E_y. For a frame F in one of the certificates C_x, C_y, we write $C_F = C_x$ or C_y depending on whether $F \in C_x$ to $F \in C_y$,and similarly $E_F = E_x$ or E_y and $\widehat{E}_F = \widehat{E}_x$ or $\widehat{E}_y$. We write C'_F for the certificate that F does not belong to and similarly $E'_F \widehat{E}'_F$. We use similar notation for boxes and witnesses.

Let

$$z = x|_{E_x} \cup y|_{E_y}$$

This is a partially defined configuration in $A^{\mathbb{Z}^2}$. Our objective is to extend z to a configuration in X. This means that we must do two things:

1. We must define the symbols z_u for $u \in (E_x \cup E_y)^c$.
2. We must define a dense certificate C compatible with z, thus showing that $z \in X$.

Both the extension of z and the definition of C will be carried out in an iterative fashion. We begin from $C = \emptyset$, which is trivially compatible with z. Frames will then be added to C one at a time in several (infinite) rounds. Initially, we add frames from C_x or C_y that require little or no modification in order to maintain the certificate properties and be compatible with z. Afterwards, we add frames from C_x, C_y that require more extensive changes, and finally we add entirely new frames, although they will also be derived, after substantial changes, from frames in C_xC_y.

When a frame F is added to C, we (partially) define any undefined symbols in z at witnesses $w \in F$, setting the appropriate component of $z_w \in \{H,T\}^{\Theta\times\Sigma}$ so as to ensure compatibility. Because we only add frames and witnesses that maintain the certificate properties, by Corollary 4.3 we never will attempt to define any component of a symbol more than once.

Once a frame is added to C and corresponding symbols defined in z, they are never changed at later stages of the construction.

5.3 Performing the gluing

Observe that $\widehat{E}_y$ is bounded, and set

$$n_0 = \min\{n \in \mathbb{N} \mid \operatorname{diam} \widehat{E}_y < \frac{1}{10} r_n\}$$

Step A: Small frames far from the boundary

> For each $1 \le n < n_0$ in turn, for every n-frame $F \in C_x \cup C_y$ whose center $u = c(F)$ satisfies $d(u, \mathbb{R}^2 \setminus \widehat{E}_F) > \frac{2}{3} r_n$, add F to C, subject to the modifications below.

Let F be an n-frame as above. We define a new frame F' (the modified version of F) with the same center and the same boxes as F, but possibly different witnesses. To make the change, we consider each witness $w \in F$ in turn, and replace it with w', as follows:

- Let $B \in F$ be the box containing w, so w is safe relative to $\mathcal{R}_{C_F}(B)$.
- By Proposition 3.1, there is a polygonal path $\gamma \subseteq \operatorname{safe}(\mathcal{R}_{C_F}(B))$ containing w, with orientation differing from $\theta(B)$ by at most $\alpha_1 = \frac{h_1}{w_1}$. We know that γ is an infinite path passing through $w \in B$, and it does not cross the long edges of B (because $\mathcal{R}_{C_F}(C)$ contains rectangles whose sides are these same long edges). So, replacing γ by $\gamma \cap B$, we can assume that $\gamma \subseteq B$ and that it connects the short ends of B.
- We define the point w' differently in each of the following cases:
  1. If $\gamma \cap \widehat{E} = \emptyset$, we set $w' = w$ (i.e. we do not "move" w at all).
  2. Otherwise, choose w' to be the point in $\gamma \cap \widehat{E}$ that is closest to the center of F. Such a point exists because $\widehat{E}$ and γ are compact.

Having added w' to each box $B \in F'$ we claim now that $C \cup \{F'\}$ is a certificate. Since the boxes in F' come from those of F they satisfy all the relevant conditions from the definition. It remains to check two things:

- The center of F' (equivalently, of F) is $r_n/2$-far from the centers of other n-frames G in C .

  If $G \in C_F$, then F, G are n-frames in the same certificate, hence they are $r_n/2$ separated.

  Otherwise, G is from the other certificate than F. In this case, each of the centers of F, G lies each in its own zone, and each is at least $\frac{2}{3}r_n$from the complement of its zone (because F, G were added to C in the present step, Step A, and this was the condition for being added). Thus, the line segment connecting $c(F), c(G)$ contains a segment of length $\frac{2}{3}r_n$ in each of the two zones, hence its total length is greater than $\frac{4}{3}r_n$, and certainly more than $\frac{1}{2}r_n$.

- Every witness from a box $B' \in C \cup \{F'\}$ is safe with respect to $\mathcal{R}_{C\cup\{F'\}}(B')$.

  Since the addition of F' to C did not introduce any new boxes at levels below n we have not affected the safety of witnesses from any level $\leq n$ in C, and C does not yet contain any frames at higher levels. So we only need to verify that the witnesses w' of F' are safe.

  Consider a witness $w' \in F'$ belonging to the section $\sigma = \sigma(w')$ of an n-box $B' \in F'$. :et w, γ be as defined above when w' was added.

  The family $\mathcal{R} = \mathcal{R}_{C\cup\{F'\}}(B')$ is derived from sections of boxes added to C from the certificate C_F of F, and those added from the other certificate.

  Since all points in γ are safe relative to $\mathcal{R}_{C_F}(B')$, and since $w' \in \gamma$, we know that w' is safe relative to $\mathcal{R}_{C_F}$.

  Thus, by part (1) of 3.1, it suffices to show that for $k < n$, if $S \in \mathcal{R}_{C\cup\{F'\}}(B') \setminus \mathcal{R}_{C_F}(B')$, then $w' \notin 20S$.

  Suppose to the contrary there exists a level-k box $B'' \in C \cup \{F'\}$ belonging to a frame $F'' \in C \cup \{F'\}$, and a section $S = (B'')^{(i)}$, such that $S \in \mathcal{R}_{C\cup F} \setminus \mathcal{R}_{C_F}$ and $w' \in 20S$.

  Claim 5.1. $d(w', \widehat{E}_{F''}) < \frac{1}{6}r_n$

  Proof of the Claim. Observe that

  - Since F'' was added to C in Step A, it intersects $\widehat{E}_{F''}$ (its own zone, which is different from that of F).

    In particular, $d(S, \widehat{E}_{F''}) \leq w_k$.
  - S has dimensions $\frac{w_k}{1000} \times h_k$ and $h_k \ll w_k$.

    In particular, since $w' \in 20S$, we have $d(w', S) \leq w_k$.

  Therefore,
  $$d(w', \widehat{E}_{F''}) \leq d(w', S) + d(S, \widehat{E}_{F''}) \leq 2w_k \ll r_n$$
  as claimed.

  □

Claim 5.2. When γ is traversed starting from w' towards the center of F, we eventually enter $\widehat{E}_{F''}$.

Proof of the Claim. Since we have assumed that $d(c(F), \mathbb{R}^2 \setminus \widehat{E}_F) > \frac{2}{3}r_n$, the previous claim implies that $d(w', c(F)) > \frac{2}{3}r_n - \frac{1}{6}r_n = \frac{1}{2}r_n$. Since $w_n = \frac{9}{10}r_n$ we conclude that the part of γ extending from w' towards $c(F)$ is at least $\frac{1}{4}r_n \gg 2r_k$ long. Next, observe that

- Since $S \in \mathcal{R}_C(B'')$, we know that $\theta(B'') = \theta(B')$
- When we travel along γ from w' towards the center of F', the slope of γ relative to the direction $\theta(F')$ is at most h_1/w_1, which may be assumed less than 1/1000. Thus, the direction of γ deviates from $\theta(F') = \theta(F'')$ at most 1 unit per 1000.
- B'' is almost radial, so when the long edges are extended in direction θ the ray passes within $\frac{1}{100}r_k$ of the center of F''.
- $w' \in 20S$, so the distance from w' to the one of the lines extending the long edges of B'' is at most $20h_k$.

Thus, we can travel along γ from w' towards $c(F)$, we will pass within $\frac{1}{100}r_k + 20h_k + \frac{1}{1000}r_k$ of the center of F'', and this distance can be assumed smaller than $\frac{1}{10}r_k$. But every point within $\frac{1}{10}r_k$ of the center of F'' lies in the zone $\widehat{E}_{F''}$, because if F'' was added to C in stage A. This proves the claim. □

To conclude, as we move along γ, we will certainly enter the zone $\widehat{E}_{F''}$. Because $\widehat{E}$ is a connected set separating the zones, this means that we also pass through $\widehat{E}$. This shows that we are not in case (1) of Step A, so we are in case (2). But the new point that we have found in $\gamma \cap \widehat{E}$ is closer to the center of F' than w' was, contradicting the definition of w' in (2).

We have shown that $C \cup \{F'\}$ is a certificate.

It remains to update z at any witness $w' \in F'$ that differs from the corresponding witness in F. Such witnesses lie in $\widehat{E}$. We have already explained why in this case, $\pi_{(\theta(w),\sigma(w))}(z_{w'})$ is not yet specified. To determine its value, we imagine witnesses moved one at a time and set $\pi_{(\theta(w),\sigma(w))}(z_{w'})$ to H or T according to the strategy in the coin-and-bucket game, started from $CBC(x, F)$ if $F \in C_x$ or from $CBC(y, F)$ if $F \in C_y$.

Step B: Large frames

For each $n \geq n_0$ in turn, add all n-frames $F \in C_x$ to C, subject to the modifications below.

(Note that we add only frames from C_x, not C_y. This is where finiteness of E_y plays a role).

Let F be an n-frame as above. We proceed as in Step A: we define a new frame F' (the modified version of F) with the same center and the same boxes as F, and for each witness $w \in F$ we define a witness $w' \in F'$ using the same procedure as in Step A, using the same path γ defined there. Having done this to all witnesses we we add F' to C.

We must verify that $C \cup \{F'\}$ is a certificate. We havea added frames at different levels than step A and all frames and boxes we added come from C_x, so separation of boxes and frames is inheritted from C_x. We also can not have disrupted the safety property of existing witnesses from Step A because we are adding only higher levels than before. What we must check is that the new witnesses added in the present stage satisfy the safety property.

So let $w' \in F'$ be a witness derived in this stage from a witness $w \in F$ belonging to section σ of a box $B \in F$. As explained in Step A, w' is safe with respect to $\mathcal{R}_{C_x}(B)$. To show that it is safe relative to $\mathcal{R}_{C\cup\{F'\}}(B)$, we must show that if $k < n$ and if $S \in \mathcal{R}_{C\cup\{F'\}}(B) \setminus \mathcal{R}_{C_x}(B)$ is the i-th section of a k-box $B'' \in C_y$ for some $k < n$, then $w' \notin 20S$.

Suppose the contrary and let k, B'', i, S is as above but $w' \in 20S$. We know that $B'' \in C_y \cap C$, and this occurs only if $k < n_0$ and B'' belongs to a frame added during Step A.

Claim 5.3. $w' \notin \widehat{E}_y$.

Proof. Proof of the Claim] By our choice of w', if $w' \notin \widehat{E}$ then γ does not intersect $\widehat{E}$ at all. So, if $w' \in \widehat{E}_y$ then $w' \notin \widehat{E}$, and, since $\widehat{E}$ separates $\widehat{E}_x, \widehat{E}_y$, we must also have $\gamma \cap \widehat{E}_x = \emptyset$. It follows that if $w' \in \widehat{E}_y$, then $\gamma \subseteq \widehat{E}_y$. On the other hand w' is an n-witness for $n \geq n_0$, so the diameter of $\widehat{E}$ (and hence of $\widehat{E} \cup \widehat{E}_y$) is less than $\frac{1}{10}r_n$, while the diameter of γ is at least $w_n = \frac{9}{10}r_n$, so $\gamma \subseteq \widehat{E}_y$ is impossible, and consequently so is $w' \in \widehat{E}_y$. □

Claim 5.4. $|i - \sigma| \leq 1$ (recall that $i - \sigma(S)$ and $\sigma = \sigma(w')$).

Proof of the Claim. Since $S \in \mathcal{R}_{C\cup\{F\}}(B)$, we know that $S \cap B^{(i)} \neq \emptyset$ (where i is the section number of S). Thus, $w' \in B^{(\sigma)}$ and $d(w', S) < \frac{40}{1000}w_k$ (because $w' \in 20S$) implies

$$d(B^{(\sigma)}, (B^{(i)})) \leq d(w', S) + \operatorname{diam} S < \frac{40}{1000}w_k + (h_k + \frac{1}{1000}w_k) \ll w_n$$

Since non-adjacent sections of B are at least $\frac{1}{1000}w_n$ apart, we may conclude that $|i-\sigma| \leq 1$. □

Let γ' denote the portion of γ starting from w' (but not including w') and continuing as far as possible towards the center $c(F)$ of F, ending in the short side of B nearest to $c(F)$. There we must consider two cases.

1. The length of γ' is less than $2r_k$.

   In this case, since $w_n \gg r_k$, the point w' is already close enough to the central short side of B, that $\sigma = 1$. By the claim above, we must have $i \in \{1, 2\}$. Let F'' denote the disk of B''. Then

   $$d(S, c(F'')) < \frac{1}{1000} w_k + \frac{1}{10} r_k < \frac{1}{3} r_k$$

   and since $d(w', S) < \frac{40}{1000} w_k$ (because $w' \in 20S$), we have

   $$d(w', c(F'')) < \frac{1}{2} r_k$$

   Since F'' was added in Step A, we know that the inner 1/2 of the disk F'' is contained in $\widehat{E}_{F''} = \widehat{E}_y$, and we conclude that $w' \in \widehat{E}_y$. But by the previous claim, this is impossible.

2. γ' extends more than $2r_k$ towards the center of B.

   Then, arguing as in Step A, there is a point on γ' closer than $r_k/10$ to $c(B'')$.

   Since B' was added in Step A, this means that $\gamma' \cap \widehat{E}_y \neq \emptyset$.

   Therefore, since $\widehat{E}$ separates $\widehat{E}_x$ from $\widehat{E}_y$, if $\gamma' \cap \widehat{E}_x \neq \emptyset$ then $\gamma' \cap \widehat{E} \neq \emptyset$.

   But $\gamma' \cap \widehat{E} = \emptyset$, since any point in $\gamma' \cap \widehat{E}$ would be closer to $c(F)$ than w', contradicting the definition of w'.

   We conclude that $\gamma' \subseteq \widehat{E}_y$, and consequently (since $w' \in \overline{\gamma'} \subseteq \widehat{E} \cup \widehat{E}_y$ and $w' \notin \widehat{E}_y$) we have $w' \in \widehat{E}$.

   Since $n \geq n_0$, it follows that $\widehat{E}_y$, and hence also γ', has diameter $< r_n/10$.

   Therefore the distance of w' (one end of γ') from the short edge of B that is closer to $c(F)$, is $< \frac{1}{10} r_n = \frac{1}{9} w_n$.

   It follows that $\sigma = \sigma(w') \leq \frac{1}{9} \cdot 1000$ so $\sigma \leq 111$.

   Therefore, since $|i - \sigma| \leq 1$, we have $i \leq 112$.

   On the other hand, B'' was added in Step A, so $d(c(F''), \widehat{E}) > \frac{2}{3} r_k$, and hence $d(c(F''), w') > \frac{2}{3} r_k$. Since $w' \in 20S$ we have $d(S, c(F'')) > \frac{2}{3} r_k - 20 \operatorname{diam}(S) > \frac{1}{2} r_k$. This clearly excludes S belonging to the first 200 sections of B'', say (because these are all closer then $\frac{1}{2} r_n$ to $c(F'')$), contradicting the property $i < 112$.

Both cases lead to a contradiction, so $w' \notin 20S$ and hence w' is safe relative to $\mathcal{R}_{C \cup \{F'\}}(B)$. We have shown that $C \cup \{F'\}$ is a certificate.

We again define z at the sites w' above so as to restore compatibility, this process is the same as in Step A.

Step C: Small frames relatively near the boundary

> For each $1 \leq n < n_0$ in turn, add to C those n-frames $F \in C_x \cup C_y$ whose center u satisfies $\frac{1}{4}r_n \leq d(u, \mathbb{R}^2 \setminus \widehat{E}_F) \leq \frac{2}{3}r_n$, subject to the modifications below.

What we keep from these frames is only their positions (i.e. their center), and completely redefine their boxes and witnesses.

First, we observe that adding frames with these centers does not violate the $r_n/2$ separation condition of n-frames. Indeed, since $n \leq n_0$, at the end of Step C all n-frames in C have come from steps A and C of the construction (since Step B only adds frames at levels higher than n_0). Frames that cam from the same certificate are separated by assumption, whereas if they came from different certificates then the centers are in their own zones and $r_n/4$-far from $\widehat{E}$, hence they are $r_n/2$ from each other (see Step A item (1)).

It remains to explain how to define the boxes and witnesses of an n-frame $F \in C_x \cup C_y$ added in Stage C.

Since $n < n_0$, the diameter of $\widehat{E}$ is at least $r_n/10$, and by assumption $d(u, \mathbb{R}^2 \setminus \widehat{E}_F) \leq \frac{2}{3}r_n$. This implies that $\widehat{E} \cap F$ has a connected component $\widehat{E}'$ with diameter at least $\frac{1}{10}r_n$. Furthermore, since $\frac{1}{4}r_n \leq d(u, \mathbb{R}^2 \setminus \widehat{E}_F)$, we know that $\widehat{E}' \subseteq F \setminus \frac{1}{4}F$.

By Lemma 4.4, we can find $N_n + 1$ rectangles R_i that are $100h_n$-separated and almost radial with respect to F, have a common orientation $\theta \in \Theta$, and such that $\widehat{E}'$ intersects both long edges of R_i.

Let $W_{>n}$ denote the set of witnesses $w \in C$ whose level k is greater than n, and such that $w \in 20S$ for some section S of one of the rectangles R_i.

Then $W_{>n}$ contains at most one witness w, because $h_{n+1} \gg r_n$, and by Lemma 4.2, the witnesses at levels $> n$ are h_{n+1}-separated, while all the rectangles R_i (and hence their sections) lie within a single r_n-ball.

Because the R_i are $100h_n$-separated, it follows that at most for all but at most one of them, all sections S of R_i satisfy $w \notin 20S$. If there is an exceptional one, throw it out.

We are left with at least N_n "good" rectangles which we denote $B_1, \ldots, B_{N_n}$; these are the new boxes in F.

Apply Proposition 3.1 to each B_i to find a path γ connecting its short edges and contained in the safe points of $\mathcal{R}_C(B_i)$. Since $\widehat{E}'$ connects the long edges of B_i we can choose $w \in \gamma \cap \widehat{E}'$; this is the new witness of B_i.

To conclude this step, we note that all witnesses w in F lie in $\widehat{E}$ so we can define $\pi_{(\theta(w),\sigma(w))}(z_w)$ to get any coin-and-bucket configuration we could want. We choose an unorientable one.

Interlude: How dense are the frames so far?

Taking stock, for $n \geq n_0$ the centers of n-frames in C are the same as the centers of n-frames in C_x, so they are $10r_n$ dense. But for $n \leq n_0$, although we have added to

C many frames from C_x, C_y, we have not added those whose centers come too close to the opposite zones. Therefore, for $1 \le n < n_0$, the centers of n-frames in C may not be $10r_n$-dense.

We cannot directly add the missing frames from levels $n \le n_0$ because, being close to the opposite zone, they may come to close to frames from the opposite zone that are alreay in C.

However, if there are frames close to the boundary, then by moving them slightly away from the boundary (possibly several times, in different directions), we can restore both separation and density. This is the final stage of the construction.

Step D: Adding frames near the boundary

> For each $1 \le n < n_0$, iterate over all sites $v_1, v_2, \ldots \in \mathbb{Z}^d$. For each v_i in turn, if it is not within $10r_n$ of the center of some n-frame in C, we will add a new n-frame to C. Its location is obtained by selecting a frame in $C_x \cup C_y \setminus C$ that is within $10r_n$ of v_i and shifting it towards v_i, as described below.

Suppose n is given and $v \in \mathbb{Z}^d$ is not within $10r_n$ of the center of any n-frame in C. Since C_x, C_y are certificates, we know that there are frames $F_x \in C_x$ and $F_y \in C_y$ whose centers are within $10r_n$ of v. Set $F = F_y$ if $v \in E_y$ and $F = F_x$ otherwise. We deal here with the latter case, the former is dealt with similarly.

Write $u = c(F)$ for the center of F. Then $d(u, \widehat{E}) < \frac{1}{4}r_n$. Indeed, we have assumed that $n < n_0$, and $d(u, \widehat{E}) \le \frac{2}{3}r_n$ because F was not added to C in Step A; these two facts, and the fact that it was not added in Step C, imply that $d(u, \widehat{E}) < \frac{1}{4}r_n$.

Let $e \in \widehat{E}$ be a point with $d(e, u) \le \frac{1}{4}r_n$.

Let ℓ denote the ray starting at u and passing through v, and let $u' \in \ell$ be the point closest to u that satisfies $d(e, u') = \frac{3}{4}r_n$. Note that $d(u', u) \ge \frac{1}{2}r_n$ because after moving a distance $t < \frac{1}{2}r_n$ along ℓ,we are at a point that is less than $\frac{1}{4}r_n + t \le \frac{1}{4}r_n + \frac{1}{2}r_n < \frac{3}{4}r_n$ from e.

We define a new n-frame F' centered u'. Its boxes and witnesses are defined similarly to the way they were defined in Step C, noting that since $d(e, u') = \frac{3}{4}r_n$ and $\operatorname{diam} \widehat{E} \ge \frac{1}{10}r_n$ we are in a position to apply Lemma 4.4 again. Also, note that $u' = c(F')$ is not $r_n/2$-close to the center of any previously added n-frame in C, because if F'' were such a frame, then its center u'' would satisfy

$$d(u'', v) \le d(u'', u') + d(u', v) \le \frac{1}{2}r_n + (d(u, v) - d(u', u)) < \frac{1}{2}r_n + (10r_n - \frac{1}{2}r_n) = 10r_n$$

Then $u'' = c(F'')$ would already be $10r_n$-close to v, contrary to assumption.

All that is left now is to add F' to C, and extend the definition of z to be compatible with it.

This concludes the construction.

6 Concluding remarks and problems

We end with some questions that arise naturally from this work. Some were already mentioned in the introduction.

Problem 6.1. Do there exist strongly irreducible $\mathbb{Z}^2$ subshifts on which $\mathbb{Z}^2$ acts freely?

Indeed, one can show that our example contains points with one direction of periodicity, so it does not provide an answer to the problem above.

One mechanism related to periodicity an freeness of the action is the complexity of individual points in the system. In this vein we can ask:

Problem 6.2. Given $c, \alpha > 0$, is there a SI $\mathbb{Z}^d$ subshift X such that

$$\forall x \in X \qquad N_n(x) \geq c2^{n^\alpha}$$

or even

$$\forall x \in X \qquad N_n(x) \geq c \cdot n^\alpha$$

where $N_n(x)$ is the number of $n \times n$ patterns in x ?

The existence of a sysmem satisfying the first condition with $\alpha > 1$ would give a positive solution to the previous problem. At the other extreme, a system that fails the second condition for $\alpha = 2$ and small enough c must contain periodic points, by results related to Nivat's conjecture.

Another natural question is:

Problem 6.3. Let $d \geq 3$. If X is a SI $\mathbb{Z}^d$ SFT and Z is an infinite free minimal $\mathbb{Z}^d$-subshift and $h_{top}(Z) < h_{top}(X)$, is there an embedding (an injective factor map) $Z \to X$?

Lightwood proved this for $\mathbb{Z}^2$. For $d \geq 3$, Bland [2] proved that an embedding exists provided there is some factor map $Z \to X$. Also note that if we drop the assumption that X is free, then a positive answer to the last problem would imply a negative one to the first problem and some cases of the second.

As noted in the introduction, a SI $\mathbb{Z}^2$ SFT contains periodic points. But the class of sofic shifts (symbolic factors of SFTs is much broader. So it is natural to ask,

Problem 6.4. Does there exist a SI $\mathbb{Z}^2$ sofic shift without periodic points?

The extending SFT need not be SI itself. Our construction is a natural candidate, since one could try to build an SFT that encodes the certificates, and the factor map to X would erase them. Our intuition is that this probably cannot be done with our example, but perhaps some modification of it would work.

Finally we repeat a well known open problem:

Problem 6.5. Do SI $\mathbb{Z}^3$-SFTs contain periodic points?

This was one of the questions we began with. As noted in the introduction, Theorem 1.1 lends some weak moral support for a negative answer, but the problem remains very much open.

References

[1] Anne Bertrand. Specification, synchronisation, average length. In *Coding theory and applications (Cachan, 1986)*, volume 311 of *Lecture Notes in Comput. Sci.*, pages 86–95. Springer, Berlin, 1988.

[2] Robert Bland. An embedding theorem for subshifts over amenable groups with the comparison property. *preprint*, 2022. https://arxiv.org/abs/2211.00215.

[3] Raimundo Briceño, Kevin McGoff, and Ronnie Pavlov. Factoring onto $\mathbb{Z}^d$ subshifts with the finite extension property. *Proc. Amer. Math. Soc.*, 146(12):5129–5140, 2018.

[4] Robert Burton and Jeffrey E. Steif. Non-uniqueness of measures of maximal entropy for subshifts of finite type. *Ergodic Theory Dynam. Systems*, 14(2):213–235, 1994.

[5] Tullio Ceccherini-Silberstein and Michel Coornaert. On the density of periodic configurations in strongly irreducible subshifts. *Nonlinearity*, 25(7):2119–2131, 2012.

[6] Angela Desai. A class of $\mathbb{Z}^d$ shifts of finite type which factors onto lower entropy full shifts. *Proc. Amer. Math. Soc.*, 137(8):2613–2621, 2009.

[7] Silvère Gangloff and Mathieu Sablik. Quantified block gluing for multidimensional subshifts of finite type: aperiodicity and entropy. *J. Anal. Math.*, 144(1):21–118, 2021.

[8] Michael Hochman and Tom Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type. *Ann. of Math. (2)*, 171(3):2011–2038, 2010.

[9] David Huczek and Sebastian Kopacz. Factoring strongly irreducible group shift actions onto full shifts of lower entropy. *preprint*, 2021. https://arxiv.org/abs/2106.10687.

[10] Samuel J. Lightwood. Morphisms from non-periodic $\mathbb{Z}^2$-subshifts. I. Constructing embeddings from homomorphisms. *Ergodic Theory Dynam. Systems*, 23(2):587–609, 2003.

[11] Samuel J. Lightwood. Morphisms from non-periodic $\mathbb{Z}^2$ subshifts. II. Constructing homomorphisms to square-filling mixing shifts of finite type. *Ergodic Theory Dynam. Systems*, 24(4):1227–1260, 2004.

[12] Ronnie Pavlov and Michael Schraudner. Entropies realizable by block gluing $\mathbb{Z}^d$ shifts of finite type. *J. Anal. Math.*, 126:113–174, 2015.

[13] E. Arthur Robinson, Jr. and Ayşe A. Şahin. Modeling ergodic, measure preserving actions on $\mathbb{Z}^d$ shifts of finite type. *Monatsh. Math.*, 132(3):237–253, 2001.

[14] David Ruelle. Statistical mechanics on a compact set with Z^v action satisfying expansiveness and specification. *Trans. Amer. Math. Soc.*, 187:237–251, 1973.