Smooth symmetries of $\times a$-invariant sets

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Abstract

We study the smooth self-maps f of $\times a$-invariant sets $X \subseteq [0,1]$. Under various assumptions we show that this forces $\log f'(x)/\log a \in \mathbb{Q}$ at many points in X. Our method combines scenery flow methods and equidistribution results in the positive entropy case, where we improve previous work of the author and Shmerkin, with a new topological variant of the scenery flow which applies in the zero-entropy case.

1 Introduction

For an integer $a \geq 2$ let T_a denote the self-map of $[0,1]$, or $\mathbb{R}/\mathbb{Z}$, given by $T_a x = ax \bmod 1$. Furstenberg famously proved that, although each of these maps individually admits a multitude of closed invariant sets, when a, b are non-commensurable, the only jointly invariant ones T_a, T_b are trivial [8, Theorem IV.1]. Here by trivial we mean either finite or the entire interval $[0,1]$, and we say that real numbers $a, b \geq 2$ are non-commensurable of $\log a/\log b \notin \mathbb{Q}$.

Furstenberg's theorem has seen many generalizations. The main direction of generalization has been to commuting actions in algebraic settings: these include commuting automorphisms of compact abelian groups [1, 2], and analogous (though still partial) measure-rigidity results in the automorphism setting as well as for higher-rank diagonal flows on homogeneous spaces [16, 4, 3]. Another generalization is to commuting diffeomorphisms on compact manifolds, see e.g. [15]. These results are distinct from the algebraic ones, although they share many common methods and are related by conjectures predicting that, in many cases, the only commuting smooth maps are those that are conjugate to algebraic ones.

This paper deals with another related phenomenon, namely, that if X is T_a-invariant and non-trivial, then very few smooth maps can map it onto

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2010 Mathematics Subject Classication. 28A80, 11K55, 11B30, 11P70 (or even into) itself, and in fact, any such map must locally behave like T_a, in the sense that it must satisfy $f' \sim a$ for many $x \in X$. We stress that no assumption of commutation is made between T_a and f. We also show that if Y is another non-trivial T_b-invariant set for $b \not\sim a$, then X cannot be mapped to Y by a smooth map.

Results of this kind have a long history for "fractal" sets, sometimes even in the Lipschitz category. For example, when X, Y are "deleted-digit" Cantor sets, i.e. defined by restricting individual digits in non-commensurable bases a and b, Falconer and Marsh showed that there is no bi-Lipschitz map taking X to Y [6]. More recent work on self-similar sets with similar flavor appears in the work of Elekes, Keleti and Mathe, and of Feng, Rao and Wang [5, 7]. Our results have some overlap with these, but we emphasize that we deal with much more general sets, including sets of entropy (or dimension) zero.

Recall that we say a set $X \subseteq [0,1]$ is *non-trivial* if it is infinite and $X \neq [0,1]$.

Theorem 1. *Let $X, Y \subseteq [0,1]$ be non-trivial and closed sets that are invariant under T_a, T_b, respectively, $a \not\sim b$. Then no C^2-diffeomorphism of $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ can map X onto Y.*

For self-maps of a single T_a-invariant set we obtain results under some additional dynamical and regularity assumptions. A set is *perfect* if it has no isolated points. If X is T_a-invariant then a point $x \in X$ is *transitive* if its orbit $\{T_a^n x\}_{n=1}^{\infty}$ is dense in X, and X is *transitive* if it contains a transitive point. It is *minimal* if every point is transitive. Minimal infinite systems are perfect. Adapting Furstenberg's terminology, we say that X is *self-restricted* if $X - X \neq [0,1] \bmod 1$, which holds in particular when X is minimal, or $\dim X < 1/2$. Finally, we say that $f \in C^1$ is *piecewise curved* if f' is piecewise strictly monotone (thus f is locally strictly concave or convex).

Theorem 2. *Let $X \subseteq [0,1]$ be a closed, perfect, transitive and non-trivial T_a-invariant set. Then there is no piecewise-curved $f \in C^1$ that maps X onto itself. Furthermore, without assuming curvedness, we have:*

1. *An affine map f with $f(X) \subseteq X$ has $f' \sim a$.*
2. *If X is minimal and $f \in C^1$, then $f(X) \subseteq X$ implies that $f'(x) \sim a$ for all $x \in X$;*
3. *If X is self-restricted and $f \in C^1$ then $f(X) = X$ implies $f'(x) \sim a$ for all x in a dense G_δ subset of X.*

In particular, in (2) and (3), if in addition f is real-analytic then f is affine.

We do not know whether $f(X) \subseteq X$ for transitive X implies similar conclusions in general.

The proofs split into two parts. The main new ingredient of this paper is a method to handle the case that X is self-restricted (though actually the main case of interested is the more special case when its entropy is zero). To explain this part it is useful to recall Furstenberg's original proof that there are no non-trivial jointly T_a, T_b-invariant sets. For such an X, the first observation is that $X - X \bmod 1$ is jointly T_a, T_b invariant (because both T_a and T_b are endomorphisms of $\mathbb{R}/\mathbb{Z}$). On the other hand, if X is infinite, then it has an accumulation point, whence 0 is an accumulation point of $X - X$. From $a \not\sim b$ it follows that $\{a^n b^m\}_{m,n\in\mathbb{N}}$ is non-lacunary, implying that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $y \in [0, \delta)$ then $\{T_a^i T_b^j y\}_{i,j\in\mathbb{N}} = \{a^i b^j y \bmod 1\}_{i,j\in\mathbb{N}}$ is ε-dense. Taken together, this shows that $X - X$ is ε-dense in $[0, 1]$ for every ε, so $X - X = [0, 1]$.

In contrast, in our setting the first stage of this argument already fails: we are assuming that X is invariant under T_a and under another map f, and while $X - X$ is still invariant under T_a, it is generally not f-invariant. The main new ingredient in our proof is to use a local version of the difference set $X - X$ which behaves well under smooth maps. This is developed in Section 2, and is motivated by the scenery flow and spectral arguments of [10, 14]. These do not make a formal appearance here, but see remark at the end of Section 2.

In the non-restricted case, and specifically when X has positive topological entropy, the theorems above are proved via analysis of invariant measures on X, adapting scenery flow methods from [10, 14]. This is also the source of the piecewise curvedness requirement.[1] To state our result for positive-entropy measures, let us say that a probability measure μ has dimension t if $\mu(E) < 1$ for all Borel sets E with $\dim E < t$, but it is supported on some set of dimension t. We write $f\mu = \mu \circ f^{-1}$ for the push-forward of μ by a map f, and note that when f is bi-Lipschitz, μ and $f\mu$ have the same dimension. Our result for measures is the following:[2]

[1]After this work was completed, Shmerkin [17] and M. Wu [18] independently proved a result on slices of products of positiv-dimension T_a- and T_b-invariant sets. this implies that C^1 is enough in Theorem 1, since our methods show that it is enough in dimension zero. The connection with Shmerkin and Wu's papers is that any C^1-embedding $f : X \to Y$ implies that the curve $y = f(x)$ intersects $X \times Y$ in a set diffeomorphic to X, while theur results imply that such an intersection mjst have dimension at most $\max\{0, \dim X + \dim Y - 1\}$, which is always $< \dim X$ when the latter is positive.

[2]We recently learned of results by Eskin, related to work of Brown and Rodriguez-Hertz, which shows that certain "general position", expanding-on-average pairs of diffeomorphisms of a compact manifold can preserve nothing but Lebesgue measure. Their

Theorem 3. *For every T_a-ergodic measure μ of dimension $0 < s < 1$, there exists $\varepsilon = \varepsilon(\mu)$ such that the following holds. For every piecewise curved f, every weak-* accumulation point of the sequence $\frac{1}{N}\sum_{n=0}^{N-1} T_a^n(f\mu)$ has dimension at least $s + \varepsilon$.*

Combined with the variational principle, this implies that if X is a non-trivial T_a-invariant set with positive dimension, then $f(X) \subseteq X$ is impossible for a piecewise curved f. Thus for a piecewise curved function f, every jointly f- and T_a-invariant ergodic probability measure is either Lebesgue or has dimension zero, and every jointly invariant set is either $[0, 1]$ or has dimension 0. This and the other results stated earlier make the following conjecture seems reasonable:

Conjecture 4. *Fix T_a and let $f \in C^\omega(\mathbb{R})$ or $f \in C^\omega(\mathbb{R}/\mathbb{Z})$, and assume that f is not affine. Then every jointly T_a- and f-invariant set is trivial.*

Of course one could also ask this for f with less smoothness, or make the same conjecture for measures.

The paper is organized as follows. We begin with the topological analysis of the zero entropy (or self-restricted) case: In Section 2 we define the local difference (or distance) set and discuss its properties, in Section 3 we discuss dimension and the relation between the local difference set and the original set, and in Sections 4 and 5 we prove Theorem 1 and (most of) Theorem 2. The last section is devoted to Theorem 3 and completing the proof of Theorem 2 (1).

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2 Localizing the difference set

As explained in the introduction, we require a "localized" version of the difference set $X - X$. For this, define the a-adic fractional part of $s > 0$ by

$$\{s\}_a = -\log_a s \bmod 1$$

i.e. if $0 \leq t < 1$ and $s = a^{-(k+t)}$ then $\{s\}_a = t \bmod 1$.

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methods do not appear to apply in our setting, where both the expansion and invertability are absent.

Definition 5. Let $X \subseteq \mathbb{R}$. For $2 \leq a \in \mathbb{N}$ and $x \in \mathbb{R}$ the *a-adic local* difference *set* of X at x is the set

$$F_{a,X}(x) = \left\{ t \in \mathbb{R}/\mathbb{Z} \;\middle|\; \begin{array}{c} \exists x_n, x'_n \in X \text{ s.t. } (x_n, x'_n) \to (x, x) \\ x_n > x'_n \text{ and } \{x_n - x'_n\}_a \to t \end{array} \;\} \right\}$$

For $X \subseteq \mathbb{R}/\mathbb{Z}$ we define $F_{a,X}(x) = F_{a,Y}(y)$ where $x = y \bmod 1$ and $X = Y \bmod 1$, and the lift Y is chosen so that reduction modulo 1 is a bijection of neighborhoods of x and y. When X is T_a-invariant, we view X as a subset of $\mathbb{R}/\mathbb{Z}$ when defining $F_{a,X}(x)$, even if X is initially given as a subset of $[0,1]$. This convention potentially enlarges $F_{a,X}(0)$ when $0, 1 \in X$.

This defines a function $F_{a,X} : \mathbb{R} \to \{\text{subsets of } \mathbb{R}/\mathbb{Z}\}$, but we sometimes think of the range as subsets of $[0,1]$, and make the identification whenever convenient. It will be convenient to write

$$F_{a,X}(X) = \bigcup_{x \in X} F_{a,X}(x)$$

We state, mostly without proof, some elementary properties of $F_{a,X}(x)$.

1. (Locality): $F_{a,X}(x) = F_{a,X \cap B_r(x)}(x)$ for any $r > 0$.

2. (Monotonicity): $Y \subseteq X$ implies $F_{a,Y}(x) \subseteq F_{a,X}(x)$. In particular if $X = \bigcup X_i$ then $F_{a,X}(X) \supseteq \bigcup F_{a,X_i}(X_i)$.

3. (Non-triviality): $F_{a,X}(x) \neq \emptyset$ if and only if x is an accumulation point of X.

4. (Closure): $F_{a,X}(x)$ is closed for all x.

5. (Semi-continuity): If $x_n \to x$ and $t_n \in F_{a,X}(x_n)$, and if $t_n \to t$ in $\mathbb{R}/\mathbb{Z}$, then $t \in F_{a,X}(x)$ (equivalently, in the space of compact subsets of $\mathbb{R}/\mathbb{Z}$, any sub-sequential limit E of $F_{a,X}(x_n)$ in the Hausdorff metric satisfies $E \subseteq F_{a,X}(x)$).

   In particular, if X is compact, then $F_{a,X}(X) = \bigcup_{x \in X} F_{a,X}(x)$ is closed.

6. (Linear transformation): For $X \subseteq \mathbb{R}$ and any $0 \neq b \in \mathbb{R}$, we have[3]

$$F_{a,bX}(bx) = F_{a,X}(x) + \log_a b \bmod 1$$

[3] We use the usual notation for arithmetic operations between sets: for $u \in \mathbb{R}$ and $W \subseteq \mathbb{R}$, $uW = \{uw : w \in W\}$ and $W + u = \{w + u : w \in W\}$.

7. (T_b-transformation): For $X \subseteq \mathbb{R}/\mathbb{Z}$ and $b \in \mathbb{N}$,

$$F_{a,T_b X}(T_b x) \supseteq F_{a,X}(x) + \log_a b \bmod 1$$

Indeed, for small $r > 0$ the map T_b acts on $B_r(x)$ as an affine map of expansion b, and sends $X \cap B_r(x)$ into a subset of $T_b X \cap B_{br}(T_b x)$. The inclusion above follows by locality and monotonicity.

For $X \subseteq [0,1]$, the same holds except possibly at points $x \in \frac{1}{b}\mathbb{Z}$, there T_b is discontinuous.

8. (C^1-transformation): If $f : \mathbb{R} \to \mathbb{R}$ is a C^1-diffeomorphism, then

$$F_{a,fX}(fx) = F_{a,X}(x) + \log_a f'(x) \bmod 1$$

Indeed, for $t \in F_{a,X}(x)$ let $x_n, x'_n \in X$ converge to x and satisfy $\{x_n - x'_n\}_a \to t$. Then by calculus, $f(x_n) - f(x'_n) = f'(\xi_n) \cdot (x_n - x'_n)$ for some point ξ_n intermediate between x_n, x'_n, so $\{f(x_n) - f(x'_n)\}_a = \{x_n - x'_n\} + \log_a f'(\xi_n)$. Since x_n, x'_n converge to x, also $\xi_n \to x$, so we have $f'(\xi_n) \to f'(x)$ and $t + \log_a f'(x) = \lim\{f(x_n) - f(x'_n)\}_a \in F_{a,f(X)}(f(x))$. The converse inclusion is shown similarly by considering f^{-1}.

Definition 5 is motivated by the spectral analysis of the scenery flow of $\times a$-invariant measures that was carried out in [10]. Indeed, property (6) means that the function $t \mapsto F_{a,tX}(0)$ is periodic (in $\log t$), and so defines an eigenfunction for the scenery flow (with values in the space of closed subsets of $\mathbb{R}/\mathbb{Z}$).

3 Dimension

We write $\dim X$ for the Hausdorff dimension of a set $X \subseteq \mathbb{R}$, and $\dim_{\mathrm{B}} X$ for its box dimension, defined as the limit of $\log N(x,r)/\log(1/r)$, $N(X,r)$ is the minimal number of r-balls needed to cover X (in the cases we apply this, the limit exists). We have the following standard facts.

1. $\dim$ and $\dim_{\mathrm{B}}$ are non-decreasing under Lipschitz maps.
2. $\dim(X \times Y) \leq \dim X + \dim_{\mathrm{B}} Y$.
3. $\dim(X \times Y) \geq \dim X + \dim Y$, with equality if $\dim X = \dim_{\mathrm{B}} X$.
4. If X is closed and T_a-invariant then $\dim_{\mathrm{B}} X$ exists and $\dim_{\mathrm{B}} X = \dim X = h_{top}(X, T_a)/\log a$, where $h_{top}(X, T_a)$ is the topological entropy of (X, T_a) [8, Section III].

It follows from the last two three that if $Y \subseteq [0,1]$ is closed and T_a-invariant, then $\dim X \times Y = \dim X + \dim Y$ for all sets X.

Proposition 6. *If $Y \subseteq [0,1]$ is T_b-invariant, then $\{b^{-t} : t \in F_{b,Y}(Y)\} \subseteq Y - Y$ and $\dim F_{b,Y}(Y) \leq 2\dim Y$.*

Proof. We can assume that $F_{b,Y}(Y) \neq \emptyset$ (i.e. Y is infinite), since otherwise the statement is trivial. Let $t \in F_{b,Y}(Y)$. We can assume that $t \neq 0$, since if it is then $b^0 \in Y - Y \bmod 1$ trivially. By definition, there exist points $y'_n, y''_n \in Y$ such that $\{y'_n - y''_n\}_b \to t$. This means that $\log_b(y'_n - y''_n) \to t \bmod 1$, or equivalently, that there exists $k_n \in \mathbb{N}$ and $0 \leq t_n < 1$ such that $y'_n - y''_n = b^{-k_n - t_n}$ and $t_n \to t \bmod 1$. Since $t \neq 0$, this means convergence is also in $\mathbb{R}$ once we identify t_n, t with elements of $[0,1)$, which we do henceforth. Since $y'_n - y''_n < b^{-k_n}$, we conclude that $T_b^{k_n} y''_n - T_b^{k_n} y'_n = b^{-t_n} \to b^{-t}$. Passing to a subsequence we can assume that $T_b^{k_n} y'_n \to y'$ and $T_b^{k_n} y''_n \to y''$, hence $b^{-t} = y'' - y' \in Y - Y$.

For the second statement, considering the $t \mapsto b^{-t}$ from $F_{b,Y}(Y)$ to $Y - Y$ is bi-Lipschitz, so it preserves dimension, and in particular its image, which is a subset of $Y - Y$, has the same dimension as $F_{b,Y}(Y)$. This gives $\dim F_{b,Y}(Y) \leq \dim(Y - Y)$. Since $Y - Y$ is the image of $Y \times Y$ under the Lipschitz map $(x,y) \mapsto x - y$, we have

$$\dim F_{b,Y}(Y) \leq \dim(Y \times Y) = 2\dim Y \qquad \square$$

Proposition 7. *Let $X \subseteq \mathbb{R}$, let $f \in C^2(\mathbb{R})$ be a diffeomorphism, and $Y = f(X)$. Then $\dim F_{b,Y}(Y) \geq \dim F_{b,X}(X) - \dim X$. The same holds if f is a piecewise C^2-diffeomorphism (with finitely many intervals where it is C^2).*

Proof. Let $g = f^{-1} : Y \to X$ and

$$Z = \bigcup_{y \in Y} \left(F_{b,Y}(y) \times \{\log_b g'(y)\}\right) \subseteq F_{b,Y}(Y) \times \mathbb{R}$$

Writing $\pi(y,t) = y + t \bmod 1$, we have $F_{b,X}(X) = \pi(Z)$, and since π is Lipschitz,

$$\dim F_{b,X}(X) \leq \dim Z$$

On the other hand $Z \subseteq F_{b,Y}(Y) \times \log_b g'(Y)$, so

$$\begin{aligned}
\dim Z &\leq \dim F_{b,Y}(Y) + \dim_{\mathrm{B}} g'(Y) \\
&\leq \dim F_{b,Y}(Y) + \dim_{\mathrm{B}} Y \\
&= \dim F_{b,Y}(Y) + \dim_{\mathrm{B}} X
\end{aligned}$$

where the line we used the fact that log preserves dimension, the newx inequality used the fact that $g' \in C^1$, so g' is Lipschitz and $\dim_{\mathrm{B}} g'(Y) \leq \dim_{\mathrm{B}} Y$, and the last step is because g is bi-Lipschitz, and hence $\dim Y = \dim X$. Combining the inequalities gives the first claim.

For the second statement, consider a finite partition of $\mathbb{R}$ into intervals $\{I_i\}_{i=1}^N$ such that $f|_{I_i} : I_i \to f(I_i)$ is a diffeomorphism except possibly at the endpoints. Let $X_i = X \cap I_i$ and $Y_i = f(X_i)$. Since $X = \bigcup X_i$ and $Y = \bigcup Y_i$ and since there are finitely many sets in these unions, we have $\dim X = \max_i \dim X_i$ and $\dim Y = \max_i \dim Y_i$. By the first part, $\dim Y_i \geq \dim F_{b,X_i}(X_i) - \dim X_i$. Since $X = \bigcup X_i$ and $Y = \bigcup Y_i$ and the unions are finite, we have $F_{b,X}(X) = \bigcup F_{b,X_i}(X_i)$ and $F_{b,Y}(Y) = \bigcup F_{b,Y_i}(Y_i)$, and again the dimension of each set is given by the maximum of the dimensions of the sets in the union. Thus,

$$\begin{aligned} \dim F(Y) &= \max_i \dim F_{b,Y_i}(Y_i) \\ &\geq \max_i (\dim F_{b,X_i}(X_i) - \dim X_i) \\ &\geq \max_i \dim F_{b,X_i}(X_i) - \dim X \\ &= \dim F_{b,X}(X) - \dim X \quad \square \end{aligned}$$

4 Proof of Theorem 1

Let $X \subseteq [0,1]$ or $\mathbb{R}/\mathbb{Z}$ be closed, infinite, T_a-invariant, and let f be a C^2-diffeomorphism of $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ such that $Y = f(X)$ is a T_b-invariant set for some $b \not\sim a$.

Suppose first that $\dim X > 0$. Then $h_{top}(X, T_a) = \log a \cdot \dim X > 0$. By the variational principle we can find a T_a-invariant and ergodic probability measure μ on X with $h(\mu, T_a) > 0$. Then by [14, Theorem 1.10], for μ-a.e. x, the point $f(x)$ equidistributes under T_b for Lebesgue measure, and in particular the T_b-orbit of $f(x)$ is dense. Since $f(X) \subseteq Y$ this means that $Y = [0,1]$. Since f is a diffeomorphism, X must also be an interval, and the only interval in $[0,1]$ invariant under T_a is $[0,1]$.

It remains to show that $\dim X > 0$.

Claim 8. $F_{b,X}(X) = [0,1]$.

Proof. X is closed and infinite it contains an accumulation point $x_0 \in X$, whence

$$E = F_{b,X}(x_0) \neq \emptyset$$

Applying T_a we have

$$F_{b,T_a^n X}(T_a^n x_0) \supseteq E + n\log_b a \bmod 1$$

Since X is T_a-invariant,

$$F_{b,X}(X) \supseteq \bigcup_{n\in\mathbb{N}} F_{b,X}(T_a^n x_0) \supseteq E + \{n\log_b a\}_{n\in\mathbb{N}} \bmod 1$$

But $b \not\sim a$, so the set $\{n\log_b a\}_{n\in\mathbb{N}}$ is dense modulo 1, hence also $E + \{n\log_b a\}_{n\in\mathbb{N}}$ and $F_{b,X}(X)$ are dense modulo 1. Since $F_{b,X}(X)$ is closed, $F_{b,X}(X) = [0,1]$. $\square$

To prove $\dim_{\mathrm{B}} X > 0$, suppose by way of contradiction that $\dim_{\mathrm{B}} X = 0$. Now, $Y \supseteq f(X)$, so by Proposition 7, if

$$\dim F_{b,Y}(Y) \geq \dim(F_{b,X}X) - \dim_B X = 1 - \dim X > 0$$

and so by Proposition 6, $\dim Y \geq \frac{1}{2}\dim F_{b,Y}(Y) > 0$. But f is bi-Lipschitz, so $\dim f(X) = \dim X = \dim Y > 0$, as desired.

Note that the C^2 assumption was used only in the positive-entropy case, and there the recent work of Shmerkin [17] and M. Wu [18] can be used to prove that C^1 is enough.

5 Proof of Theorem 2

We prove Theorem 2 under the assumption that X is self-restricted, i.e. $X - X \neq [0,1] \bmod 1$. This covers the case $\dim X = 0$ and the case of minimal X.

Proof of part (2): Let $X \subseteq [0,1]$ be an infinite, minimal T_a-invariant set. Given $x, y \in X$ there is a sequence $n_k \to \infty$ such that we have $T_a^{n_k} x \to y$. Since

$$F_{a,X}(x) \subseteq F_{a,T_a^{n_k}X}(T_a^{n_k}x) = F_{a,X}(T_a^{n_k}x)$$

for all k, by semi-continuity of $F_{a,X}(\cdot)$ we conclude that $F_{a,X}(x) \subseteq F_{a,X}(y)$. Since $x, y \in X$ were arbitrary, $F_{a,X}(x)$ is independent of $x \in X$. We denote this set by E. Since X is infinite there is an accumulation point $x_0 \in X$, so $E = F_{a,X}(x_0) \neq \emptyset$.

Suppose that I is a non-empty open interval and $f : I \to \mathbb{R}$ a C^1-embedding such that $f(X \cap I) \subseteq X$. Let $x \in X \cap I$, we must show that

$\alpha = \log_a f'(x) \in \mathbb{Q}$. Indeed suppose $\alpha \notin \mathbb{Q}$. Then, writing $y = f(x)$, by the last paragraph we have

$$\begin{aligned} E &= F_{a,X}(y) \\ &\supseteq F_{a,f(X)}(y) \\ &= F_{a,X}(x) + \alpha \bmod 1 \\ &= E + \alpha \bmod 1 \end{aligned}$$

Thus E is closed, non-empty and invariant under $t \mapsto t + \alpha \bmod 1$. Since α is irrational this implies $E = [0,1]$.

Now, for $x \in X$ we have $F_{a,X}(x) \subseteq \log_b(X - X \bmod 1)$ by Proposition 6. Thus $\log_b(X - X)$ has non-empty interior, so the same is true of $X - X$, and since $X - X \bmod 1$ is T_a-invariant this implies that

$$X - X = [0,1] \bmod 1$$

But, since X is minimal, this is impossible by [8, Theorem III.1].

Proof of part (3): Suppose that X is perfect and transitive, and self-restricted, and that $f(X) = X$ for some local diffeomorphism $f \in C^1$ (the curvedness hypothesis is only needed for the non-restricted case). The proof is very similar to the minimal case. Indeed, since X is perfect, $F_{a,X}(x) \neq \emptyset$ for all $x \in X$, in particular at transitive points. By the same argument as above, $F_{a,X}(x)$ takes the same value for all transitive points $x \in X$. Let $W \subseteq X$ denote the set of transitive points, which is well-known to be is a residual set in X.

Let $I \subseteq [0,1]$ be a non-trivial interval such that, writing $J = f(I)$, the restriction $f : X \cap I \to X \cap J$ is bijective. Clearly $W \cap I$ is residual in $X \cap I$. At the same time, $f : X \cap I \to X \cap J$ is a homeomorphism, and $W \cap J$ is residual in $X \cap J$, so $f^{-1}(W \cap J)$ is residual in $X \cap I$. Consequently, $W \cap f^{-1}(W) \cap X \cap I$ is residual in $X \cap I$. Now, for any x in this set, both x and $f(x)$ are transitive. Arguing as in the minimal case we conclude that if $\log_a f'(x) \notin \mathbb{Q}$ then $F_{a,X}(x) = [0,1]$, hence $X - X = [0,1] \bmod 1$. But this is impossible by restriction of X.

By the last paragraph, the set of $x \in X$ such that $f'(x) \in \{a^s : s \in \mathbb{Q}\}$ is residual, i.e. contains a dense G_δ. On the other hand this set is just $(f')^{-1}(\{a^s : s \in \mathbb{Q}\}) = \bigcup_{s \in \mathbb{Q}} (f')^{-1}(a^s)$, and by continuity of f' this is an F_σ-set. Thus it contains a dense open set.

It remains to note that if f is real-analytic, than our conclusion shows that $f'(x)$ belongs to the countable set $\{a^s : s \in \mathbb{Q}\}$ for an uncountable number of x, hence f' takes on some rational power a^s of a on a convergent sequence, and being itself real-analytic, $f' \equiv a^s$. Thus f is affine, and has the stated form.

Proof of main statement of theorem: Let $X \subseteq [0,1]$ be a closed perfect, transitive and non-trivial T_a-invariant set and $f \in C^1$ piecewise curved. If $\dim X = 0$ then X is self-restricted, so $f(X) = X$ implies that $f'(x) \in \{a^s : s \in \mathbb{Q}\}$ for uncountably many $x \in X$; this is impossible because f' is piecewise strictly monotone, and so takes every value at most countably many times.

It remains to deal with the case that $\dim X > 0$. By the variational principle, there exists a T_a-ergodic probability measure μ on X with $\dim \mu = \dim X$. If $f(X) = X$ then $f\mu$ is also supported on X, and by Theorem 3, we obtain (by averaging $T_a^n f\mu$ along some subsequence of times) a measure on X of dimension $> \dim X$, which is impossible.

Proof of part (1): Let $f(x) = rx + t$. If $\dim X = 0$, let $x \in X$ with $F_{a,X}(x) \neq \emptyset$. Then $F_{a,X}(f^n x) = F_{a,X}(x) + n \log_a r \bmod 1$, and if $\log_a r \notin \mathbb{Q}$ this means that $F_{a,X}(X)$ contains a dense subset of $[0,1]$ and so is equal to $[0,1]$. By Proposition 6 this is inconsistent with X being self-restricted. Thus $\log_a r \in \mathbb{Q}$ as claimed.

In the case $\dim X > 0$, we provide a similar proof in the next section using measure theoretic tools.

6 Proof of Theorem 3 and part Theorem 2 part (1)

We provide a proof sketch, since a full proof would be lengthy, and the results in [17, 18] can be used to give alternative proofs of the application to Theorems 1 and 2. We rely heavily on the scenery flow methods from [13, 14] and additive combinatorics methods from [11] and refer the reader to those papers for definitions and notation.

Let μ be a T_a-ergodic measure μ of dimension $0 < s < 1$ and $f \in C^2$ piecewise curved. Let $\nu = f\mu$ and let $\nu' = \lim \frac{1}{N_k} \sum_{n=0}^{N_k - 1} T_a^n(f\mu)$ for some sequence $N_k \to \infty$. We claim that there exists $\varepsilon = \varepsilon(\mu) > 0$ such that $\dim \nu' > \dim \mu + \varepsilon$.

It is well known (e.g. [10]) that μ generates an ergodic fractal distribution (EFD; [9, Definition 1.2]) P supported on measures of dimension s, which are supported (up to a bounded scaling and translation) on X. See e.g. [10, Section 2.2]. Because X is porous, P-a.e. measure is $(1-\varepsilon')$ -entropy porous along any sequence $[n^{1+\tau}]$ of scales, in the sense of [12, Section 6.3], for some ε' depending only on $\dim X$.

Since f is a piecewise diffeomorphism, for μ-a.e. x the measure $\nu = f\mu$

$\log a$-generates $S^*_{\log f'(x)}P$ at $y = f(x)$ (see e.g. the proof of [14, Lemma 4.16]).

Let π denote the (partially defined) operation of restricting a measure to $[0,1]$ and normalizing it to a probability measure. It now follows, as in [14, Theorem 5.1], that if $\frac{1}{N_k}\sum_{n=0}^{N_k-1} T_a^n\nu \to \nu'$ then there is an auxiliary probability space $(\Omega, \mathcal{F}, Q)$ and functions $y, t : \Omega \to \mathbb{R}$ and $\eta : \Omega \to \mathcal{P}([0,1])$, such that t_ω, $\omega \sim Q$ is distributed like $\log f'(x)$, $x \sim \mu$, and η_ω, $\omega \sim Q$ is distributed according to P, and such that

$$\nu' = \int \pi(S_{t_\omega}(\eta_\omega * \delta_{y_\omega}))dQ(\omega)$$

(the formula above differs from [14, Theorem 5.1] in the scaling S_{t_ω} of the integrand; the scaling comes from the fact that at ν-a.e. point $y = f(x)$, the measure ν generates $S^*_t P$, where $t = \log f'(x)$. Also, the theorem in [14] refers to ν' arising from the orbit of a single ν-typical point, not as above, but in fact the averaged version above follows from the pointwise one).

Re-interpreting the last equation and the properties of t, η stated before it, we find that ν' can be generated in the following way: choose x according to μ_ω, independently choose a P-typical measure η, scale η by $f'(x)$, and translate by a random amount y (whose distribution depends on x, η). Changing the order with which we choose x and η, we find that for P-typical η there is a probability measure θ_η on the group of affine maps of $\mathbb{R}$ such that we can represent ν' as

$$\begin{aligned}\nu' &= \int\int \pi(T\eta)\, d\theta_\eta(T)\, dP(\eta) \\ &= \int \pi(\theta_\eta * \eta)\, dP(\eta)\end{aligned}$$

Furthermore, choosing $T \sim \theta$, the distribution of the contraction ratio of T is the same as the distribution of $f'(x)$ for $x \sim \mu$. Since f was assumed piecewise curved, f' is locally bi-Lipschitz, so the image of μ under $x \mapsto f'(x)$ has the same dimension as μ, so by the above, $\dim\theta_\eta \geq \dim\nu_\omega = s$. Also, recall that η has uniform entropy dimension s (in the sense of [11, Definition 5.1]; this is an immediate conseqence of the definition of the measures η, of the definition of Kolmogorov-Sinai entropy, and of the ergodic theorem). It follows from [12, Theorem 9] that $\dim_{\mathrm{H}}\theta_\eta * \eta \geq \dim\eta + \varepsilon = s + \varepsilon$ for some $\varepsilon = \varepsilon(s)$, hence

$$\dim_{\mathrm{H}}\nu' \geq \int \dim_{\mathrm{H}}\theta_\eta * \eta\, dP(\eta) \geq s + \varepsilon$$

which is what we wanted to show.

We now turn to the case that $f(x) = rx + t$ is affine, $f(X) \subseteq X$ and $\dim X > 0$. Fix a dimension-maximizing T_a-invariant and ergodic measure μ on X. Choose a typical point x and consider the EFD P_n $\log a$-generated at $f^n x$. Evidently this is $P_n = S_{n \log_a r} P_0$. Assuming $\log_a r \notin \mathbb{Q}$, we can average and pass to a weak* limit, and find that X supports a measure of the form

$$\nu = \int \int_0^{\log a} \pi(S_t(\eta * \delta_{y_{\eta,t}})) dt dP(\eta)$$

We now again apply [12, Theorem 9] to conclude that $\dim \nu > \dim \mu = \dim X$, a contradiction, which shows that $\log_a r \in \mathbb{Q}$.

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