SLOW ENTROPY AND DIFFERENTIABLE MODELS FOR INFINITE-MEASURE PRESERVING $\mathbb{Z}^k$ ACTIONS

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In memory of Dan Rudolph

Abstract. We define "slow" entropy invariants for $\mathbb{Z}^d$ actions on infinite measure spaces, which measures growth of itineraries at subexponential scales. We use this to construct infinite-measure preserving $\mathbb{Z}^2$ actions which cannot be realized as a group of diffeomorphisms of a compact manifold preserving a Borel measure, contrary to the situation for $\mathbb{Z}$-actions, where every infinite-measure preserving action can be realized in this way.

1. Introduction

Let $T = (T^u)_{u\in\mathbb{Z}^k}$ be a finite-measure preserving (f.m.p.) $\mathbb{Z}^k$-action on a Lebesgue space $(\Omega, \mathcal{B}, \mu)$. We always assume the action is ergodic and free, and for simplicity assume that the total mass is $\mu(\Omega) = 1$. It is a classical problem to determine when such an action has a differentiable model, i.e. when it is isomorphic to the action of a group of diffeomorphisms on a compact manifold preserving a Borel measure (and, more specifically, when it has a smooth model, i.e. a differentiable model in which the measure is absolutely continuous with respect to the volume). It is well known that entropy presents various obstructions: a f.m.p. $\mathbb{Z}$-action has a differentiable model if and only if the entropy is finite (whether this suffices for a smooth model is a longstanding open question). Sufficient conditions for a f.m.p. $\mathbb{Z}^k$ action to have a differentiable model are not known when $k \geq 2$, but a necessary condition is that the entropy must be 0, and other obstructions of entropy type have also been identified, which we shall discuss further below.

In this paper we investigate the existence of differentiable models for infinite-measure preserving (i.m.p.) actions, that is, actions such as above but with μ an infinite σ-finite measure. For $\mathbb{Z}$-actions this question is trivial: *every* ergodic i.m.p. $\mathbb{Z}$-action has a differentiable model. Indeed, by a theorem of Krengel, such actions have a two set generator [14], and hence one can transfer the measure to a horseshoe, giving a differentiable version of the action (the existence of smooth models is again open, for a discussion of the non-singular case, see [4, Section 7]). The main result of this paper is that, for higher rank i.m.p. actions, existence of differentiable models is not automatic:

Theorem 1.1. *There exist ergodic i.m.p. $\mathbb{Z}^2$-actions without a differentiable model.*

The mechanism which underlies Theorem 1.1, as well as the classical results for f.m.p. actions mentioned above, is, briefly, the following (see Section 3 for more details). For a

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compact metric $d(\cdot,\cdot)$ on Ω, write

$$\text{sep}(\Omega, d, \varepsilon) = \max\left\{ N \;\middle|\; \begin{array}{c} \exists\, x_1,\ldots,x_N \in \Omega \text{ such that} \\ d(x_i,x_j) \geq \varepsilon \text{ for } 1 \leq i < j \leq N \end{array} \right\}$$

For a $\mathbb{Z}^k$-action T on Ω and $n \in \mathbb{N}$, denote the Bowen metric by

$$d_n^\infty(x,y) = \sup_{\|u\| \leq n} d(T^u x, T^u y)$$

If the action is by Lipschitz maps and (Ω, d) has finite Minkowski (box) dimension, as is the case for actions by diffeomorphisms on compact manifolds, then it is simple to show that $\text{sep}(\Omega, d_n^\infty, \varepsilon)$ grows at most exponentially in n for every $\varepsilon > 0$.

In contrast, dynamical invariants of entropy type give, for small $\varepsilon > 0$, lower bounds for the growth of $\text{sep}(\Omega, d_n^\infty, \varepsilon)$ as $n \to \infty$. For example, if a f.m.p. $\mathbb{Z}^k$ action has positive entropy then, when ε is sufficiently small, $\text{sep}(\Omega, d_n^\infty, \varepsilon) \geq c\exp(cn^k)$ for some $c > 0$, and hence, when $k > 1$, this is an obstruction to differentiable realization. A finer invariant, introduced by Katok and Thouvenot under the name "slow entropy" [12], has the property that when its value for a f.m.p. action is $> \alpha$, one is guaranteed for small ε that $\text{sep}(\Omega, d_n^\infty, \varepsilon) \geq c\exp(n^\alpha)$ along some subsequence as $n \to \infty$.

The proof of Theorem 1.1 follows a similar strategy. The first step is to define a "slow entropy" invariant $\rho_{slow}(T,\mu)$ of i.m.p. $\mathbb{Z}^k$-actions. The definition is a variation on the standard name-counting definition of entropy, using a stretched-exponential scale. Since the details are important for the discussion that follows, we give the complete definition here.

Let $\mathcal{P}$ be a finite, measurable partition of Ω. We restrict the discussion to *co-finite* partitions, i.e. we assume that all atoms but one have finite measure. The union of the finite-measure atoms is called the *core*.[1] Given $n \in \mathbb{N}$ let

$$Q_n = \{u \in \mathbb{Z}^k \,:\, \|u\|_\infty \leq n\}$$

For $x, y \in \Omega$, the usual way to compare the partial orbits of x, y on Q_n is by the Hamming distance between their $\mathcal{P}$-itineraries, i.e., the proportion of $u \in Q_n$ such that $T^u x, T^u y$ belong to different atoms of $\mathcal{P}$. However, since the core of $\mathcal{P}$ has finite measure and $\mu(\Omega) = \infty$, the ergodic theorem tells us that the orbits of x, y typically spend a 0-fraction of their time in the core, and hence almost all their time in the common atom of infinite measure. Consequently the Hamming distance tends to 0 as $n \to \infty$, and μ-a.a. pair x, y is Hamming-asymptotic. Instead, we introduce a Hamming-like distance "relative" to the visits to the core:

$$d_{\mathcal{P},n}(x,y) = \frac{\#\{u \in Q_n \,:\, T^u x, T^u y \text{ are in different } \mathcal{P}\text{-atoms}\}}{\#\{u \in Q_n \,:\, T^u x \text{ or } T^u y \text{ are in the core of } \mathcal{P}\}} \tag{1.1}$$

With the convention $0/0 = 0$ this is a pseudo-metric on Ω (Lemma 2.3). We denote the $d_{\mathcal{P},n}$-diameter of $E \subseteq A$ by

$$\text{diam}(E, d_{\mathcal{P},n}) = \sup_{x,y \in E} d_{\mathcal{P},n}(x,y)$$

Next, given $\varepsilon > 0$, the usual definition of entropy counts the number of sets of $d_{\mathcal{P},n}$-diameter ε that are needed to cover all but an ε-fraction of the space. Since in our setting $\mu(\Omega) = \infty$, this does not makes sense. Instead we fix a reference set $A \subseteq \Omega$ of positive and finite measure,

[1]In order to apply what follows to f.m.p. actions, assume instead that all atoms of $\mathcal{P}$ are finite, and then the core is the entire space.

and define the ε-covering number by

$$N(A, d_{\mathcal{P},n}, \varepsilon) = \min \left\{ N \;\middle|\; \begin{array}{c} \exists\, E_1, \ldots, E_N \subseteq \Omega \text{ with } \operatorname{diam}(E_i, d_{\mathcal{P},n}) \leq \varepsilon \\ \text{and } \mu(A \cap \bigcup_{i=1}^N E_i) \geq (1-\varepsilon)\mu(A) \end{array} \right\} \tag{1.2}$$

Note that $N(A, d_{\mathcal{P},n}, \varepsilon)$ is monotone in ε, and we may define

$$\rho_{slow}(T, \mu, A, \mathcal{P}) = \lim_{\varepsilon \searrow 0} \left(\limsup_{n\to\infty} \frac{\log(\log N(A, d_{\mathcal{P},n}, \varepsilon))}{\log n} \right) \tag{1.3}$$

Note that by dividing by $\log n$ rather than $\log |Q_n| = k\log(2n+1)$ we have normalized ρ_{slow} so that $0 \leq \rho_{slow}(T, \mu) \leq k$. We shall see later that $\rho_{slow}(T, \mu, A, \mathcal{P})$ does not depend on the choice of A (Corollary 2.12), and we denote it $\rho_{slow}(T, \mu, \mathcal{P})$. Thus, roughly speaking, $\rho_{slow}(T, \mu, \mathcal{P}) = \alpha$ means that, when ε is small, $N(A, d_{\mathcal{P},n}, \varepsilon)$ grows like 2^{n^α} along some subsequence. Finally, the *slow entropy* of $(\Omega, \mathcal{B}, \mu, T)$ is

$$\rho_{slow}(T, \mu) = \sup \rho_{slow}(T, \mu, \mathcal{P})$$

where the supremum is over co-finite partitions $\mathcal{P}$.

The quantity $\rho_{slow}(T, \mu)$ is clearly an isomorphism invariant, and it reduces to Katok-Thouvenot slow entropy when applied to f.m.p. $\mathbb{Z}^k$-actions (to do so we drop the co-finiteness condition on partitions). It also shares several basic features with entropy. In particular, we shall see that it can be computed from a generating co-finite partition.

Returning to our original problem, the second step in the proof of Theorem 1.1 is to relate $\rho_{slow}(T, \mu)$ to the growth of $\operatorname{sep}(\Omega, d_n^\infty, \varepsilon)$ when Ω is endowed with a metric. One might expect, as in the f.m.p. case, that $\rho_{slow}(T, \mu) > 1$ would imply super-exponential growth, and hence be an obstruction to differentiable realization. While it is possible that this is true, we have not been able to prove it. Instead, we introduce the following notion:

Definition 1.2. An action $(\Omega, \mathcal{B}, \mu, T)$ has uniform slow entropy α if $\rho_{slow}(T, \mu, \mathcal{P}) = \alpha$ for every non-trivial co-finite partition $\mathcal{P}$.

Equivalently, there is no factor with lower slow entropy. The condition of uniform slow entropy is similar to uniform entropy dimension as defined by Dou, Huang and Park [6] (entropy dimension is an invariant of f.m.p. $\mathbb{Z}$-actions which measures subexponential growth of the entropy of partitions refined along subsequences, but appears not to be equivalent to slow entropy). uniform α-slow entropy may be seen as generalizing completely positive entropy, since f.m.p. $\mathbb{Z}^k$-actions of completely positive entropy have uniform slow entropy k.

Theorem 1.3. *Suppose that $(\Omega, \mathcal{B}, \mu, T)$ is an ergodic $\mathbb{Z}^k$-action by Lipschitz maps on a complete metric space of finite box dimension, preserving a Borel measure. If the action does not have uniform slow entropy then $\rho_{slow}(T, \mu) \leq 1$.*

This is proved in Section 3.1. Following this we derive Theorem 1.1 by constructing, via cutting and stacking, a $\mathbb{Z}^2$-action which does not have uniform slow entropy and with $\rho_{slow}(T, \mu) > 1$.

Having defined slow entropy for i.m.p. actions, let us now say a few words and make some speculation about its relation to entropy theory. In fact, the literature already contains at least four definitions of entropy for i.m.p. actions: Krengel entropy [13], Parry entropy [16], the entropy of Danilenko and Rudolph [5], and Poisson entropy [11]. These are not believed to be equivalent, although inequivalence has only been established between Krengel and Poisson entropies [10].

Since ρ_{slow} does not aim to capture exponential growth of orbits, it should not be compared to invariants which do. Instead one may ask whether it is related to one of the existing entropy theories in the same way that Katok-Thouvenot slow entropy is related to Kolmogorov-Sinai entropy. That is, if the definition of ρ_{slow} is modified to measure growth of orbits at the exponential scale, does the resulting invariant coincide with one of the existing entropy theories for i.m.p. actions?

The obvious modification to make is to replace the ratio in the limit (1.3) with the quantity $|Q_n|^{-1}\cdot\log(N(A,\mathcal{P},n,\varepsilon))$. If we do so, however, the resulting invariant ρ_{exp} is trivial, assigning the value is 0 to every i.m.p. action. One way to understand this fact is simply that i.m.p. actions are best understood as analogues of zero entropy systems, and that the subexponential scale is the appropriate one with which to study orbit growth for such actions.

Instead, there is a more interesting modification which involves a re-scaling of time along orbits. The idea presented next is very close to the entropy theory for cross sections developed recently by N. Avni [2], which deals with countable probability-preserving equivalence relations endowed with a cocycle into an amenable group. In our setting such a relation arises by restricting the orbit relation and orbit cocycle to a set A of finite measure, and the mean ergodic theorem for cross sections in the sense of [2, Theorem 2.6] follows from the ratio ergodic theorem [9]. However, we shall present a more concrete version of the idea adapted to our notation.

As we have already observed, when one considers the partial orbit $(T^u x)_{u\in Q_n}$ for a large n, the frequency of visits to the core is asymptotically negligible, and furthermore this frequency depends on x. Instead, one can choose Q_n in a manner depending on x so that the number of visits is approximately constant. More precisely, for $x\in\Omega$ and $n\in\mathbb{N}$ define $m=m(x,n)$ to be the smallest integer such that there are at least $|Q_n|$ elements $u\in Q_m$ for which $T^u x$ is in the core of $\mathcal{P}$. Define a pseudo-metric $\widetilde{d}_{\mathcal{P},n}$ on Ω by

$$\widetilde{d}_{\mathcal{P},n}(x,y)=d_{\mathcal{P},m(x,n)}(x,y)+d_{\mathcal{P},m(y,n)}(x,y)$$

Thus, we are comparing portions of the itineraries of x,y which have a similar number $|Q_n|$ of visits to the core.

Proceeding as before but using the metric $\widetilde{d}_{\mathcal{P},n}$, we can now define invariants $\widetilde{\rho}_{exp}$ and $\widetilde{\rho}_{slow}$ using, respectively, exponential and stretched-exponential scales.

It turns out that $\widetilde{\rho}_{exp}$ is, for $\mathbb{Z}$-actions, nothing other than Krengel entropy. Let us recall the definition. Assuming the action is generated by the transformation $T:\Omega\to\Omega$ and A is a set of positive finite measure, let μ_A denote the normalized restriction of μ to A and let $T_A:A\to A$ be the first return map to A, i.e. $T_A x=T^{r(A,x)}x$ where $r(A,x)=\min\{i>0\,:\,T^i x\in A\}$. Krengel entropy is then defined by

$$h_{Kr}(T,\mu)=\mu(A)h(T_A,\nu_A)$$

(this is independent of A by Abramov's formula).

The equivalence follows from Avni's work, and we shall not prove it here. We note however that $\widetilde{\rho}_{exp}$ can be taken as a definition of Krengel entropy for $\mathbb{Z}^k$ actions, $k>1$; a new definition is necessary because there is no good notion of an induced map in the higher rank case. Avni's machinery works, in fact, for general discrete amenable groups (and some non-discrete ones) and it would give a definition of Krengel entropy for actions of those groups if and when a ratio ergodic theorem becomes available for them.

Thus, $\widetilde{\rho}_{slow}$ is to Krengel entropy as Katok-Thouvenot slow entropy is to Kolmogorov-Sinai entropy. This raises the question, what is the relation between ρ_{slow} and $\widetilde{\rho}_{slow}$? One might expect that the relation should in some way involve the recurrence properties of the action, since the difference in their definitions is a re-scaling of time along orbits which regularizes the frequency of returns to the partition's core.

One almost trivial relation between recurrence properties and slow entropy is the following. The recurrence set of x to A, up to "time" n, is

$$R_n(A,x) = \{u \in Q_n \,:\, T^u x \in A\} \tag{1.4}$$

Define

$$\alpha(A,x) = \limsup_{n\to\infty} \frac{\log |R_n(A,x)|}{\log |Q_n|} \tag{1.5}$$

Hence $0 \leq \alpha \leq 1$, and α is the largest number such that $|R_n(A,x)| \approx |Q_n|^{\alpha+o(1)}$ along some subsequence. By the Chacon-Ornstein lemma for $\mathbb{Z}^k$ actions [9], $\alpha(A,\cdot)$ is invariant, i.e. $\alpha(A,x) = \alpha(A,T^u x)$ for all $u \in \mathbb{Z}^k$, so by ergodicity $\alpha(A,x)$ is a.s. independent of x. Also, given another set B of finite measure, the ratio ergodic theorem implies that $|R_n(A,x)|/|R_n(B,x)| \to \mu(A)/\mu(B)$, hence the a.s. value of $\alpha(A,x)$ is also independent of A. This justifies defining the *recurrence dimension* $\alpha(T,\mu)$ of the action to be this value. We then have the following combinatorial bound:

Proposition 1.4. *$\rho_{slow}(T,\mu) \leq k\alpha(T,\mu)$.*

We shall see in Section 3.3 that strict inequality is possible, but in view of the discussion above, one might expect a more precise relationship such as $\rho_{slow}(T,\mu) = \alpha(T,\mu)\cdot\widetilde{\rho}_{slow}(T,\mu)$, or perhaps at least an inequality between these quantities. Alternatively, one might expect at least that $0 < \widetilde{\rho}_{exp}(T,\mu) < \infty$ implies $\rho_{slow}(T,\mu) = k\cdot\alpha(T,\mu)$. A related phenomenon was observed in the work of Galatolo, Kim and Park [8], who studied the relation of recurrence rates and Krengel entropy in i.m.p. $\mathbb{Z}$-actions. We leave this matter for future investigation, but note that, in any event, ρ_{slow} seems better suited to our application than $\widetilde{\rho}_{slow}$.

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2. Slow entropy

2.1. The ratio ergodic theorem.

For basic background on ergodic theory and infinite ergodic theory, see [17] and [1], respectively. We recall the following basic fact, which was proved in [9]:

Theorem 2.1. *If $(\Omega,\mathcal{B},\mu,(T_u)_{u\in\mathbb{Z}^k})$ is an ergodic i.m.p. action, then for every $f,g \in L^1(\mu)$ with $\int g\,d\mu \neq 0$,*

$$\lim_{n\to\infty} \frac{\sum_{u\in Q_n} f(T^u x)}{\sum_{u\in Q_n} g(T^u x)} = \frac{\int f\,d\mu}{\int g\,d\mu} \qquad \mu\text{-a.e.}$$

In particular if A, B have positive finite measure then, taking $f = 1_A$ and $g = 1_B$, we have

$$\lim_{n\to\infty} \frac{|\{u \in Q_n \,:\, T^u x \in A\}|}{|\{u \in Q_n \,:\, T^u x \in B\}|} = \frac{\mu(A)}{\mu(B)}$$

Thus, the relative frequency with which an orbit visits A and B is equal to their relative masses. Note that for finite μ this result is an easy consequence of the pointwise ergodic theorem, which states that the frequency of visits to *each* of the sets is asymptotically equal to the mass of the set. This reasoning is not valid in the infinite measure case; indeed, the frequency of visits to a finite measure set is asymptotically zero. To see this, let A be fixed and choose a sequence of sets B_k with $\mu(B_k)\to\infty$. Then for each n,

$$\limsup_{n\to\infty}\frac{|\{u\in Q_n\,:\,T^ux\in A\}|}{|Q_n|}\leq\lim_{n\to\infty}\frac{|\{u\in Q_n\,:\,T^ux\in A\}|}{|\{u\in Q_n\,:\,T^ux\in B_k\}|}=\frac{\mu(A)}{\mu(B_k)}$$

Taking $k\to\infty$ we find that the frequency of visits to A is zero.

A related result, which implies the ratio ergodic theorem, is the Chacon-Ornstein lemma [3] which was proved for $\mathbb{Z}^k$-actions in [9]:

Theorem 2.2. *Let $(\Omega,\mathcal{B},\mu,T)$ be an i.m.p. $\mathbb{Z}^k$-action. Then for any non-negative $0\neq f\in L^1(\mu)\cap L^\infty(\mu)$,*

$$\lim_{n\to\infty}\frac{\sum_{u\in Q_{n+1}\setminus Q_n}f(T^ux)}{\sum_{u\in Q_n}f(T^ux)}=0\qquad\mu\text{-a.e.}$$

2.2. **The metric.** Let $(\Omega,\mathcal{B},\mu,T)$ be an i.m.p. $\mathbb{Z}^k$-action. For a co-finite partition $\mathcal{P}=\{P_1,\ldots,P_r\}$ we assume by convention that P_1 is the infinite atom, so $\mu(P_i)<\infty$ for $i=2,\ldots,r$ and $\bigcup_{i=2}^r P_i$ is the core of $\mathcal{P}$.

For $x\in\Omega$, write $\mathcal{P}(x)=i$ if $x\in P_i$, and given $n\in\mathbb{N}$ denote by $x_{\mathcal{P},n}\in\{1,\ldots,r\}^{Q_n}$ the $(\mathcal{P},n)$-name x, i.e., the coloring $u\mapsto\mathcal{P}(T^ux)$ of Q_n. It is convenient to introduce a distance $d(a,b)$ for $a,b\in\{1,\ldots,r\}^{Q_n}$ given by

$$d(a,b)=\frac{|\{u\in Q_n\,:\,a_u\neq b_u\}|}{|\{u\in Q_n\,:\,a_u\neq 1\text{ or }b_u\neq 1\}|}\tag{2.1}$$

using again the conventions that $0/0=0$. Then $d(a,b)\leq 1$, and the distance $d_{\mathcal{P},n}(\cdot,\cdot)$ from the introduction is nothing other than $d_{\mathcal{P},n}(x,y)=d(x_{\mathcal{P},n},y_{\mathcal{P},n})$.

Lemma 2.3. *d is a metric and $d_{\mathcal{P},n}$ is a pseudo-metric.*

Proof. The second statement follows from the first. Positivity and symmetry of d are immediate, so it remains to show, for $a,b,c\in\{1,\ldots,r\}^{Q_n}$, that $d(a,c)\leq d(a,b)+d(b,c)$.

We can assume that $b\neq a$, since otherwise the inequality is trivial. Let $A=\{u\in Q_n\,:\,a_u\neq 1\}$ and define B,C similarly using b,c, respectively. Clearly

$$d(a,b)\quad=\quad\frac{|A\setminus B|+|\{u\in B\,:\,a_u\neq b_u\}|}{|A\cup B|}$$

and similarly for $d(b,c)$.

Suppose first that $B\subseteq A\cup C$. We then have Since $|A\cup C|\geq|A\cup B|$, the inequality $d(a,c)\leq d(a,b)+d(b,c)$ will follow from the inequality for the numerators of the corresponding expressions above for $d(a,b),d(b,c)$, i.e. from

$$\begin{aligned}|\{u\in Q_n\,:\,a_u\neq c_u\}|\quad&\leq\quad|A\setminus B|+|\{u\in B\,:\,a_u\neq b_u\}|\;+\\&\qquad+|C\setminus B|+|\{u\in B\,:\,c_u\neq b_u\}|\end{aligned}$$

But this is clear since, for each $u\in Q_n$ which contributes to the left hand side, if $u\notin B$ then u contributes twice to the right hand side (to both $|A\setminus B|$ and $|C\setminus B|$), and if $u\in B$ it contributes at least once (since $a_u\neq c_u$ implies we can have both $a_u=b_u$ and $c_u=b_u$).

In the general case, let $B' = B \cap (A \cup C)$. Then the analysis above shows that

$$\begin{aligned} d(a,c) \quad \le \quad & \frac{1}{|A \cup B'|}\left(|A \setminus B'| + |\{u \in B' \,:\, a_u \neq b_u\}|\right) \; + \\ & + \frac{1}{|C \cup B'|}\left(|C \setminus B'| + |\{u \in B' \,:\, c_u \neq b_u\}|\right) \end{aligned}$$

The expression for $d(a,b) + d(b,c)$ is obtained from the right hand side by adding $|B \setminus A|$ to both numerator and denominator of the first term on the right, and similarly adding $|B \setminus C|$ to both the numerator and denominator of the second term. This has the effect of increasing them, and we obtain the triangle inequality. □

2.3. **The invariant.** For the sake of completeness we present a slight generalization of the invariants given in the introduction, aimed at accommodating other growth scales. A *growth function* $\rho : [0,\infty)^{\mathbb{N}} \to \mathbb{R}$ is a function of real-valued sequences $\underline{s} = s_1, s_2, \ldots$ satisfying

(a) Tail property: If $s_i = t_i$ for all sufficiently large i, then $\rho(\underline{s}) = \rho(\underline{t})$;
(b) Scaled monotonicity: If $s_i \le c t_i$ for $c > 0$ then $\rho(\underline{s}) \le \rho(\underline{t})$.

The principle example we shall be interested in is

$$\rho_{slow}(\underline{s}) = \limsup_{n\to\infty} \frac{\log(\log s_n)}{\log n}$$

We shall also refer to

$$\rho_{exp}(\underline{s}) = \limsup_{n\to\infty} \frac{\log s_n}{n}$$

Given a growth function $\rho(\cdot)$ and $\mathcal{P}$, ε and A as above, let

$$\rho(T,\mu,A,\mathcal{P}) = \lim_{\varepsilon \searrow 0}\left(\limsup_{n\to\infty} \rho\left((N(A, d_{\mathcal{P},n}, \varepsilon))_{n=1}^{\infty}\right)\right) \tag{2.2}$$

We shall show that this is independent of A (Corollary 2.12), and denote it $\rho(T,\mu,\mathcal{P})$ for short. Finally, define the ρ-entropy of the action by

$$\rho(T,\mu) = \sup \rho(T,\mu,\mathcal{P})$$

where the supremum is over co-finite partitions $\mathcal{P}$.

2.4. **Analysis.** We begin the analysis with some elementary facts about covering numbers. Given a pseud-metric d' on Ω it will be convenient to define the quantities $\operatorname{diam}(E, d')$ and $N(A, d', \varepsilon)$ in the same way as in the introduction, where the definitions were given for $d' = d_{\mathcal{P},n}$. Note that $N(A, d', \varepsilon)$ is non-decreasing as $\varepsilon \searrow 0$.

Lemma 2.4. *Let d^1, d^2 be pseudo-metrics on Ω. Let $a \ge 1$ and $\delta > 0$, and suppose that $A_0 \subseteq A$ satisfies $\mu(A_0) > (1-\delta)\mu(A)$ and $d^2(x,y) \le a d^1(x,y) + \delta$ for $x, y \in A_0$. Then $N(A, d^2, a\varepsilon + \delta) \le N(A, d^1, \varepsilon)$.*

Proof. Let $m = N(A, d^1, \varepsilon)$ and let $E_1, \ldots, E_m$ be an optimal (ε, d^1)-cover of A. Set $E'_i = E_i \cap A_0$, so that

$$\mu(A \setminus \bigcup_{i=1}^{m} E'_i) \le \mu(A \setminus \bigcup_{i=1}^{m} E_i) + \mu(A \setminus A_0) \le (\varepsilon + \delta)\mu(A) \le (a\varepsilon + \delta)\mu(A)$$

Since $E'_i \subseteq A_0$, and using the inequality between $d^2$and d^1, we have

$$\operatorname{diam}(E'_i, d^2) \le a \operatorname{diam}(E'_i, d^1) + \delta \le a \operatorname{diam}(E_i, d^1) + \delta < a\varepsilon + \delta$$

Therefore, $E_1', \dots, E_m'$ is a $(d^2, a\varepsilon+\delta)$-almost cover of A, so $N(A, d^2, a\varepsilon+\delta) \le m = N(A, d^1, \varepsilon)$. □

We now turn to the analysis of ρ-entropy. Recall that a partition $\mathcal{R}$ refines a partition $\mathcal{P}$ if every atom of $\mathcal{R}$ is a subset of an atom of $\mathcal{R}$.

Lemma 2.5. *If $\mathcal{R}$ refines $\mathcal{P}$ then $N(A, d_{\mathcal{P},n}, \varepsilon) \le N(A, d_{\mathcal{R},n}, \varepsilon)$ and consequently $\rho(T, \mu, A, \mathcal{P}) \le \rho(T, \mu, A, \mathcal{R})$.*

Proof. By the previous lemma it suffices to show that $d_{\mathcal{P},n}(x,y) \le d_{\mathcal{R},n}(x,y)$ for all $x, y \in \Omega$. Consider the intermediate partition

$$\mathcal{S} = \{P_1, R \,:\, R \in \mathcal{R} \text{ and } R \cap P_1 = \emptyset\}$$

Note that $\mathcal{S}$ refines $\mathcal{P}$. We first claim that

$$d_{\mathcal{P},n}(x,y) \le d_{\mathcal{S},n}(x,y)$$

Indeed, in the definition of these quantities, the expressions in the denominator agree for $\mathcal{P}$ and $\mathcal{S}$ since they have the same core. On the other hand the numerator corresponding to $\mathcal{P}$ is no greater than that of $\mathcal{S}$, because any $u \in Q_n$ which satisfies $\mathcal{S}(T^u x) = \mathcal{S}(T^u y)$ then it certainly also satisfies $\mathcal{P}(T^u x) = \mathcal{P}(T^u y)$, since $\mathcal{S}$ refines $\mathcal{P}$. It now remains to show that

$$d_{\mathcal{S},n}(x,y) \le d_{\mathcal{R},n}(x,y)$$

This follows from the fact that, when one compares the ratio defining the left and right hand sides, one finds that each $u \in Q_n$ which contributes to the numerator of the right hand side but not the left, also contributes the same amount to the denominator of the right hand side, but not the left.

The last statement of the lemma is immediate from the definitions and monotonicity of $N(A, d_{\mathcal{P},n}, \varepsilon)$ in ε. □

Recall that for $F \subseteq \mathbb{Z}^k$, the F-refinement of $\mathcal{P}$ is the partition $\mathcal{P}^F = \bigvee_{u \in F} T^u \mathcal{P}$, where $T^u \mathcal{P} = \{T^u P \,:\, P \in \mathcal{P}\}$. This is the coarsest partition which refines $T^u \mathcal{P}$ for all $u \in F$. Note that $\mathcal{P}^F$ is co-finite if $\mathcal{P}$ is.

Lemma 2.6. *If $0 \in F \subseteq \mathbb{Z}^k$ is finite and $\mathcal{R} = \mathcal{P}^F$, then $N(A, d_{\mathcal{R},n}, \varepsilon) \le N(A, d_{\mathcal{P},n}, \varepsilon/|F|)$, and in particular $\rho(T, \mu, A, \mathcal{P}) = \rho(T, \mu, A, \mathcal{R})$.*

Proof. By Lemma 2.4, it suffices to show that $d_{\mathcal{R},n}(x,y) \le |F| \cdot d_{\mathcal{P},n}(x,y)$ for all $x, y \in \Omega$. Write

$$U = \{u \in Q_n \,:\, T^u x \notin P_1 \text{ or } T^u y \notin P_1\}$$

for the union of the times in Q_n at which x or y visit the core of $\mathcal{P}$. Denote the set of $u \in Q_n$ where the $(\mathcal{P}, n)$-names differ by

$$D = \{u \in Q_n \,:\, \mathcal{P}(T^u x) \ne \mathcal{P}(T^u y)\}$$

By definition $d_{\mathcal{P},n}(x,y) = |D|/|U|$. Let U', D' be defined similarly using $\mathcal{R}$ instead of $\mathcal{P}$ (recall that $R_1 \in \mathcal{R}$ is the infinite atom of $\mathcal{R}$). Notice that $D' = (D+Q_n) \cap Q_n$ and $U' = (U+Q_n) \cap Q_n$, where $U + Q_n = \{u + v \,:\, u \in U\,,\, v \in Q_n\}$ and similarly for the other expression. We clearly have $|D'| \le |F| \cdot |D|$, and since $0 \in F$ we also have $|U'| \ge |U|$. Thus

$$d_{\mathcal{R},n}(x,y) = \frac{|D'|}{|U'|} \le |F| \cdot \frac{|D|}{|U|} = |F| \cdot d_{\mathcal{P},n}(x,y)$$

as desired.

For the second part of the lemma, notice that the first part implies that $\rho(T,\mu,A,\mathcal{P}) \geq \rho(T,\mu,A,\mathcal{R})$, while the reverse inequality was proved in the previous lemma. □

For co-finite partitions $\mathcal{P} = \{P_1,\ldots,P_r\}$ and $\mathcal{R} = \{R_1,\ldots,R_r\}$ of the same size, define the distance

$$\Delta(\mathcal{P},\mathcal{R}) = \frac{\mu(\bigcup_{i=1}^{r}(P_i\Delta R_i))}{\mu(\bigcup_{i=2}^{r}(P_i\cup Q_i))}$$

This is finite when $\mathcal{P},\mathcal{R}$ are co-finite (we continue to assume that P_1, R_1 are the infinite atoms). One can show that this is a metric in a similar manner to the proof of Lemma 2.3 but we shall not need this.

Lemma 2.7. *If $\mathcal{P} = \{P_1,\ldots,P_r\}, \mathcal{R} = \{R_1,\ldots,R_r\}$ are co-finite partitions and $\Delta(\mathcal{P},\mathcal{R}) < \varepsilon$, then $N(A, d_{\mathcal{P},n}, 6\varepsilon) \leq N(A, d_{\mathcal{R},n}, \varepsilon)$ for sufficiently large n, and visa versa. In particular, $\rho(T,\mu,A,\mathcal{P}) \leq \rho(T,\mu,A,\mathcal{R})$ and visa versa.*

Proof. By the ratio ergodic theorem, there is an n_0 and a set $A' \subseteq A$ with $\mu(A') > (1-\varepsilon)\mu(A)$, such that for $n > n_0$ and $x \in A'$,

$$\frac{|\{u \in Q_n \,:\, T^u x \in \bigcup_{i=1}^{r}(P_i\Delta R_i)\}|}{|\{u \in Q_n \,:\, T^u x \in \bigcup_{i=2}^{r}(P_i\cup R_i)\}|} < \varepsilon$$

which easily implies

$$d(x_{\mathcal{P},n}, x_{\mathcal{R},n}) \leq \frac{\varepsilon}{1-\varepsilon} < 2\varepsilon$$

It follows that

$$d_{\mathcal{P},n}(x,y) \leq d(x_{\mathcal{P},n}, x_{\mathcal{R},n}) + d(x_{\mathcal{R},n}, y_{\mathcal{R},n}) + d(y_{\mathcal{R},n}, y_{\mathcal{P},n}) \leq d_{\mathcal{R},n}(x,y) + 4\varepsilon$$

The inequality $N(A, d_{\mathcal{P},n}, 6\varepsilon) \leq N(A, d_{\mathcal{R},n}, \varepsilon)$ now follows from Lemma 2.4, and the last claim by applying $\rho(\cdot)$ and using monotonicity in ε. □

Write $\sigma(\mathcal{P})$ for the smallest sub-σ-algebra in $\mathcal{B}$ with respect to which $\mathcal{P}$ is measurable.

Lemma 2.8. *If $\mathcal{P}_n$ is a refining sequence of co-finite partitions such that $\sigma(\bigvee_{n=1}^{\infty}\mathcal{P}_n^{\mathbb{Z}^k}) = \mathcal{B} \bmod \mu$, then $\sup_{\mathcal{R}} \rho(T,\mu,A,\mathcal{R}) = \sup_n \rho(T,\mu,A,\mathcal{P}_n)$. In particular if $\mathcal{P}$ generates then $\sup_{\mathcal{R}} \rho(T,\mu,A,\mathcal{R}) = \rho(T,\mu,A,\mathcal{P})$*

Proof. Write $\beta = \sup_{\mathcal{R}} \rho(T,\mu,A,\mathcal{R})$. Fix $\varepsilon > 0$ and a co-finite partition $\mathcal{R}$ such that $\rho(T,\mu,A,\mathcal{R}) \geq \beta - \varepsilon$. Also let $\delta > 0$. Since $\rho(T,\mu,A,\mathcal{P}_n)$ is non-decreasing in n it follows that for each atom $R_i \in \mathcal{R}$ and all large enough n there is an $r = r(n,i)$ and $P_{n,i} \in \sigma(\mathcal{P}_n^{Q_r})$ such that $\mu(R_i\Delta P_{n,i}) < \delta$. It follows that we can choose an n and r such that there is a partition $\mathcal{P}$ which is coarser than $\mathcal{P}_n^{Q_r}$, and such that $\Delta(\mathcal{P},\mathcal{R}) < \delta$. Hence

$$\begin{aligned}
\rho(T,\mu,A,\mathcal{P}_n) &= \rho(T,\mu,A,\mathcal{P}_n^{Q_r}) \\
&\geq \rho(T,\mu,A,\mathcal{P}) \\
&\geq \rho(T,\mu,A,\mathcal{P},\delta) \\
&\geq \rho(T,\mu,A,\mathcal{R},6\delta)
\end{aligned}$$

where the equality is by Lemma 2.6, the first inequality is by Lemma 2.5, the second equality is by definition, and the last inequality is by Lemma 2.7. Taking the limit as $\delta \to 0$ gives

$$\rho(T,\mu,A,\mathcal{P}_n) \geq \rho(T,\mu,A,\mathcal{R}) \geq \beta - \varepsilon$$

The claim follows. □

Lemma 2.9. *There is a set $\Omega_0 \subseteq \Omega$ with $\mu(\Omega \setminus \Omega_0) = 0$ such that for every $0 < \varepsilon < 1$ and every $u \in \mathbb{Z}^k$, if $x, y \in \Omega$ then*

$$|d_{\mathcal{P},n}(x,y) - d_{\mathcal{P},n}(T^u x, T^u y)| < \varepsilon \tag{2.3}$$

for all large enough n

Proof. Let Ω_0 denote the set of points for which the Chacon-Ornstein lemma (Theorem 2.2) holds for the function $1_{\Omega\setminus P_1}$. We claim this is the desired set. Let $x, y \in \Omega_0$, let

$$U \quad (x) = \quad \{v \in \mathbb{Z}^k \,:\, \mathcal{P}(T^v x) = 1\}$$

and similarly $U(y)$, and let $U = U(x) \cup U(y)$. By our choice of Ω_0, if n is large enough then

$$\begin{aligned}\frac{|U \cap (Q_n \Delta (Q_n + u))|}{|U \cap Q_n|} &\leq \frac{|U(x) \cap (Q_n \Delta (Q_n + u)|}{|U(x) \cap Q_n|} + \frac{|U(y) \cap (Q_n \Delta (Q_n + u)|}{|U(y) \cap Q_n|} \\ &< \varepsilon\end{aligned}$$

since two ratios on the right hand side are just the ratios in the Chacon-Ornstein theorem applied to $1_{\Omega\setminus P_1}$ and the points x, y. This implies that, in the expressions for $d_{\mathcal{P},n}(x,y)$ and $d_{\mathcal{P},n}(T^u x, T^u y)$, the numerators differ by at most a multiplicative factor of $(1\pm\varepsilon)$, and similarly the denominators. Using the fact that $(1+\varepsilon)/(1-\varepsilon) \leq 1+4\varepsilon$ for $0 < \varepsilon < 1$ and the fact that $d_{\mathcal{P},n} \leq 1$, we obtain

$$|d_{\mathcal{P},n}(x,y) - d_{\mathcal{P},n}(T^u x, T^u y)| \leq 4\varepsilon$$

the lemma follows. □

Corollary 2.10. *Let $\varepsilon > 0$ and $u \in \mathbb{Z}^k$. Then $N(A, \mathcal{P}, n, \varepsilon) \leq N(T^u A, \mathcal{P}, n, \varepsilon/2)$ for all sufficiently large n.*

Proof. Write $d^u_{\mathcal{P},n}(x,y) = d_{\mathcal{P},n}(T^u x, T^u y)$. This is a pseudo-metric on A and by the previous lemma we can find $n_0 \in \mathbb{N}$ and $A_0 \subseteq A$ with $\mu(A_0) > (1-\varepsilon)\mu(A)$ and such that (2.3) holds for $n > n_0$ and $x, y \in A_0$. It follows from Lemma 2.4 that $N(A, d_{\mathcal{P},n}, \varepsilon) \leq N(A, d^u_{\mathcal{P},n}, \varepsilon/2)$, and the result follows since $N(A, d^u_{\mathcal{P},n}, \varepsilon/2) = N(T^u A, d_{\mathcal{P},n}, \varepsilon/2)$. □

Lemma 2.11. *If $A \subseteq B$ are sets of positive finite μ-measure then, for all large enough n,*

$$N(A, \mathcal{P}, n, c_1\varepsilon) \leq N(B, \mathcal{P}, n, \varepsilon) \leq c_2 N(A, \mathcal{P}, n, \varepsilon/c_3)$$

where c_1, c_2, c_3 are constants which do not depend on n. In particular, $\rho(T, \mu, A, \mathcal{P}) = \rho(T, \mu, B, \mathcal{P})$.

Proof. Fix $0 < \varepsilon < 1$ and let $E_1, \ldots, E_m$ be a collection of size $m = N(B, d_{\mathcal{P},n}, \varepsilon)$ which is an ε-almost cover of B. Then

$$\mu(A \setminus \bigcup_{i=1}^m E_i) \leq \mu(B \setminus \bigcup_{i=1}^m E_i) = \varepsilon\mu(B) = \varepsilon\frac{\mu(B)}{\mu(A)} \cdot \mu(A)$$

so $E_1, \ldots, E_m$ is an $\varepsilon\mu(B)/\mu(A)$-almost cover of A. It follows that $N(A, d_{\mathcal{P},n}, \frac{\mu(B)}{\mu(A)}\varepsilon) \leq N(B, d_{\mathcal{P},n}, \varepsilon)$.

For the other inequality we argue as follows. By ergodicity we have $\mu(\Omega \setminus \bigcup_{u\in\mathbb{Z}^k} T^u A) = 0$, so there is a finite set $F \subseteq \mathbb{Z}^k$ such that $\mu(B \setminus \bigcup_{u\in F} T^u A) < \frac{\varepsilon}{2}\mu(B)$. Write $\widetilde{A} = \bigcup_{u\in F} T^u A$, and note that $\mu(\widetilde{A}) \leq |F| \cdot \mu(A)$, so

$$\frac{\mu(B \cap \widetilde{A})}{\mu(\widetilde{A})} \geq \frac{(1-\varepsilon/2)\mu(B)}{|F|\mu(A)} \geq \frac{1}{2|F|}$$

By the previous corollary, for each $u \in F$ and large enough n we have

$$N(T^u A, d_{\mathcal{P},n}, \frac{\varepsilon}{4|F|}) \leq N(A, d_{\mathcal{P},n}, \frac{\varepsilon}{8|F|})$$

so for large enough n we have

$$N(\widetilde{A}, d_{\mathcal{P},n}, \frac{\varepsilon}{4|F|}) \leq |F| \cdot N(A, d_{\mathcal{P},n}, \frac{\varepsilon}{8|F|})$$

Applying the first part of the current lemma to the containment $B \cap \widetilde{A} \subseteq \widetilde{A}$, we find that

$$N(B \cap \widetilde{A}, d_{\mathcal{P},n}, \frac{\varepsilon}{2}) \leq N(\widetilde{A}, d_{\mathcal{P},n}, \frac{\mu(B \cap \widetilde{A})}{\mu(\widetilde{A})} \cdot \frac{\varepsilon}{2}) \leq N(\widetilde{A}, d_{\mathcal{P},n}, \frac{\varepsilon}{4|F|})$$

Since $\mu(B \cap \widetilde{A}) \geq (1 - \varepsilon/2)\mu(B)$ it follows that

$$N(B, d_{\mathcal{P},n}, \varepsilon) \leq N(B \cap \widetilde{A}, d)(\frac{\varepsilon}{2})$$

combining the last three inequalities gives the desired result ☐

Corollary 2.12. *If A, B are sets of finite measure then $\rho(T, \mu, \mathcal{P}, A) = \rho(T, \mu, \mathcal{P}, B)$.*

Proof. Let $C = A \cup B$. Since $A \subseteq C$ and $B \subseteq C$, the previous lemma and monotonicity of covering numbers in ε implies that $\rho(T, \mu, \mathcal{P}, A) = \rho(T, \mu, \mathcal{P}, C)$ and $\rho(T, \mu, \mathcal{P}, B) = \rho(T, \mu, \mathcal{P}, C)$, and the conclusion follows. ☐

2.5. **Connections with recurrence.** In the previous sections we defined ρ-entropy using name counts. In this section we give a slightly simpler characterization in terms of the complexity of recurrence patterns.

Let $\rho(\cdot)$ be a growth function and $A \subseteq \Omega$ a set of positive and finite measure. For $x \in A$, recall that the pattern of returns to A "up to time n" is

$$R_n(A, x) = \{u \in \mathbb{Z}^k \,:\, \|u\| \leq n \text{ and } T^u x \in A\}$$

In order to compare return patterns of $x, y \in A$, for $n \in \mathbb{N}$ introduce the distance $d_{A,n}(x, y)$ by

$$d_{A,n}(x, y) = \frac{|R_n(A, x) \Delta R_n(A, y)|}{|R_n(A, x) \cup R_n(A, y)|}$$

Next, let $N_n(A, \varepsilon) = N(A, d_{A,n}, \varepsilon)$ and set

$$\widetilde{\rho}(T, \mu) = \sup_A \left(\lim_{\varepsilon \searrow 0} \left(\limsup_{n \to \infty} \rho\left((N_n(A, \varepsilon))_{n=1}^{\infty}\right) \right) \right)$$

where the supremum is over measurable $A \subseteq \Omega$ of positive finite measure.

Proposition 2.13. *$\widetilde{\rho}(T, \mu) = \rho(T, \mu)$ for i.m.p. actions and zero-entropy f.m.p. actions. $\widetilde{\rho}_{slow}(T, \mu) = \rho_{slow}(T, \mu)$ in all cases.*

Proof. Note that $d_{A,n}(x, y) = d_{\{A, \Omega \setminus A\}, n}(x, y)$, where the right hand side is as in equation (1.1). This implies that $\widetilde{\rho}(T, \mu) \leq \rho(T, \mu)$ for all actions.

For the reverse inequality, when the action is i.m.p. or f.m.p. with entropy zero, choose A such that $\mathcal{P} = \{A, \Omega \setminus A\}$ generates for the action (in the former case this can be done by Krengel's theorem [14], in the latter by Krieger's generator theorem [15]). By Lemma 2.8 we have

$$\lim_{\varepsilon \searrow 0} \left(\limsup_{n \to \infty} \rho\left((N_n(A, \varepsilon))_{n=1}^{\infty}\right) \right) = \rho(T, \mu, A, \{A, \Omega \setminus A\}) = \rho(T, \mu)$$

and hence $\widetilde{\rho} \geq \rho$.

For ρ_{slow}, it remains to prove equality for positive-entropy f.m.p. actions. In this case observe that one can always find a set A such that the partition $\{A, \Omega \setminus A\}$ has positive entropy; hence $\widetilde{\rho}_{slow}((N(A, d_{A,n}, \varepsilon))_{n=1}^{\infty}) = 1$ for all small enough $\varepsilon > 0$, and so $\widetilde{\rho}_{slow}(T, \mu) \geq 1 = \rho_{slow}(T, \mu)$. The reverse inequality was established at the beginning of the proof. □

Recall from the introduction that

$$\alpha(T, \mu) = \limsup_{n \to \infty} \frac{\log |R_n(A, x)|}{\log |Q_n|}$$

which is independent of A and a.s. independent of x. We show next that $\rho_{slow}(T, \mu) \leq \alpha(T, \mu)$.

Proof of Proposition 1.4. Using the trivial binomial bound $\binom{n}{m} \leq n^m$, for $\delta > 0$ we find that the number of subsets $E \subseteq Q_n$ with $|E| \leq |Q_n|^\delta$ is at most

$$\binom{|Q_n|}{\lceil |Q_n|^\delta \rceil} \leq |Q_n|^{\lceil |Q_n|^\delta \rceil} \leq |Q_n|^{2|Q_n|^\delta} = 2^{2|Q_n|^\delta \log |Q_n|}$$

Given a set $A \subseteq \Omega$ of finite measure and $\varepsilon \geq 0$, by definition of $\alpha = \alpha(T, \mu)$ there is an n_0 and a subset $A_\varepsilon \subseteq A$ such that $\mu(A_\varepsilon) > (1 - \varepsilon)\mu(A)$, and if $n > n_0$ and $x \in A_\varepsilon$ then $|R_n(A, x)| \leq |Q_n|^{\alpha+\varepsilon}$. Fix $n > n_0$. For each $E \subseteq Q_n$ of size $|E| \leq |Q_n|^{\alpha+\varepsilon}$ let $A_\varepsilon^E = \{x \in A_\varepsilon : R_n(A, x) = E\}$. Clearly $\operatorname{diam}(A_\varepsilon^E, d_{A,n}) = 0$. On the other hand the estimate above tells us that one can cover A_ε by $2^{2|Q_n|^\alpha \log |Q_n|}$ sets of this form. Therefore, for $n > n_0$ we have

$$N(A, d_{A,n}, \varepsilon) \leq 2^{2|Q_n|^{\alpha+\varepsilon} \log |Q_n|}$$

Plugging this into the definition we find that $\widetilde{\rho}(T, \mu) \leq \alpha + \varepsilon$, and hence by the previous proposition the same is true for $\rho_{slow}(T, \mu)$. Since $\varepsilon > 0$ was arbitrary, this proves the claim. □

3. Lipschitz actions and examples

3.1. **Lipschitz actions, dimension and slow entropy.** A map $f : \Omega \to \Omega$ is Lipschitz if there is a constant C such that $d(fx, fy) \leq Cd(x, y)$. The smallest constant with this property is denoted $\operatorname{Lip}(f)$. Note that

$$\operatorname{Lip}(f \circ g) \leq \operatorname{Lip}(f) \cdot \operatorname{Lip}(g)$$

Note also that diffeomorphisms of Riemannian manifolds are Lipschitz maps.

Lemma 3.1. *Suppose $\mathbb{Z}^k$ acts on a metric space (Ω, d) by Lipschitz maps. Then there is a constant C such that $\operatorname{Lip} T^u \leq C^{\|u\|}$.*

Proof. Let $T_1, \ldots, T_k$ be generators of the action and

$$C = \sup_{i=1,\ldots,k} \{\operatorname{Lip}(T_i), \operatorname{Lip}(T_i^{-1})\}$$

Then for $u = (u_1, \ldots, u_k) \in \mathbb{Z}^k$,

$$\begin{aligned}
\operatorname{Lip}(T^u) &= \operatorname{Lip}(T^{u_1} T^{u_2} \ldots T^{u_n}) \\
&\leq \prod_{i=1}^k \operatorname{Lip}(T^{u_i}) \\
&\leq \prod_{i=1}^k (\operatorname{Lip} T_i)^{|u_i|} \\
&\leq C^{\sum |u_i|}
\end{aligned}$$

the claim follows. □

Recall that if (Ω, d) is a compact metric space then the ε-separation number $\text{sep}(\Omega, d, \varepsilon)$ is the size of the largest $E \subseteq \Omega$ such that $d(x, y) \geq \varepsilon$ for distinct $x, y \in E$. The upper Minkowski (box) dimension (Ω, d) is

$$\text{bdim}(\Omega, d) = \limsup_{\varepsilon \searrow 0} \frac{\log \text{sep}(\Omega, d, \varepsilon)}{\log \varepsilon}$$

so $\text{bdim}(\Omega, d) < \alpha$ implies that $\text{sep}(\Omega, d, \varepsilon) \leq c \cdot (1/\varepsilon)^{\alpha}$. Note that if Ω is a compact manifold of dimension m and d is a Riemannian metric, then $\text{bdim}(\Omega, d) = m$.

Finally, recall that when $\mathbb{Z}^k$ acts on a metric space (Ω, d) then the Bowen metric on Ω is

$$d_n^{\infty}(x, y) = \sup_{\|u\| \leq n} d(T^u x, T^u y) \tag{3.1}$$

Lemma 3.2. *Suppose that (Ω, d) has finite box dimension and that $\mathbb{Z}^k$ acts on it by Lipschitz maps. Then there is a constant C such that* $\text{sep}(\Omega, d_n, \varepsilon) \leq C^n/\varepsilon^C$.

Proof. Let C_1 be a constant such that $\text{Lip}(T^u) \leq C_1^{\|u\|}$, and let C_2 be a constant such that $\text{sep}(\Omega, d, \varepsilon) \leq C_2 (1/\varepsilon)^{C_2}$. Notice that if $d_n^{\infty}(x, y) \geq \varepsilon$ then there is some u with $\|u\| \leq n$ such that $d(T^u x, T^u y) \geq \varepsilon$, so

$$d(x, y) \geq \frac{\varepsilon}{C_1^{\|u\|}} \geq \frac{\varepsilon}{C_1^n}$$

Therefore if $E \subseteq \Omega$ is a $(d_n^{\infty}, \varepsilon)$-separated set it is also $(d, \varepsilon/C_1^n)$-separated, hence

$$\text{sep}(\Omega, d_n^{\infty}, \varepsilon) \leq \text{sep}(\Omega, d, \varepsilon/C_1^n) \leq C_2 \cdot C_1^{C_2 n}/\varepsilon^{C_2}$$

the claim follows. □

We now turn to the proof of Theorem 1.3.

Lemma 3.3. *If $(\Omega, \mathcal{B}, \mu, T)$ is an ergodic with non-uniform slow entropy and $\rho_{slow}(T, \mu) > \beta$, then there is a co-finite partition $\mathcal{P}$ with core A such that $\rho_{slow}(T, \mu, \mathcal{P}) > \beta$ and $\rho_{slow}(T, \mu, \mathcal{P}) > \rho_{slow}(T, \mu, \{\Omega \setminus A, A\})$.*

Proof. Since the action has non-uniform slow entropy, there is a co-finite partition $\mathcal{P}' = \{P_1', \dots, P_m'\}$ such that $\rho_{slow}(T, \mu, \mathcal{P}') < \rho_{slow}(T, \mu)$. Let A denote the core of $\mathcal{P}'$ and let $\mathcal{P}'' = \{\Omega \setminus A, A\}$. Since $\mathcal{P}'$ is a refinement of $\mathcal{P}''$, by Lemma 2.5 we have $\rho_{slow}(T, \mu, \mathcal{P}'') < \rho_{slow}(T, \mu)$.

Let $\mathcal{P}_n$ be a refining sequence of co-finite partitions with common core A and which separate points in A. By ergodicity, the hypothesis of Lemma 2.8 is satisfied, and we can choose an n such that the partition $\mathcal{P}''' = \mathcal{P}_n$ satisfies $\rho_{slow}(T, \mu, \mathcal{P}''') > \rho_{slow}(T, \mu, \mathcal{P}'')$ and also $\rho_{slow}(T, \mu, \mathcal{P}''') > \beta$. This is the desired partition. □

We now begin the proof of Theorem 1.3. Suppose that $(\Omega, \mathcal{B}, \mu, T)$ is an ergodic $\mathbb{Z}^k$-action by Lipschitz maps with respect to the compact, separable metric d on Ω. Assuming $\rho_{slow}(T, \mu) > 1$ and the action does not have uniform slow entropy, we will show that $\text{bdim}(\Omega, d) = \infty$.

Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a co-finite partition with core A and write $\mathcal{A} = \{\Omega \setminus A, A\}$. Assume that $\rho_{slow}(T, \mu, \mathcal{P}) > 1$ and $\rho_{slow}(T, \mu, \mathcal{P}) > \rho_{slow}(T, \mu, \mathcal{A})$, as we may by the previous lemma. Choose β, γ and $\varepsilon > 0$ such that $\gamma > 1$ and

$$\rho_{slow}(A, d_{\mathcal{P},n}, \varepsilon) > \gamma > \beta > \rho_{slow}(A, d_{\mathcal{A},n}, \varepsilon)$$

Recall that a finite Borel measure μ on a complete separable metric space is inner regular, i.e.

$$\mu(C) = \sup\{\mu(K) : K \subseteq C \text{ and } K \text{ is compact}\}$$

Applying this to the restriction of μ to A, define a partition $\mathcal{R}$ as follows. Let $R_1 = P_1$, so $\mathcal{R}$ and $\mathcal{P}$ have the same core A, and replace each finite-measure atom $P_i \in \mathcal{P}$ by sets K_i and $P_i \setminus K_i$, where K_i is a compact set satisfying

$$\mu(K_i) > (1 - \frac{\varepsilon}{8|\mathcal{P}|})\mu(P_i) \tag{3.2}$$

Note that $\mathcal{R}$ refines $\mathcal{P}$, so

$$\rho_{slow}(T, \mu, \mathcal{R}, \varepsilon) > \gamma$$

Since the K_i are compact there is a $\tau > 0$ such that

$$\min\{d(x, y) : x \in K_i \ , \ y \in K_j\} > \tau \qquad \text{for all } i \neq j$$

Also, write

$$K = \bigcup_{i=1}^{m} K_i$$

By the ratio ergodic theorem, we may choose an n_0 and a set $A_0 \subseteq A$ of measure $\mu(A_0) > (1 - \frac{\varepsilon}{4})\mu(A)$ such that, for $n > n_0$ and $x \in A_0$,

$$\frac{\sum_{u \in Q_n} 1_{P_i \setminus K_i}(T^u x)}{\sum_{u \in Q_n} 1_A(T^u x)} < \frac{\mu(P_i \setminus K_i)}{\mu(A)} + \frac{\varepsilon}{8|\mathcal{P}|} \qquad i = 2, \ldots, m \tag{3.3}$$

Lemma 3.4. *Let $n > n_0$ and $x, y \in A_0$, and suppose that $d_{\mathcal{A},n}(x, y) < \varepsilon/4$ and $d_{\mathcal{R},n}(x, y) \geq \varepsilon/2$. Then $d_n^\infty(x, y) \geq \tau$.*

Proof. Let $U = \{u \in Q_n \ : \ T^u x \in A \text{ or } T^u y \in A\}$. We then have

$$\begin{aligned} d_{\mathcal{R},n}(x, y) &= \frac{\{u \in U \ : \ T^u x \notin A \text{ or} T^u y \notin A\}}{|U|} + \\ &\quad + \frac{\{u \in Q_n \ : \ \mathcal{R}(T^u x) \neq \mathcal{R}(T^u y) \text{ and } T^u x, T^u y \in A\}}{|U|} \end{aligned}$$

The first term is just $d_{\mathcal{A},n}(x, y)$, so

$$\frac{\{u \in Q_n \ : \ \mathcal{R}(T^u x) \neq \mathcal{R}(T^u y) \text{ and } T^u x, T^u y \in A\}}{|U|} > d_{\mathcal{R},n}(x, y) - \frac{\varepsilon}{4} \geq \frac{\varepsilon}{4}$$

Now the set in the numerator of the left hand side can be written as a sum of two terms, the first consisting of those u for which $T^u x \in A \setminus K$ or $T^u y \in A \setminus K$, and the second of those for which both $T^u x, T^u y \in K$. Using (3.3), we see that the first term contributes at most $\varepsilon/4$. It follows that

$$\frac{|\{u \in Q_n \ : \ \mathcal{R}(T^u x) \neq \mathcal{R}(T^u y) \text{ and } T^u x, T^u y \in K\}|}{|U|} > 0$$

Thus there is at least one $u \in Q_n$ such that $T^u x \in K_i$ and $T^u y \in K_j$ for some $i \neq j$. Hence $d(T^u x, T^u y) > \tau$, and so $d_n^\infty(x, y) > \tau$. $\square$

Fix n and let $E_1, \ldots, E_s$ be a minimal $(d_{\mathcal{A},n}, \varepsilon/4)$-almost cover of A, i.e. $\text{diam}(E_i, d_{\mathcal{A},n}) < \varepsilon/4$ and $\mu(A \setminus \bigcup_{i=1}^s E_i) > (1 - \varepsilon/4)\mu(A)$. Since $\rho_{slow}(A, d_{\mathcal{A},n}, \varepsilon/4) < \beta$, for large enough n we will have $s < 2^{n^\beta}$.

Let $E_i' = E_i \cap A_0$. For each $i = 1, \ldots, s$, if $\text{sep}(E_i', d_{\mathcal{R},n}, \varepsilon/2) < 2^{n^{(1+\gamma)/2}}$ then we can cover E_i' with $2^{n^{(1+\gamma)/2}}$ $d_{\mathcal{R},n}$-balls of radius $\varepsilon/2$. If this held for all $i = 1, \ldots, s$ we would have a collection of at most $2^{n^\beta} \cdot 2^{n^{(1+\gamma)/2}}$ sets of $d_{\mathcal{R},n}$-diameter ε which completely cover $\bigcup_{i=1}^s E_i'$, and since

$$\mu(A \setminus \bigcup_{i=1}^{s} E_i') \leq \mu(A \setminus A_0) + \mu(A \setminus \bigcup_{i=1}^{s} E_i) \leq \varepsilon/2$$

this would be a $(d_{\mathcal{R},n}, \varepsilon)$-cover of A. Since $2^{n^\beta} \cdot 2^{n^{(1+\gamma)/2}} < 2^{n^\gamma}$ for all large enough n and since $\rho_{slow}(A, d_{\mathcal{R},n}, \varepsilon) > \gamma$, there must be arbitrarily large n for which the above fails, i.e. there is a set $E \subseteq A_0$ of $d_{\mathcal{A},n}$-diameter $\leq \varepsilon/4$ and $\text{sep}(E, d_{\mathcal{R},n}, \varepsilon/2) \geq 2^{n^{(1+\gamma)/2}}$. The last condition means that there is a collection $I \subseteq E$ of size $|I| \geq 2^{n^{(1+\gamma)/2}}$ such that $d_{\mathcal{R},n}(x,y) \geq \varepsilon/2$ for distinct $x, y \in I$. Applying Lemma 3.4, we have $d_n^\infty(x,y) \geq \tau$ for $x, y \in I$. Thus $\text{sep}(\Omega, d_n, \tau) \geq \text{sep}(I, d_n, \tau) > 2^{n^\gamma}$, and since this holds for infinitely many n and since $(1+\gamma)/2 > 1$, we conclude from Lemma 3.1 that $\text{bdim}(\Omega, d) = \infty$.

This concludes the proof of Theorem 1.3.

As a final remark, note that the proof above would have been much simplified if we could find a compact set K_1 such that $\mu(P_1 \setminus K_1)$ is very close to 0. However, since $\mu(P_1) = \infty$, which may not happen; that is, "most" of the mass of P_1 may accumulate near the core. The assumption of non-uniform slow entropy in Theorem 1.3 allows us to avoid this problem by "relativising" the problem to A.

3.2. **Cutting and stacking.** Our next task is to construct a $\mathbb{Z}^2$ action which does not have uniform slow entropy and with $\rho_{slow} > 1$. We shall combine several methods. The first is cutting and stacking, though a better name in the multidimensional case might be cutting and tiling. We shall only require the rank-1 version, which we describe next.

Begin with an abstract atomless σ-finite Lebesgue space of infinite measure, which we refer to as the *pool*. We shall define an action on increasing subsets of the pool, eventually arriving at the desired action on a part or all of the pool. It is convenient also to imagine that, as we define our subset and action, we also color the points we use, thus defining a partition of the pool according to the colors. We shall use this informal coloring procedure, since a more formal definition does not seem to add much.

Recall that any measurable subset A of a Lebesgue space with measure $w > 0$ can be identified, in a measurable and measure-preserving fashion, with an interval of length w in the real line. An *arrangement of radius r and width $w \subseteq \mathbb{R}^2$* is an assignment $u \mapsto A_u$ from $u \in Q_r$ to pairwise disjoint subsets A_u of the pool. The A_u have measure w and we identify them with intervals of length w. Each A_u is colored monochromatically by a color which may depend on u. If $x \in A_u$ we call u the position of x, and we say that x is in the arrangement if $x \in \bigcup_{u \in Q_r} A_u$.

Given an arrangement as above there is a partial measure-preserving action T on $\bigcup_{u \in Q_r} A_u$, defined as follows. If $x \in A_u$ and $v \in \mathbb{Z}^2$ is such that $u + v \in Q_r$, then $T^v x \in A_{u+v}$ is the point corresponding to x when the intervals A_u and A_{u+v} are identified by a translation (this is the reason we identify A_u with intervals: it provides canonical isomorphisms between them).

The cutting and tiling construction is a recursive procedure in which, at the i-th stage, one has an arrangement $\mathcal{A}_i = (A_{i,u})_{u \in Q_{r(i)}}$ of radius $r(i)$ and width $w(i)$ in which each $A_{i,u}$ is monochromatic, along with the associated partial actions. One begins with some collection of arrangements of radius 0 (recall that $|Q_0| = 1$). Assuming we have carried out the construction up to step i, one constructs $\mathcal{A}_{i+1}$ using the following two steps:

Cutting: Choose an integer m and partition each interval $A_{i,u}$ into m equal subintervals $A_{i,u,j}$, $1 \leq j \leq m$, each of length $w(i+1) = w(i)/m$. For each such j we obtain a new arrangement $\mathcal{A}_{i,j} = (A_{i,u,j})_{u \in Q_{r(i)}}$ of radius $r(i)$ and width $w(i+1)$.

Tiling: Choose $r(i+1)$ and a function $\psi : \{1, \ldots, m\} \to Q_{r(i+1)-r(i)}$ such that $\|\psi(j_1) - \psi(j_2)\| \geq 2r(i)$ for all $1 \leq j_1 < j_2 \leq m$. For each $1 \leq j \leq m$, translate $\mathcal{A}_{i,j}$ to $\psi(j)$, thus defining $\mathcal{A}_{i+1}$ at the sites $\varphi(j) + u$, $u \in Q_{r(i)}$. More precisely, for $u \in Q_{r(i)}$ set $A_{i+1,\psi(j)+u} = A_{i,u,j}$. Note that $\varphi(j) + u \in Q_{r(i+1)}$, and no conflicts occur, due to our assumptions on ψ. Finally, to the remaining $u \in Q_{r(i+1)}$ assign new intervals taken from the pool, and assign a color to each interval.

Let $\Omega_i = \bigcup_{u \in Q_{r(i)}} A_{i,u}$ and $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, and let $\mathcal{P}$ denote the measurable partition of Ω into monochromatic sets. Note that the partial action defined on Ω_{i+1} extends the partial action on Ω_i, so on Ω we have a well-defined measure-preserving partial action T. We would like conditions which ensure that T is an a.e.-defined action. To this end, let ε_i denote the total mass of points in Ω_i which in in $\mathcal{A}_{i+1}$ are located within distance i of the boundary of $Q_{r(i+1)}$. If $\sum \varepsilon_i < \infty$, then by Borel-Cantelli lemma, for every $v \in \mathbb{Z}^k$, a.e. point $x \in \Omega$ belongs at some stage to an arrangement in which is located at distance at least $\|v\|$ from the boundary, and hence $T^v x$ is defined. Thus T is a $\mathbb{Z}^2$-action defined a.e. on Ω.

The following fact is standard:

Proposition 3.5. *Rank-one cutting and stacking constructions produce ergodic actions.*

3.3. **A rank-1 construction.** As a preliminary step in the proof of Theorem 1.1 we construct a certain action $(\Omega_0, \mathcal{B}_0, \mu_0, T_0)$ in the manner described above. We use only two colors, 0 and 1. The construction will be determined by the sequence $r(i) \to \infty$, upon which we place a few requirements. Write

$$s(i) = r(i+1) - r(i)$$

and

$$m(i) = s(i)^{1/3}$$

We assume $1 \leq r(1) < r(2) < \ldots$ grows quickly enough that

(1) $m(i)$ is an integer and $m(i) > 2r(i)$,
(2) $\prod_{j \leq i} r(j) = r(i)^{1+o(1)}$

Begin at step $i = 1$ with a single arrangement of radius $r(0) = 0$, width 1, and color 1. At later steps we will only add mass colored 0.

For the inductive step, suppose we have already constructed the arrangement $\mathcal{A}_i = (A_{i,u})_{u \in Q_{r(i)}}$ of width $w(i)$. Define

$$\Gamma_i = Q_{s(i)} \cap m(i)\mathbb{Z}^2$$

Note that

$$|\Gamma_i| = (2\frac{s(i)}{m(i)} + 1)^2 = r(i+1)^{3/2+o(1)}$$

Cut $\mathcal{A}_i$ into $|\Gamma_i|$ sub-arrangements of equal width $w(i+1) = w(i)/|\Gamma_i|$, which we denote $\mathcal{A}_{i,u}$, $u \in \Gamma_i$. Let $\psi_i : \{1, \ldots, |\Gamma_i|\} \to \Gamma_i$ be a fixed bijection, and form $\mathcal{A}_{i+1}$ from this data as described in the previous section, coloring all new intervals with the color 0. Note that ψ_i satisfies the requirements in the tiling stage by property (1) of the growth of $r(i)$.

Let $(\Omega_0, \mathcal{B}_0, \mu_0, T_0)$ denote the resulting action, which is ergodic by Proposition 3.5. Let $A \subseteq \Omega_0$ denote the set of points colored 1, and note that $\mu(A) = 1$ since we did not add any more points with this color after the first step.

It will be useful to have a more direct description of the coloring of $Q_{r(i)}$ derived from $\mathcal{A}_i$. For sets $U, V \subseteq \mathbb{Z}^k$ define $U + V = \{u + v \,:\, u \in U\,,\, v \in V\}$, and similarly for more than two summands. Also, abbreviate $U + v = U + \{v\}$. Let

$$\Gamma_i^* = \Gamma_1 + \ldots + \Gamma_i$$

Then $u \in Q_{r(i)}$ is colored 1 if and only if

$$u \in \Gamma_1 + \ldots + \Gamma_{i-1} = \Gamma_{i-1}^*$$

Note that, because of the super-exponential growth of $m(i)$, such a u has a unique representation as $u = \sum_{j=1}^{i} u_j$ with $u_j \in \Gamma_j$. We also have estimates on $|\Gamma_i^*|$:

Claim 3.6. $|\Gamma_i^*| = r(i-1)^{3/2+o(1)}$

Proof. On the one hand,

$$|\Gamma_1 + \ldots + \Gamma_{i-1}| \geq |\Gamma_{i-1}| = r(i)^{3/2+o(1)}$$

On the other hand,

$$|\Gamma_1 + \ldots + \Gamma_{i-1}| \leq \prod_{j=1}^{i-1} |\Gamma_j| = \prod_{j=1}^{i-1} r(j+1)^{3/2+o(1)} = r(i)^{3/2+o(1)}$$

where the last equality is by property (2) of the growth of $r(i)$. Combining these gives the claim. □

Claim 3.7. $\mu_0(\Omega_0) = \infty$.

Proof. The total number of intervals in $\mathcal{A}_i$ is $|Q_{r(i)}| = r(i)^{2+o(1)}$. The number of intervals in $\mathcal{A}_i$ which already appeared in $\mathcal{A}_{i-1}$ is $|\Gamma_i^*| = r(i)^{3/2+o(1)}$. Since the ratio tends to ∞ as $i \to \infty$, we have $\mu_0(\Omega_i \setminus \Omega_{i-1}) \to \infty$, so $\mu_0(\Omega) = \infty$. □

We do not require the next claim, since without it we can simply pass to the factor of $(\Omega_0, \mathcal{B}_0, \mu_0, T_0)$ corresponding to the smallest invariant σ-algebra containing A. Therefore we only briefly indicate the proof.

Claim 3.8. The partition $\mathcal{P} = \{\Omega_0 \setminus A, A\}$ generates for T_0.

Proof. We claim that if $u \in Q_{r(i)}$ and $x \in A_{i,u}$ then u can be recovered from the $(\mathcal{P}, 2r(i))$-name of x. Indeed, let it is easy to see that

$$R_{2r(i)}(A, x) = \Gamma_i^* - u$$

Since Γ_i^* is symmetric about the axes, we can recover u from $R_{2r(i)}(A, x)$ by

$$u = -\frac{1}{|R_{2r(i)}(A, x)|} \sum_{v \in R_{2r(i)}(A,x)} v$$

It follows that $\bigvee_{u \in Q_{2r(i)}} T^u \mathcal{P}$ partitions Ω_i into intervals of length $w(i)$, and since $w(i) \to 0$, we find that $\bigvee_{u \in \mathbb{Z}^2} T^u \mathcal{P}$ separates points. □

Claim 3.9. $\rho_{slow}(T_0, \mu_0) = 0$.

Proof. Let us estimate $N(A, d_{A,r}, \varepsilon)$ (see section 2.5). Fix r in the range

$$m(i-1) - 2r(i-1) \leq n < m(i) - 2r(i)$$

If $x \in A$ then $R_n(A, x)$ is completely determined by the position of x in $\mathcal{A}_i$, since by the construction distances between occurrences of sub-arrangements of $\mathcal{A}_i$ in $\mathcal{A}_j$ for any $j \geq i$ are at least $m(i)-2r(i)$, which is greater than n). As we already observed, the possible positions of x in $\mathcal{A}_i$ are the vectors $u\in\Gamma_i^*$. Hence

$$\begin{aligned}N(A,d_{A,n},\varepsilon) &\leq |\Gamma_i^*|\\ &\leq r(i)^{3/2+o(1)}\\ &\leq m(i-1)^{9/2+o(1)}\\ &\leq (m(i-1)-2r(i-1))^{9/2+o(1)}\\ &\leq n^{9/2+o(1)}\\ &\leq 2^{o(1)n}\end{aligned}$$

so $\rho_{slow}((N(A,d_{N,r},\varepsilon))_{r=1}^{\infty})=0$ for every $\varepsilon>0$. Since $\{\Omega_0\setminus A,A\}$ generates, it follows that $\rho_{slow}(T_0,\mu_0)=0$. □

Essentially the same computation gives:

Claim 3.10. If $x\in\Omega_i$ then $|R_{2r(i)}(A,x)|=|Q_{2r(i)}|^{3/4+o(1)}=r(i)^{3/2+o(1)}$.

In particular this implies that $\alpha(T,\mu_0)\geq 3/4$ (equality also holds but we shall not need this). Combined with the previous claim, this shows that there exist $\mathbb{Z}^k$-actions such that $\rho(T,\mu)\leq k\alpha(T,\mu)$.

3.4. **Symbolic systems.** It will be useful now to introduce the language of symbolic representation. Let Σ be a finite set considered as a discrete topological space. The full $\mathbb{Z}^2$-shift over Σ is the product space $\Sigma^{\mathbb{Z}^2}$, endowed with the product topology and the Borel σ-algebra, which we suppress in our notation. The shift action S is the continuous $\mathbb{Z}^2$ actions given by

$$(S^u x)_v = x_{u+v}$$

We use the same symbol S to represent the shift action on full shifts over different alphabets.

Given an integer m and a map $\pi:\Sigma^{Q_m}\to\Delta$ to another finite set Δ of symbols, there is an induced map $\overline{\pi}:\Sigma^{\mathbb{Z}^2}\to\Delta^{\mathbb{Z}^2}$ defined by

$$(\overline{\pi}x)_u = \pi((S^u x)|_{Q_m})$$

This is a factor map, i.e. $S^u\overline{\pi}=\overline{\pi}S^u$. Given an invariant measure ν on $\Sigma^{\mathbb{Z}^2}$ the push-forward $\overline{\nu}=\overline{\pi}\nu$ of ν is S invariant, and $\overline{\pi}$ is then a factor map between the measure preserving systems $(\Sigma^{\mathbb{Z}^2},\nu,S)$ and $(\Delta^{\mathbb{Z}^2},\overline{\nu},S)$.

3.5. **Completion of the construction.** Let $(\Omega_0,\mathcal{B}_0,\mu_0,T_0)$ be the action constructed in Section 3.3. We now augment it to obtain an action with non-uniform slow entropy >1. Informally, we shall color each point in A randomly by one of two colors a and b. In order to formalize this it is convenient to represent the actions symbolically.

First, we identify $(\Omega_0,\mathcal{B}_0,\mu_0,T_0)$ with a shift invariant measure on $\{0,1\}^{\mathbb{Z}^2}$ obtained by pushing μ_0 through the map $f:\Omega_0\to\{0,1\}^{\mathbb{Z}^2}$ which codes for the partition $\{A,\Omega\setminus A\}$,

$$f(x)_u = 1_A(T^u x)$$

Thus from now on $\Omega_0=\{0,1\}^{\mathbb{Z}^2}$, μ_0 is a shift-invariant measure on Ω_0, and $T_0=S$, the shift.

Next, let a,b be new symbols and let μ_1 denote the product (Bernoulli) measure on $\Omega_1=\{a,b\}^{\mathbb{Z}^2}$ whose marginals are the uniform $(\frac{1}{2},\frac{1}{2})$ measure on $\{a,b\}$. The probability measure μ_1 is S-invariant and ergodic.

Let $\Omega_2=\{0,1\}^{\mathbb{Z}^2}\times\{a,b\}^{\mathbb{Z}^2}$, and consider the product measure $\mu_2=\mu_0\times\mu_1$, which we regard as a shift-invariant measure on Ω_2 in the obvious manner.

Claim 3.11. (Ω_2, μ_2, S) is ergodic.

Proof. This follows e.g. from [7]: μ_2 is a Bernoulli measure and therefore mildly mixing, so its product with any ergodic i.m.p. action, and in particular (Ω_2, μ_2, S) is ergodic. □

We next "erase" the symbol a, b from sites which are not in A. Let $\pi : \{0,1\} \times \{a,b\} \to \{0, a, b\}$ be the map

$$\pi(\sigma, \tau) = \begin{cases} 0 & \text{if } \sigma = 0 \\ \tau & \text{if } \sigma = 1 \end{cases}$$

and let $\mu_3 = \overline{\pi}\mu_2$. Clearly, (μ_0, S) is a factor of (μ_3, S), obtained from the symbol-wise map $\pi'(0) = 0$ and $\pi'(a) = \pi'(b) = 1$.

Identify the set $A \subseteq \Omega_0$ with the set $(\overline{\pi}')^{-1}A \subseteq \Omega_3$, which we also denote by A. Let $\mathcal{R} = \{\Omega_3 \setminus A, A\}$; then clearly

$$\rho_{slow}(S, \mu_3, \mathcal{R}) = \rho_{slow}(S, \mu_0, \{\Omega_0 \setminus A, A\}) = 0$$

On the other hand,

Claim 3.12. $\rho_{slow}(S, \mu_3) \geq 3/2$.

Proof. Let $\mathcal{P} = \{P_0, P_a P_b\}$ denote the partition with elements are the cylinder sets $P_\sigma = \{x \in \Omega_3 : x_0 = \sigma\}$, $\sigma \in \{0, a, b\}$, so $\mathcal{P}$ generates, and note that $\mathcal{R} = \{P_0, P_a \cup P_b\}$. Our aim is to estimate $N(A, d_{\mathcal{P},n}, \varepsilon)$ from below for $n = 2r(i)$.

Indeed, fix $n = 2r(i)$ and consider the partition of A induced by $\mathcal{P}^{Q_n}$, which refines $\mathcal{R}^{Q_n}$. Let $A_1, \ldots, A_M \subseteq A$ denote the intersection of A with the atoms of $\mathcal{P}^{Q_n}$. Since $N(A, d_{\mathcal{P},n}, \varepsilon) \leq C$ implies $N(A_j, d_{\mathcal{P},n}, \varepsilon) \leq C$ for some j, it suffice to show for every j that this is impossible if $C \leq 2^{r(i)^{3/2+o(1)}}$.

Fix $1 \leq j \leq M$, let $x_j \in A_j$, and note that $R_n(A, x_j) \subseteq Q_r$ depends on j but not on the choice of $x_j \in A_j$. By Claim 3.10, we have $|R_n(A, x_j)| = |Q_n|^{3/4+o(1)}$.

From the construction of μ_3 it is easy to see that the $(\mathcal{P}, n)$-names arising from points in A_j consist of all colorings of Q_r such that $Q_r \setminus R_n(A, x_j)$ is colored 0 and $R_n(A, x_j)$ is colored by a and b. Furthermore these colorings are equally likely with respect to $\mu_3|_{A_j}$. Thus, $N(A_j, d_{\mathcal{P},n}, \varepsilon) = N(\{a,b\}^{R_n(A,x_j)}, d, \varepsilon)$, where d is the standard Hamming distance and the measure on $\{a,b\}^{R_n(A,x_j)}$ is the product measure with uniform marginals. It is well-known, however, that there is a lower bound of the form $2^{(1-\delta)|R_n(A,x_j)|}$, where $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. This gives $N(A_j, d_{\mathcal{P},r}, \varepsilon) \geq 2^{r^{3/2+o(1)}}$, which is the desired result. □

In summary, we have shown that (Ω_3, μ_3, S) has slow entropy $\geq 3/2$ but is does not have uniform slow entropy, because with respect to $\mathcal{R}$ its slow entropy is 0. Thus by Theorem 1.3 it has no differentiable model. This establishes Theorem 1.1.

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