Partitions, measurable partitions and disintegration

Partitions and σ-algebras

Let $(X,\mathcal{F})$ be a standard Borel space. A partition ξ of X always means a partition into measurable sets. $\xi(x)\in\xi$ is the element containing x. A set $A\in\mathcal{B}$ is ξ-saturated if $x\in A$ impiles $\xi(x)\subseteq A$. Thus $A=\bigcup_{x\in A}\xi(x)$.

A partition ξ defines the σ-algebra of measurable ξ-satured sets,

$$\mathcal{F}_\xi=\{A\in\mathcal{F}\,:\,A \text{ is } \xi\text{-saturated}\}$$

Example 1. If ξ is the partition into points then $\mathcal{F}_\xi$is the full σ-algebra. If $X=[0,1]^2$ and ξ is the partition into vertical lines then $\mathcal{F}_\xi=\{A\times[0,1]\,:\,A\subseteq Borel([0,1])\}$.

One way to define a partition is as the atoms of a countably generated sub-σ-algebra: if $\mathcal{A}=\sigma(A_1,A_2,\ldots)\subseteq\mathcal{F}$ write $A^0=A$ and $A^1=X\setminus A$, and for $i\in\{0,1\}^{\mathbb{N}}$ write

$$A^i=\bigcap_{k=1}^{\infty}A^{i_K}$$

Then define

$$\begin{aligned}\xi_{\mathcal{A}} &= \{A^i\,:\,i\in\{0,1\}^{\mathbb{N}}\}\\ &= \{A\in\mathcal{F}\,:\,\forall i\in\mathbb{N}\in\mathcal{F}\,,\,A\subseteq C_i \text{ or } A\subseteq X\setminus C_i\}\\ &= \{A\in\mathcal{F}\,:\,\forall B\in\mathcal{F}\,,\,A\subseteq B \text{ or } A\subseteq X\setminus B\}\end{aligned}$$

This construction is related to the previous one by

$$\mathcal{F}_{\xi_{\mathcal{A}}}=\mathcal{A}$$

and if ξ is a partition and $\mathcal{F}_\xi$ is countably generated then also $\xi=\xi_{\mathcal{F}_\xi}$. (These statements are exercises).

Example 2. If $X=\{0,1\}^{\mathbb{N}}$ and $\mathcal{A}$ is the σ-algebra determined by coordinates $2,3,4,\ldots$, then $\xi_{\mathcal{A}}$ is the partition whose sets are pairs of points differeing int heir forst coordinate.

Example 3. In general $\mathcal{F}_\xi$ is not countably generated. For example if $T:X\to X$ is a measurable automorphism then the partition ξ into orbits consists of countable, hence measureble, sets, and $\mathcal{F}_\xi$ is the σ-algebra of T-invariant sets. If μ is a non-atomic ergodic invariant measure for ξ then every orbit has measure 0 and every invariant set has measure 0 or 1, and to $\xi_{\mathcal{F}_\xi}$ contains an atom of measure 1, which cannot be an orbit.

Example 4. Let $A\in SL_d(\mathbb{Z})$ be a hyperbolic matrix, $V\leq\mathbb{R}^d$ the span of the expanding eigendirections, and ξ the partition of $\mathbb{T}^d$ such that $x\sim y$ if and only if

$x = y + v \bmod 1$ for some $v \in V$. This is called the *partition into unstable leaves.* One can characterize it dynamically by the property that $x \sim y$ if and only if $d(T_A^{-n}x, T_A^{-n}y) \to 0$. In general, $\mathcal{B}_\xi$ is not countably generated. For example for hyperbolic $A \in SL_2(\mathbb{Z})$, the expanding eignespace is a line of irrational slope, and ξ is the orbit ralation on the torus of the flow in thsi direction, which is ergodic. Therefore by the previous example, it is not countably generated.

Definition 5. A partition $\xi \subseteq \mathcal{B}$ is *measurable* if $\mathcal{B}_\xi$ is countably generated.

Adding a measure to the picture

Now let μ be a probability measure on $(X, \mathcal{F})$. We identify partitions and σ-algebras that differ on a zero-measure set.

Specifically, we say that $\xi = \xi' \bmod \mu$ if there is a set $X_0 \subseteq X$ of full measure such that $\xi|_{X_0} = \xi'|_{X_0}$. This implies that $\mathcal{F}_\xi = \mathcal{F}_{\xi'} \bmod \mu$.

We say that σ-algebras $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{F}$ satisfy $\mathcal{A} \subseteq \mathcal{A}' \bmod \mu$ if for every $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}'$ with $\mu(A \triangle A') = 0$. If both $\mathcal{A} \subseteq \mathcal{A}' \bmod \mu$ and $\mathcal{A}' \subseteq \mathcal{A} \bmod \mu$ then $\mathcal{A} = \mathcal{A}' \bmod \mu$.

When $\mathcal{A}, \mathcal{A}'$ are countably generated, if $\mathcal{A} = \mathcal{A}' \bmod \mu$ then $\xi_\mathcal{A} = \xi_{\mathcal{A}'} \bmod \mu$ (Find generating sequences $\{C_i\}$ and $\{C_i'\}$ for $\mathcal{A}, \mathcal{A}'$ respectiely with $\mu(C_i \triangle C_i') = 0$ and take $X_0 = X \setminus \bigcup (C_i \triangle C_i')$).

Lemma 6. *For every σ-algebra $\mathcal{A} \subseteq \mathcal{F}$ there is a countably generated sub-σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A} = \mathcal{A}' \bmod \mu$.*

Proof. Let $d(A, B) = \mu(A \triangle B)$. This is a separable metric on $\mathcal{F}$ (this uses the fact that $(X, \mathcal{F})$ is standard Borel), so $\mathcal{A} \subseteq \mathcal{F}$ is separable as well; choose a countable dense sequence $A_1, A_2, \ldots \subseteq \mathcal{A}$ and let $\mathcal{A}' = \sigma(\{A_i\})$. It is an exercise to show that $\mathcal{A} = \mathcal{A}' \bmod \mu$. □

Remark 7. This shows that the property of being countably generated is not preserved under equality mod μ! But of course, the property of being countably generated mod μ, is.

Measure-valued integration

Given a measurable space $(X, \mathcal{B})$, a family $\{\nu_x\}_{x \in X}$ of probability measures on $(Y, \mathcal{C})$ is measurable if for every $E \in \mathcal{C}$ the map $x \mapsto \nu_x(E)$ is measurable (with respect to $\mathcal{B}$). Equivalently, for every bounded measurable function $f : Y \to \mathbb{R}$, the map $x \mapsto \int f(y)\, d\nu_x(y)$ is measurable.

Given a measure $\mu \in \mathcal{P}(X)$ we can define the probability measure $\nu = \int \nu_x d\mu(x)$ on Y by

$$\nu(E) = \int \nu_x(E)\, d\mu(x)$$

For bounded measurable $f : Y \to \mathbb{R}$ this gives

$$\int f\, d\nu = \int (\int f\, d\nu_x)\, d\mu(x)$$

and the same holds for $f \in L^1(\nu)$ by approximation (although f is defined only on a set E of full ν-measure, we have $\nu_x(E) = 1$ for μ-a.e. x, so the inner integral is well defined μ-a.e.).

Example 8. Let X be finite and $\mathcal{B} = 2^X$. Then

$$\int \nu_x \, d\mu(x) = \sum_{x \in X} \mu(x) \cdot \nu_x$$

Any convex combination of measures on Y can be represented this way, so the definition above generalizes convex combinations.

Example 9. Any measure μ on $(X, \mathcal{B})$ the family $\{\delta_x\}_{x \in X}$ is measurable since $\delta_x(E) = 1_E(x)$, and $\mu = \int \delta_x \, d\mu(x)$ because

$$\mu(X) = \int 1_E(x) d\mu(x) = \int \nu_x(E) \, d\mu(x)$$

In this case the parameter space was the same as the target space.

In particular, this representation shows that Lebesgue measure on $[0,1]$ is an integral of ergodic measures for the identity map.

Example 10. $X = [0,1]$ and $Y = [0,1]^2$. For $x \in [0,1]$ let ν_x be Lebesgue measure on the fiber $\{x\} \times [0,1]$. Measurability is verified using the definition of the product σ-algebra, and by Fubini's theorem

$$\nu(E) = \int \nu_x(E) d\mu(x) = \int_0^1 \int_0^1 1_E(x,y) dy \, dx = \int \int_E 1 dx dy$$

so ν is just Lebesgue measure on $[0,1]^2$.

One could also represent ν as $\int \nu_{x,y} \, d\nu(x,y)$ where $\nu_{x,y} = \nu_x$. Written this way each fiber measure appears many times.

Disintegration

We now reverse the procedure above and study how a measure may be decomposed as an integral of other measures. Specifically, we will study the decomposition of a measure with respect to a partition.

Example 11. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\xi = \{P_1, \ldots, P_n\}$ be finite or countable partition of it. For simplicity assume also that $\mu(P_i) > 0$. let μ_x^ξ denote the conditional measure on $\xi(X)$, i.e. $\mu_x = \frac{1}{\mu(\mathcal{P}(x))} \mu|_{\mathcal{P}(x)}$. Then it is easy to check that $\mu = \int \mu_x \, d\mu(x)$.

Our goal is to give a similar decomposition of a measure with respect to an infinite (usually uncountable) partition of X. Then the partition elements $E \in \mathcal{E}$ typically have measure 0, and the formula $\frac{1}{\mu(E)} \mu|_E$ no longer makes sense. As in probability theory one can define the conditional probability of an event E given that $x \in E$ as the conditional expectation $\mathbb{E}(1_E|\mathcal{P})$ evaluated at

x (conditional expectation is reviewed in the Appendix). This would appear to give the desired decomposition: define $\mu_x(E) = \mathbb{E}(1_E|\mathcal{E})(x)$. For any countable algebra this does give a countably additive measure defined for μ-a.e. x. The problem is that $\mu_x(E)$ is defined only for a.e. x but we want to define $\mu_x(E)$ for all measurable sets. Overcoming this problem is a technical but nontrivial chore which we do not undertake here, but which gives the following result.

Theorem 12. *Let $(X, \mathcal{F})$ be a standard Borel space, ξ a measurable partition and $\mathcal{E} = \mathcal{B}_\xi \subseteq \mathcal{B}$ the corresponding countably generated sub-σ-algebra and ξ. Then there is an $\mathcal{E}-$measurable family $\{\mu_y^\xi\}_{y\in X} \subseteq \mathcal{P}(X)$ such that μ_y^ξ is supported on $\mathcal{E}(y)$ and*

$$\mu = \int \mu_y^\xi \, d\mu(y)$$

i.e. for every $f \in L^2(\mu)$, we have $f \in L^1(\mu_x^\xi)$ for μ-a.e. x and

$$\int f d\mu = \int \left(\int f d\mu_x^\xi \right) d\mu(x)$$

Furthermore if $\{\mu'_y\}_{y\in X}$ is another such system then $\mu_y^\xi = \mu'_y$ a.e.

Note that $\mathcal{E}$-measurability has the following consequence: For μ-a.e. y, for every $y' \in \xi(y)$ we have $\mu_{y'} = \mu_y$ (and, since since $\mu_y(\xi(y)) = 1$, it follows that $\mu_{y'}^\xi = \mu_y^\xi$ for μ_y^ξ-a.e y').

Definition 13. The representation $\mu = \int \mu_y^\xi \, d\mu(y)$ in the proof is often called the *disintegration* of μ over $\mathcal{E}$ (or ξ).

The Martingale theorem

Assume that $(X, \mathcal{F})$ is endowed with a countable family $\{f_n\}$ of bounded measurable test functions generating $\mathcal{F}$ (e.g. when X is compact, a dense set of continuous functions). We define convergence of measures $\mu_n \to \mu$ on X by $\int f_k d\mu_n \to \int f_k d\mu$ for all k.

Let $\xi_1 \preceq \xi_2 \preceq \xi_3 \ldots$ be measurable partitions and $\xi_\infty = \bigvee \xi_n$ the coarsest common refinement, given by $\xi_\infty(x) = \bigcap_{n=1}^\infty \xi_n(x)$. Then ξ_∞ is measurable.

Theorem 14 ("Forward" Martingale theorem)**.** *If $\xi_1 \preceq \xi_2 \preceq \xi_3 \ldots$ are measurable partitions, and $\xi_\infty = \bigvee \xi_n$, then*

$$\mu_x^{\xi_n} \to \mu_x^{\xi_\infty} \qquad \mu\text{-a.e. } x$$

Let $\xi_1 \succeq \xi_2 \succeq \xi_3 \succeq \ldots$ be measurable partitions, let $\mathcal{B}_\infty = \bigcap \mathcal{B}_{\xi_n} \bmod \mu$ be a countably generated σ-algebra and $\xi_\infty = \xi_{\mathcal{B}_\infty}$; we denote $\xi_\infty = \bigwedge \xi_n$, and note that it is measurable, but defined mod μ (because $\mathcal{B}_\infty$ is only defined mod μ).

Theorem 15 ("Backward" Martingale theorem)**.** *Let $\xi_1 \succeq \xi_2 \succeq \xi_3 \succeq \ldots$ be measurable partitions, let $\mathcal{B}_\infty = \bigcap \mathcal{B}_{\xi_n} \bmod \mu$ be a countably generated σ-algebra and $\xi_\infty = \xi_{\mathcal{B}_\infty}$. Then*

$$\mu_x^{\xi_m} \to \mu_x^{\xi_\infty} \qquad \mu\text{-a.e. } x$$

Remark 16. Note that in general, $\xi_\infty(x) \neq \bigcup \xi_n(x)$. For example let $X = \{0,1\}^{\mathbb{N}}$ with the product measure μ with marginal $(\frac{1}{2}, \frac{1}{2})$. Let ξ_n denote the partition according to coordinates $n, n+1, n+2, \ldots$. Then $\mathcal{B}_\infty = \bigcap \mathcal{B}_{\xi_n}$ is trivial mod μ, by the Kolmogorov $0,1$-law, and ξ_∞ is trivial (consists of a set of full measure and a nullset). On the other hand, for every x, $\bigcup \xi_n(x)$ is a countable set consisting of all which eventually agree with x