Automorphisms of tori

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ as a group and as a topological space. The metric is the "flat" metric

$$d(\mathbb{Z}^d + u, \mathbb{Z}^d + v) = \min\{\|w\| : w + u - v \in \mathbb{Z}^d\}$$

We identify $\mathbb{T}^d$ with $[0,1)^d$, which is a fundamental domain for the projection $\mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$, or with $[0,1]^d$ from which $\mathbb{T}^d$ arises also by identifying opposite sides.

A character of $\mathbb{T}$ is a funciton of the form $e_n : x \mapsto \exp(2\pi i \langle x, n\rangle)$ where $n \in \mathbb{Z}^d$ and $x \cdot n$ denotes inner product. These are well defined and are group homomorphisms from $\mathbb{T}$ to the unit circle. It is easy to see that they are continuous and separate points, and a product of characterisis again a character. Hence the linear span of the characters is dense in the sup norm on $C(\mathbb{T}^d)$, and dense in $L^2(\mu)$ for any measure μ on $\mathbb{T}^d$. Also, characters are orthogonal with respect to the volume, since for each fixed y,

$$\begin{aligned}\int e_n(x)\overline{e_m}(x)dx &= \int e_n(x+y)\overline{e_m}(x+y)d\mu(x)dx \\ &= \int \exp(2\pi i \langle y, n\rangle)e_n(x)\exp(2\mu i \langle y, -m\rangle)\overline{e_m}(x)dx \\ &= \exp(2\pi i \langle y, n-m\rangle)\int e_n\overline{e_m}\end{aligned}$$

Thus $\langle y, n-m\rangle \in \mathbb{Z}$ for all $y \in \mathbb{R}^d$ which can occur only if $m = n$. Thus, $\{e_m\}_{m\in\mathbb{Z}^d}$ forms an orthonormal basis for $L^2(\mathbb{T}^d, vol)$ (they are of unit modulus, hence for normalized volume have unit length), and every $f \in L(vol)$ can be written uniquely as

$$f = \sum_{n\in\mathbb{Z}^d} a_n e_n \qquad \text{with } \sum a_n^2 = \|f\|_2^2$$

Let $A \in SL_d(\mathbb{Z})$. Then A maps $\mathbb{Z}^d$ injectively to itself, and A induces a group automorphism of $\mathbb{T}^d$ to itself: indeed, it induces a well-defined map because $\mathbb{Z}^d + u = \mathbb{Z}^d + v$ if and only if $u - v \in \mathbb{Z}$ if and only if $Au - Av = A(u-v) \in \mathbb{Z}$ if and only if $A(\mathbb{Z}^d + u) = A(\mathbb{Z}^d + v)$. In particular, Au is a representative of $A(\mathbb{Z}^d + u)$. The fact that A acts on $\mathbb{T}^d$ as a group automorphism is now trivial to check.

A preserves the volume measure on $\mathbb{T}^d$. One can see this either using the fact that its Jacobian is A, hence has determinant 1; or from uniqueness of Haar measure, since the volume is invariant and so is the push-forward of the volume by A.

Proposition 1. *$A \in SL_d(\mathbb{Z})$ acts ergodically on $\mathbb{T}^d$ if and only if it has no roots of unity among its eigenvalues.*

Proof. Suppose that A has no roots of unity among its eigenvalues. Then for $u \in \mathbb{Z}^d$ and $0 \neq n \in \mathbb{N}$ we have $A^n u \neq u$, for otherwise A^n would have

1 as an eivenvalue, implying a root of unity of order n is an eigenvalue of A. Now, to show that A is ergodic we must show that there are no invariant non-constant vectors in $L^2(vol)$. Suppose then that f is A-invariant and write $f = \sum_{n\in\mathbb{Z}^d} a_n e_n$. Then using $e_n \circ A(x) = e_n(A^* x)$, we have

$$\sum a_n e_n = f = Af = \sum a_n \cdot A e_n = \sum a_n e_{A^* n}$$

By uniqueness of the expassion we conclude that $a_{A^* n} = a_n$ for all n. Since $\{A^* n\}_{n\in\mathbb{N}}$ is infinite for $n \neq 0$, it follows from $\sum |a_n|^2 < \infty$ then $a_n = 0$ except when $n = 0$. Thes $f = a_0 e_0$ is constant.

Conversely, if A has a root of unity then A^n has 1 as an eigenvalue for some $n \in \mathbb{N}$. Hence there is a rational vector $u \neq \emptyset$ satisfying $A^n u = u$, and we can assume it is an integer vector. Then $\sum_{i=0}^{n-1} e_{(A^*)^i u}$ is invariant and non-trivial. □

Lemma 2. *Let $\beta \neq 0$, let*

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and let $B = \beta I + U$ be a $d \times d$ matrix in Jordan form with a single Jordan block and eigenvalue β. Then for any every $\varepsilon > 0$, for every large enough n we have for all $u \in \mathbb{R}^d$,

$$e^{-\varepsilon n} \cdot |\beta|^{-n} \cdot \|u\| \leq \|B^n u\| \leq e^{\varepsilon n} \cdot |\beta|^n \cdot \|u\|$$

Proof. It is formally equivalent to prove the inequalities with a multiplicative error depending only on β and the dimension of the matrix.

Since βI and U commute, and $U^d = 0$, by the binomial theorem

$$B^n = \sum_{\ell=0}^{n} \binom{n}{\ell} \beta^{n-\ell} U^\ell = \sum_{\ell=0}^{d-1} \binom{n}{\ell} \beta^{n-\ell} U^\ell$$

hence

$$\|B^n\| \leq \sum_{\ell=0}^{d-1} n^\ell \|U\|^\ell \beta^{n-\ell} = O_{d,\beta}\left(|\beta|^n \cdot e^{\varepsilon n}\right)$$

This gives the right inequality. By the same reasoning applied to B^{-1} we have

$$\left\|B^{-n}\right\| = O_{d,\beta}\left(|\beta|^{-n} e^{\varepsilon n}\right)$$

giving the left inequality. □

If $A \in SL_d(\mathbb{Z})$ write $\mathbb{R}^d = \oplus V_i$ with $A|_{V_i}$ a jordan block with eigenvalue α_i. Let $x = \sum x_i$ with $x_i \in V_i$. Define

$$\|x\|_A = \sum \|x_i\|_2$$

This is a norm on $\mathbb{R}^d$ so is uniformly equivalent to the Euclidean norm. Then

$$\|A^n x\|_A = \max_i \|A^n x_i\|$$

and $A^n x_i$ obeys the bounds in the previous lemma.

Proposition 3. *If A has no roots of unity, then $h_{vol}(A) = \sum m_i \max\{0, \log \lambda_i\}$.*

Proof. The norm $\|\cdot\|_A$ descends to $\mathbb{T}^d$, we define Boqwen balls with respect to it. Clearly $vol(B^n(x,\varepsilon)) = vol(B^n(0,\varepsilon)$. Let $\varepsilon > 0$ and define

$$B_+^n = \{x = \sum x_i \,:\, x_i \in V_i \text{ and } \|x_i\| < \varepsilon e^{\varepsilon n} \max\{1, \lambda_i^n\}^{-1}\}$$

If $x = \sum x_i$ as above and $\|A^n x\|_A < \varepsilon$ then $\|A^n x_i\| < \varepsilon$ and by the lemma, assuming n is large enough, we have $e^{-\varepsilon n}|\alpha_i|^n \|x_i\| < \|A^n x_i\| < \varepsilon$, hence $\|x_i\| < \varepsilon e^{\varepsilon n}|\alpha_i|^{-n} \leq \varepsilon e^{\varepsilon n} \max\{1, |\alpha_i|^n\}^{-1}$. It follows that for all alrge enough n,

$$B^n(0,\varepsilon) \subseteq B_+^n$$

Similarly, define

$$B_-^n = \{x = \sum x_i \,:\, x_i \in V_i \text{ and } \|x_i\| < \varepsilon e^{-\varepsilon n} \max\{1, \lambda_i^n\}^{-1}\}$$

If $x = \sum x_i \in B_-^n$ then for each i, for large enough n by the lemma, $\|A^n x_i\| \leq e^{\varepsilon n}|\alpha_i|^n \|x_i\| < \varepsilon$ so

$$B_n^- \subseteq B^n(0,\varepsilon)$$

Now,

$$\begin{aligned} vol(B_+^n) &= c \cdot \varepsilon^k e^{\varepsilon k n} \prod \max\{1, \lambda_i\}^{-n m_i} \\ vol(B_-^n) &= c \cdot \varepsilon^k e^{\varepsilon k n} \prod \max\{1, \lambda_i\}^{-n m_i} \end{aligned}$$

Taking logarithms and dividing by n, and letting $n \to \infty$ we find that

$$\left| \limsup_{n\to\infty} -\frac{1}{n} \log vol(B^n(x,\varepsilon) - \sum m_i \max\{0, \ln \lambda_i\} \right| \leq c \cdot \varepsilon$$

Letting $\varepsilon \to 0$ and using the Brin-Katok lemma, the proposition is proved. □

Proposition 4. *If $A \in SL_d(\mathbb{Z})$ and μ is an ergodic invariant measure on $\mathbb{T}^d$ then*

$$h_\mu(T_A) \leq \sum m_i \max\{0, \ln \lambda_i\}$$

Proof. Let $\varepsilon, \delta > 0$ and set

$$E_n = \{x \in \mathbb{T}^n \,:\, e^{-n(h+\delta)} < \mu(B^n(x,\varepsilon)) < e^{-n(h-\delta)}\}$$

When ε is small enough, for large enough n we have $\mu(E_n) > \frac{1}{2}$. Choose a maximal set $F_n \subseteq E_n$ with

$$B^n(x,\varepsilon/2) \cap B^n(y,\varepsilon/2) = \emptyset \qquad \text{for distinct } x, y \in E_n$$

Then

$$|F_n| \leq \min\{vol(B^n(x,\varepsilon) \,:\, x \in E_n\}^{-1} \leq c \cdot (\frac{2}{\varepsilon})^k e^{\varepsilon kn} \prod \max\{1, \lambda_i\}^{nm_i}$$

But by maximality, $\bigcup_{x\in F_n} B^n(x,\varepsilon)$ covers E_n so $\mu(\bigcup_{x\in F_n} B^n(x,\varepsilon)) \geq 1/2$ so

$$|F_n| \geq \frac{1}{2} e^{-n(h+\delta)}$$

Sombining these two inequalities, which hold for all large enough n, arbitrarily small ε and arbitrary $\delta > 0$, gives the result. □

Definition 5. $A \in SL_d(\mathbb{R})$ is *hyperbolic* if none of its eigenvalues has modulus 1.

For a 2×2 matrix, if it has no roots of unity, it is hyperbolic, with eigenvalues $\lambda > 1$ and $0 < \lambda^{-1} < 1$.

Theorem 6. *Let μ be an invariant measure on $\mathbb{T}^2$ for a hyperbolic 2×2 matrix A. Then μ is exact dimensional and its dimension is $2h_\mu(A)/\ln\lambda$.*

Proof. Let

$$B^{\pm n}(x,\varepsilon) = \{y \,:\, d(A^i x, A^i y) < \varepsilon \text{ for } -n \leq i \leq n\}$$

Then by a variant of Brin-Katok,

$$\lim_{\varepsilon\to 0} \limsup -\frac{1}{2n} \log B^{\pm n}(x,\varepsilon) = \lim_{\varepsilon\to 0} \liminf -\frac{1}{2n} \log B^{\pm n}(x,\varepsilon) = h_\mu(A)$$

Now note that there is a constant c such that

$$B^{\pm n}(x, c^{-1}\varepsilon) \subseteq B(x, \varepsilon\lambda^{-n}) \subseteq B^{\pm n}(x, c\varepsilon)$$

It follows that

$$2\limsup -\frac{1}{2n} \log B^{\pm n}(x, c^{-1}\varepsilon) \leq \limsup \frac{\ln B(x,\varepsilon\lambda^n)}{\ln \varepsilon\lambda^n} \leq 2\limsup -\frac{1}{2n} \log B^{\pm n}(x, c\varepsilon)$$

Letting $\varepsilon \to 0$ we obtain

$$\limsup \frac{\ln B(x,r)}{\ln r} = 2h_\mu(T)$$

For the lininf the calculation is the same. □

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