The Brin-Katok "local" entropy formula

Given a transformation $T : X \to X$ let

$$d^n(x,y) = \max_{0 \le i \le n-1} d(T^i x, T^i y)$$

and

$$B^n(x,\varepsilon) = \{y \in X \,:\, d^n(x,y) < \varepsilon\}$$

As usual write

$$\alpha^n = \bigvee_{i=0}^{n-1} T^{-i}\alpha$$

Theorem 1 (Brin-Katok)**.** *Let (X,T) be a topological dynamical system with metric d, and μ an ergodic T-invariant measure with entropy h. Then*

$$\lim_{\varepsilon \searrow 0}\left(\limsup_{n\to\infty} \frac{-\log \mu(B^n(x,\varepsilon))}{n}\right) = \lim_{\varepsilon \searrow 0}\left(\liminf_{n\to\infty} \frac{-\log \mu(B^n(x,\varepsilon))}{n}\right) = h$$

Fix $\varepsilon > 0$. For a partition α with atoms of diameter $< \varepsilon$ we have $\alpha^n(x) \subseteq B^n(x,\varepsilon)$, hence for μ-a.e. x,

$$\limsup_{n\to\infty} \frac{-\log \mu(B^n(x,\varepsilon))}{n} \le \limsup_{n\to\infty} \frac{-\log \mu(\alpha^n(x))}{n} = h_\mu(T,\alpha) \le h_\mu(T)$$

Letting $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0}\left(\limsup_{n\to\infty} \frac{-\log \mu(B^n(x,\varepsilon))}{n}\right) \le h_\mu(T)$$

For the other direction, fix $\rho > 0$; it suffices to show that

$$\mu\left(x \,:\, \lim_{\varepsilon\to 0}\left(\liminf_{n\to\infty} \frac{-\log B^n(x,\varepsilon)}{n}\right) < h_\mu(T) - \rho\right) = 0$$

Let $\alpha = \{A_1, \ldots, A_k\}$ be a partition with $\mu(\partial A_i) = 0$ and

$$h_\mu(\alpha, T) > h_\mu(T) - \rho/4$$

Fix ε for the moment and set

$$E_\varepsilon = \bigcup_{A\in\alpha} (\partial A)^{(\varepsilon)}$$

and note that $\mu(E_\varepsilon) \to 0$ as $\varepsilon \to 0$. We suppress the dependence ε from now on.

Let

$$I_n(x) = \{0 \le i \le n-1 \,:\, T^i x \notin E_\varepsilon\}$$

and

$$\gamma_n(x) = \bigcap_{i\in I_n} T^{-i}\alpha(x)$$

Lemma 2. $B^n(x,\varepsilon) \subseteq \gamma_n(x)$

Proof. If $y \in B^n(x,\varepsilon)$ and $T^i x \notin E_\delta$ for $1 \le i \le n-1$, then $d(T^i y, T^i x) < \varepsilon$ and $d(x, \partial(T^{-i}\alpha(x))) \ge \varepsilon$, so $T^i y \in T^{-i}\alpha(x)$, which implies $y \in \gamma_n(x)$. □

Thus is suffices for us to show that for μ-a.e. x,

$$\lim_{\varepsilon\to\infty} \left(\liminf_{n\to\infty} \frac{-\log \gamma_n(x)}{n} \right) \ge h_\mu(T) - \rho$$

Let

$$\beta = \{A_1 \cap E_\varepsilon, \ldots, A_k \cap E_\varepsilon, X \setminus E_\varepsilon\}$$

Note that $h_\mu(T,\beta) \le \sum_{B\in\beta} -\mu(B)\log\mu(B)$, and since the number of atoms of β is fixed but one of them is $X \setminus E_\varepsilon$and $\mu(X \setminus E_\varepsilon) \to 1$ as $\varepsilon \to 0$, we have $h_\mu(T,\beta) \to 0$ as $\varepsilon \to 0$.

Now, note that

$$\gamma_n(x) \cap \beta^n(x) \subseteq \alpha^n(x)$$

Suppose for a moment that $\{\gamma_n(x)\}_{x\in X}$ is a partition (which it is not!). We then could argue as follows: Let

$$\begin{aligned}
\mathcal{U}_n &= \{\alpha^n(x) \,:\, \mu(\alpha^n(x)) < 2^{-h_\mu(T,\alpha)-\rho/4)}\} \\
\mathcal{V}_n &= \{\beta^n(x) \,:\, \mu(\beta^n(x)) > 2^{-n(h_\mu(T,\beta)+\rho/4)}\} \\
\mathcal{W}_n &= \{\gamma_n(x) \,:\, \mu(\gamma^n(x)) > 2^{-n(h_\mu(T)-\rho)}\} \\
\mathcal{Z}_n &= \{\gamma_n(x) \cap \beta^n(x) \,:\, \alpha^n(x) \in \mathcal{U}_n \,,\; B^n(x) \in \mathcal{V}_n \text{ and } \gamma^n(x) \in \mathcal{W}_n\}
\end{aligned}$$

Note that μ-a.e. x has $\alpha^n(x) \in \mathcal{U}_n$ and $\beta^n(x) \in \mathcal{V}_n$ for all large enough n, so it is enough to show that μ-typically, $\gamma_n(x) \cap \beta^n(x) \in \mathcal{Z}_n$ for only finitely many n. To see this note that $|\mathcal{W}_n| \le 2^{n(h_\mu(T)-\rho)}$ and $|\mathcal{V}_n| \le 2^{n(h_\mu(\beta)+\rho/4)}$, and if $D \in \mathcal{Z}_n$ then $\mu(D) < 2^{-n(h_\mu(T)-\rho/4)}$, so

$$\mu(\cup\mathcal{Z}_n) \le 2^{n(h_\mu(T)-\rho)} \cdot 2^{n(h_\mu(T,\beta)+\rho/4)} \cdot 2^{-n(h_\mu(T,\alpha)-\rho/4)} < 2^{-n(\rho/4-h_\mu(\beta))}$$

Since $h_\mu(T,\beta) < \rho/4$ for all small enough ε, for such ε the probabilities above are summable, and the claim follows by Borel-Cantelli.

Since $\{\gamma_n(x)\}_{x\in X}$ isn't a partition, this argument fails, spcifically, the conclusion $|\mathcal{W}_n| \le 2^{n(h_\mu(T)-\rho)}$ does not follow from $\mu(C) \ge 2^{-n(h_\mu(T)-\rho)}$ for $C \in \mathcal{W}_n$. To get around, this define

$$\Gamma_n = \{\gamma_n(x) \,:\, x \in X \text{ and } |I_n(x)| > n(1-2\mu(E_\varepsilon))\}$$

By the ergodic theorem applied to 1_{E_ε}, for μ-a.e. x,

$$\gamma_n(x) \in \Gamma_n \qquad \text{for all large enough } n$$

Lemma 3. *If $A_1, \ldots, A_N$ are sets in a probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$, and if $\mathbb{P}(A_i) > c$ and each $\omega \in \Omega$ belongs to at most k of the sets, then $N \le k/c$.*

Proof. By assumption $\sum_{i=1}^{N} 1_{A_i} \leq k$ so

$$k \geq \int \sum_{i=1}^{N} 1_{A_i} d\mathbb{P} = \sum_{i=1}^{N} \mu(A_i) > Nc$$

□

Since those elements of Γ_n containing x are intersections of at least $n(1 - 2\mu(E_\varepsilon))$ of the sets $T^{-i}\alpha(x)$, $0 \leq i < n$, it follows that

Each x belongs to at most $\binom{n}{(1-2\mu(E_\varepsilon))n} = \binom{n}{2\mu(E_\varepsilon)n} \leq 2^{nH(2\mu(E_\varepsilon))}$ elements of Γ_n

Now define

$$\mathcal{W}_n = \{C \in \Gamma_n : \mu(C) > 2^{-n(h_\mu(T)-\rho)}\}$$

so by the lemma,

$$|\mathcal{W}_n| \leq 2^{n(h_\mu(\alpha)-\rho)} \cdot 2^{nH(2\mu(E_\varepsilon))}$$

With this definition of $\mathcal{W}_n$, and since $H(2\mu(E_\varepsilon)) \to 0$ as $\varepsilon \to 0$, the previous argument goes through unchanged.