Final exam in "Fractal Geometry and Dynamics" (80852)

Instructions

- You have **3 weeks** to complete this exam.

  You can **start at any time**;

  You must submit the solution up to 3 weeks from when you start but **no later than August 30, 2012**.

  You can **submit a hardcopy** to Orly in the student office or **an electronic copy** by email directly to me (mhochman@math.huji.ac.il).

- You may **not** discuss the exam with other people.

- It is **ok** to use course notes, books etc. but the solution you submit must be **your own work**.

- Please write legibly!

Part 1

Solve 3 of the following 4 problems (11% each).

a) Let $\varphi_i : [0,1] \to [0,1]$ be the (nonlinear) contractions $\varphi_1(x) = x/(4+2x)$ and $\varphi_2(x) = 1/(2+x)$. Let $\Phi = \{\varphi_1, \varphi_2\}$ and let K be the attractor of the IFS Φ. Show that $\log 2/\log 9 \le \dim K \le 1/2$. (Can you give better bounds?).

b) Compute the dimension α of the set of $x \in [0,1]$ whose binary digits do not have a limiting frequency, i.e.

$$X = \{\sum_{n=1}^{\infty} a_n 2^{-n} \; : \; a_n \in \{0,1\}\,,\ \liminf \frac{1}{n}\sum_{i=1}^{n} a_i < \limsup \frac{1}{n}\sum_{i=1}^{n} a_i\}$$

Is there an α-regular measure giving X positive measure?

c) Let v_x be the vertical line passing through $(x,0)$ and h_y the horizontal line passing through $(0,y)$. Consider the McMullen carpet X with the pattern

$$\begin{pmatrix} & \blacksquare & \\ \blacksquare & & \blacksquare \end{pmatrix}$$

Show that $X \cap v_x$ contains at most two points for every $x \in \mathbb{R}$, and hence has dimension 0, but that $\dim X \cap h_y = 1/(2\log_2 3)$ for Lebesgue-a.e. $y \in [0,1]$. (Can you propose a construction with $\dim X \cap h_y = \alpha$ for Lebesgue-a.e. y, and a given value of $\alpha \in [0,1]$?).

d) Show that if $X \subseteq [0,1]^d$ is compact and $\dim X = d$, then $[0,1]^d \in \mathcal{G}_X$ (where $\mathcal{G}_X$ is the gallery determined by X).

Part 2

Complete the proofs of 2 of the following 3 sections (33% each).

2.A. How does the dimension of a self-similar set depend on the IFS?

For an IFS Φ we write X_Φ for its attractor. Recall that Φ satisfies strong separation if $\varphi X_\Phi \cap \psi X_\Phi = 0$ for distinct $\varphi, \psi \in \Phi$.

Proposition 1. *Let $\Phi = \{\varphi_i\}_{i \in \Lambda}$ be an IFS consisting of contracting similarities on $\mathbb{R}^d$ and $\alpha = \overline{\mathrm{Mdim}} X_\Phi$. Show that for every $\varepsilon > 0$ there is an IFS $\Phi' \subseteq \bigcup_{k=1}^{\infty} \bigcup_{i_i,\ldots,i_k \in \Lambda^k} \varphi_{i_1 \ldots i_k}$ satisfying strong separation and $\dim X_{\Phi'} \geq \alpha - \varepsilon$.*

Prove the proposition. (Suggestion: consider an appropriate subset of the $\varphi_{i_1 \ldots i_k}$ with contraction approximately equal to some small r).

Corollary 2. *For Φ as above, $\dim X_\Phi = MdimX_\Phi$.*

Prove the corollary (Note that we know that $\dim X_\Phi = \mathrm{Mdim}\, X_\Phi$ if the open set condition holds; the point here is that there is no separation assumption).

Now fix the dimension d of $\mathbb{R}^d$ and the index set Λ ($|\Lambda| \geq 2$), and consider the space $\mathcal{F}$ of all IFSs $\Phi = \{\varphi_i\}_{i \in \Lambda}$ consisting of (contracting) similitudes. $\mathcal{F}$ can be identified with $|\Lambda|$-tuples of similitudes and the space of similitudes can be identified with $(0,1) \times O(d) \times \mathbb{R}^d$, where $O(d)$ is the space of orthogonal $d \times d$ matrices and (r, U, a) corresponds to the similitude $x \mapsto rUx + a$. Thus $\mathcal{F}$ inherits a topology from the product $((0,1) \times O(d) \times \mathbb{R}^d)^\Lambda$.

Lemma 3. *Let $\Phi_n \to \Phi$ and suppose that Φ satisfies strong separation. Then so does Φ_n for all large enough n.*

Prove the lemma.

Theorem 4. *Let $\delta : \mathcal{F} \to [0, \infty)$ be the map $\Phi \mapsto \dim X_\Phi$. Show that δ is lower semi-continuous, i.e. that if $\Phi_n \to \Phi$ then $\liminf_{n \to \infty} \delta(\Phi_n) \geq \delta(\Phi)$.*

Prove the theorem

Let $F \subseteq \mathbb{R}^d$ be the one-dimensional Sierpinski gasket and let $F_t = \pi_t F$, where $\pi_t(x, y) = tx + y$. We know from Marstrand's theorem that $\dim F_t = 1$ for Lebesgue-a.e. t. We can now prove that the set of t satisfying this is large also in the sense of Baire category:

Corollary 5. *Show that there is a dense G_δ set of t such that $\dim F_t = 1$.*

Prove the corollary (Suggestion: it is enough to show that $\{t : \dim F_t > 1 - \varepsilon\}$ contains an open dense set).

Remark. For a general compact set $E \subseteq \mathbb{R}^2$ the set $\{t : \dim \pi_t E = \min\{1, \dim E\}\}$ has full measure (Marstrand) but does not need to contain a dense G_δ. Also, the corollary is true for any self-similar set, although the proof given above does not work except when the contractions of the IFS do not include rotations (why?).

2.B. Dimension conservation for homogeneous fractals

Prove the following theorem:

Theorem 6. *Let $Z \subseteq [0,1]^2$ be compact and homogeneous. Let $\pi(x,y) = x$. Prove that there exists a point $x \in \pi Z$ with* $\dim(Z \cap \pi^{-1}(x)) \geq \dim Z - \operatorname{Mdim} \pi Z$.

Suggestion: there is more than one way to prove this, but you may want to use the following facts (if you do you should prove them):

1. The family of all minisets of vertical fibers of X is a gallery: i.e. the family

$$\mathcal{F} = \{\emptyset \neq Y \subseteq [0,1]^2 \cap \big(r(\pi^{-1}(y) \cap X) + a\big) \ : \ y \in \mathbb{R} \ , \ r \geq 1 \ , \ a \in \mathbb{R}^2\}$$

   is a gallery.

2. If μ_n are probability measures on sets of the form $X \cap \pi^{-1}(I_n) \neq \emptyset$ for some intervals I_n with $|I_n| = 2^{-n}$, then any weak-* accumulation point $P \in \mathcal{P}(\Phi)$ of $P_n = \frac{1}{n}\sum_{k=1}^{n} T_F^n \delta_{(0,\mu_n)}$ has the property that $\operatorname{supp}\nu \in \mathcal{F}$ for P-a.e. ν.

2.C. Badly approximable numbers in self-similar sets

A number $x \in \mathbb{R}$ is called *badly approximable* if there is a constant $c > 0$ such that

$$|x - \frac{m}{n}| \geq \frac{c}{n^2} \text{ for all } m, n \in \mathbb{Z}$$

We will denote the set of badly approximable numbers by Γ. (If you have not seen it before this definition may seem arbitrary, but Γ is an important object in Diophantine approximation.It is, in a sense, a generalization of the set of algebraic numbers: Liouville showed that any irrational algebraic number α as the property above. Elements of Γ are also characterized as those numbers whose partial quotients in the continued fraction expansion of x are bounded).

In this problem you will prove the following theorem:

Theorem 7. *Let $X \subseteq \mathbb{R}$ be a self-similar set of positive dimension. Then* $\dim(X \cap \Gamma) = \dim X$*; in particular $X \cap \Gamma \neq \emptyset$.*

For each k let $\mathcal{A}_k$ be a disjoint family of compact subsets of $\mathbb{R}$ and write

$$d_k = \max_{A \in \mathcal{A}_k} |A|$$

We say that $\{\mathcal{A}_k\}_{k=0}^{\infty}$ is *treelike* if it satisfies the following properties:

1. $\mathcal{A}_0 = \{A_0\}$ consists of a single set.
2. For every $k \geq 0$ and $U, V \in \mathcal{A}_k$, either $U = V$ or $U \cap V = \emptyset$.
3. For every $k \geq 1$ and every $U \in \mathcal{A}_k$ there exists $V \in \mathcal{A}_{k-1}$ with $U \subseteq V$.
4. For every $k \geq 1$ and every $U \in \mathcal{A}_k$ there exists $W \in \mathcal{A}_{k+1}$ with $W \subseteq U$.

5. $d_k \to 0$ as $k \to \infty$ (here $|A| = \operatorname{diam}(A)$.

Denote $\bigcup \mathcal{A}_k = \bigcup_{E \in \mathcal{A}_k} E$. Clearly $\emptyset \neq \bigcup \mathcal{A}_{k+1} \subseteq \bigcup \mathcal{A}_k$, so

$$A_\infty = \bigcap_{k=0}^{\infty} \bigcup \mathcal{A}_k$$

is a non-empty compact set. Notice that $\mathcal{A}_k$ is totally disconnected and that the family $\{U \cap A_\infty \,:\, U \in \bigcup_{k=0}^{\infty} A_k\}$ forms a basis of closed and open sets for the induced topology on A_∞.

Let μ be a finite measure on $\mathbb{R}$ and suppose that

$$\mu(A) > 0 \qquad \text{for every } A \in \bigcup_{k=0}^{\infty} \mathcal{A}_k$$

Note that then $A_\infty \subseteq \operatorname{supp}\mu$. Define

$$\Delta_k = \min_{U \in \mathcal{A}_k} \frac{\mu(\bigcup\{V \in \mathcal{A}_{k+1} \,:\, V \subseteq U\})}{\mu(U)}$$

Note that $0 < \Delta_k < \infty$ for all k.

Proposition 8. *Suppose that $\{\mathcal{A}_k\}$ is treelike and μ, Δ_k are as above. Then there exists a finite measure ν with supp$\nu = A_\infty$ such that for any $x \in A_\infty$,*

$$\dim(\nu, x) \geq \dim(\mu, x) - \limsup_{k\to\infty} \frac{\sum_{i=0}^{k} \log \Delta_i}{\log d_k}$$

In particular, for any open set U intersecting A_∞,

$$\dim(U \cap A_\infty) \geq \inf_{x \in U} \dim(\mu, x) - \limsup_{k\to\infty} \frac{\sum_{i=0}^{k} \log \Delta_i}{\log d_k}$$

Prove the proposition (Hint: Let $\mathcal{B}$ denote the σ-algebra of Borel sets on A_∞ and $\mathcal{B}_0 \subseteq \mathcal{B}$ the sub-algebra generated by the sets $U \cap A_\infty$, $U \in \mathcal{A}_k$, $k \geq 0$. Define a finitely additive measure ν_0 on $\mathcal{A}$ by the condition that if $U \in \mathcal{A}_k$ then the mass of $U \cap A_\infty$ is distributed among the sets $V \cap A_\infty \subseteq U$, $V \in \mathcal{A}_{k+1}$ in the same proportion as the μ-mass of U is distributed among the sets $V \subseteq U$, $V \in \mathcal{A}_{k+1}$. Extend ν_0 to $\mathcal{B}$ to obtain ν. Show that ν has the desired properties.).

Proposition 9. *Let X be a self-similar set for an IFS $\Phi = \{\varphi_i\}$ on $\mathbb{R}$ satisfying the open set condition. Let μ be the self-similar measure defined by the probability vector $p_i = r_i^\alpha$, where r_i is the contraction ratio of φ_i and $\alpha = \dim X = \operatorname{sdim} X$. Show that there exist constants $c_2 > c_1 > 1$ such that for every interval I centered in X,*

$$c_1 |I|^\alpha \leq \mu(I) \leq c_2 |I|^\alpha$$

in particular there are constants $c_1', c_2' > 0$ such that for $x \in X$,

$$c_1' t^\alpha < \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} < c_2' t^\alpha$$

Prove the Proposition (we proved the upper bound in the first inequality in class, you only need to justify the lower bound).

From now on we work in $\mathbb{R}$. Let X, μ, α, c_1, c_2 be as in the proposition.

Lemma 10. *For all sufficiently small $\beta > 0$ (depending on μ) the following holds. Let $x \in X$, $r > 0$ and let $y \in \mathbb{R}$. Then there is a point $x' \in X \cap B_r(x)$ such that $B_{\beta r}(x') \subseteq B_r(x) \setminus B_{\beta r}(y)$.*

Prove the lemma.

Proposition 11. *There exists a (small) $c > 0$, depending on μ, so that for every sufficiently small $\delta > 0$ the following holds. Let $x \in X$, $r > 0$ and let $Y \subseteq \mathbb{R}$ be a set with $|y - y'| \geq \delta r/c$ for distinct $y, y' \in Y$. Then there is a subset $X' \subseteq X$ such that the family $\mathcal{B} = \{B_{\delta r}(x')\}_{x' \in X'}$ is pairwise disjoint, $\bigcup \mathcal{B} \subseteq B_r(x) \setminus \bigcup_{y \in Y} B_{\delta r}(y)$, and $\mu(\bigcup \mathcal{B}) > c \cdot \mu(B_r(x))$.*

Prove the proposition (Suggestion: Apply the Besicovitch lemma to the family $\{B_{\delta r}(z) : z \in X, B_{\delta r}(z) \subseteq B_r(x)\}$ and use the previous lemma. It may simplify things to write out the statement for $r = 1$ first).

Let

$$\Gamma_c = \{x \in \mathbb{R} : |x - \frac{m}{n}| \geq \frac{c}{n^2} \text{ for all } m, n \in \mathbb{Z}\}$$

so $\Gamma = \bigcup_{c>0} \Gamma_c$. In order to prove that $\dim \Gamma \cap X = \dim \Gamma$ for every self-similar set X satisfying the open set condition, it is enough to show that $\dim X \cap \Gamma_c \to \dim X$ as $c \searrow 0$.

Fix X and μ. Let c be the constant from Proposition 11, and let $\delta = 1/M^2$ for a sufficiently large $M \in \mathbb{N}$ that the proposition applies. Construct by induction on $k \geq 0$ a treelike system $\{\mathcal{A}_k\}$, with $\mathcal{A}_k$ consisting of disjoint balls of radius c/M^{2k} centered in X, and such that $\Delta_k \geq c$. Start with an arbitrary $x_0 \in X$ and $\mathcal{A}_0 = \{B_c(x_0)\}$. For the induction step, assume that we have defined $\mathcal{A}_k$ already, and apply Proposition 11 to each ball $B_r(x) \in \mathcal{A}_k$ and the set

$$Y_k = \{\frac{m}{n} : M^k < n \leq M^{k+1}\}$$

(**use elementary algebra to show that Y_k has the required properties.**) Take $\mathcal{A}_{k+1}$ to be the collection of resulting balls.

Verify that $\{\mathcal{A}_k\}$ has the stated properties and is treelike, and that $A_\infty \subseteq X \cap \Gamma_{c/M^2}$. .

Use Proposition 8 to prove that $\dim A_\infty \to \dim X$ as the parameter $M \to \infty$.

Derive Theorem 7 (you may use Proposition 1 also if you did not prove it).

Remark 12. There are a few special cases where one can also show that $X \setminus \Gamma$ is large (in fact has positive $\dim X$-dimensional Hausdorff measure); for example this is the case for the middle-1/3 Cantor set.