

Convergence of sets and measures

Let X be a compact metric space. Recall the definition of convergence in $2^X = \{A \subseteq X : \emptyset \neq A, A \text{ is compact}\}$.

Definition 1. $A_n \rightarrow A$ in 2^X if for every $\varepsilon > 0$ we have $A_n \subseteq A^{(\varepsilon)}$ and $A \subseteq A_n^{(\varepsilon)}$ for all large enough n .

Lemma 2. $A_n \rightarrow A$ if and only if the following two conditions hold:

i) If $x \in X$ then there exists $x_n \in A_n$ with $x_n \rightarrow x$.

ii) If $x_{n_k} \in A_{n_k}$ and $x_{n_k} \rightarrow x$ then $x \in A$.

Proof. Suppose $A_n \rightarrow A$. To verify (i), let $x \in X$ and let $x_n \in A_n$ denote a point minimizing the distance from x to A_n . Since $A \subseteq A_n^{(\varepsilon)}$ for all large enough n we conclude that $d(x, x_n) = d(x, A_n)$ is eventually smaller than ε for every $\varepsilon > 0$, so $d(x, x_n) \rightarrow 0$. To verify (ii), let x_{n_k} be as in (ii). Then since eventually $A_n \subseteq A^{(\varepsilon)}$ also $x_{n_k} \in A^{(\varepsilon)}$ eventually, so, since A is closed, $\lim x_{n_k} \in A$.

Now suppose $A_n \not\rightarrow A$. Then either (i') there is an $\varepsilon > 0$ and infinitely many n with $A \not\subseteq A_n^{(\varepsilon)}$, or (ii') there is $\varepsilon > 0$ and infinitely many n with $A_n \not\subseteq A^{(\varepsilon)}$. We show that either (i) or (ii) fail. It is enough to show that (i') implies that (i) fail and (ii') implies that (ii) fails.

If (i') holds then there is an $x \in A$ with $d(x, A_n) \geq \varepsilon$ infinitely often, which clearly contradicts (i).

If (ii') holds, then there is a sequence $n_k \rightarrow \infty$ and $x_{n_k} \in A_{n_k} \setminus A^{(\varepsilon)}$. By passing to a subsequence we can assume $x_{n_k} \rightarrow x$. Since $d(x_{n_k}, A) > \varepsilon$ we have $d(x, A) \geq \varepsilon$ so $x \notin A$, contradicting (ii). \square

Next we turn to measures. Let $\mathcal{P}(X) = \{\text{Borel probability measures on } X\}$. We define weak (or weak-*) convergence $\mu_n \rightarrow \mu$ in $\mathcal{P}(X)$ by

$$\mu_n \rightarrow \mu \iff \forall f \in C(X) \quad \int f d\mu_n \rightarrow \int f d\mu$$

This convergence is induced by a metric: choose a countable set $\{f_n\} \subseteq C(X)$ of functions bounded by 1, whose space is dense in $C(X)$. Then set

$$d(\mu, \nu) = \sum 2^{-n} |\int f_n d\mu - \int f_n d\nu|$$

Another compatible metric is

$$d_L(\mu, \nu) = \sup\{|\int f d\mu - \int f d\nu| : f \text{ is Lipschitz with constant } \leq 1\}$$

The topology defined by this convergence is compact (we will not prove this).

Example 3. Let $X = [0, 1]$ and $\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{k/n}$. For every $f \in C(X)$,

$$\int f d\mu_n = \frac{1}{n} \sum_{k=1}^n f(k/n) \rightarrow \int f(x) dx = \int f dLeb$$

where the limit is from the definition of the Riemann integral. Thus $\mu_n \rightarrow Leb$.

Example 4. Let $X = [-1, 1]$ and $\mu_n = \frac{n}{2} Leb|_{[-1/n, 1/n]}$ (note that μ_n is indeed normalized). Then for $f \in C(X)$,

$$\int f d\mu_n = \frac{n}{2} \int_{-1/n}^{1/n} f dx \rightarrow f(0) = \int f d\delta_0$$

so $\mu_n \rightarrow \delta_0$.

Definition 5. The (topological) support of $\mu \in \mathcal{P}(X)$ is

$$\text{supp } \mu = X \setminus \bigcup \{U \subseteq X : U \text{ is open and } \mu(U) = 0\}$$

Equivalently, if \mathcal{U} is a basis for the topology of X , $\text{supp } \mu = X \setminus \bigcup \{U \in \mathcal{U} : \mu(U) = 0\}$. Choosing \mathcal{U} to be countable, as we can since X is compact metric, this implies

$$\mu(\text{supp } \mu) \geq \mu(X) - \sum \{\mu(U) : U \in \mathcal{U} \text{ and } \mu(U) = 0\} = \mu(X)$$

so μ is indeed supported on $\text{supp } \mu$.

$\text{supp } \mu$ does not determine μ or even the measure class of μ .

Example 6. Let $\mathbb{Q} \cap [0, 1] = \{q_n\}_{n=1}^\infty$. Let $\mu = \sum 2^{-n} \delta_{q_n}$. Then every open interval $I \subseteq [0, 1]$ has $\mu(I) > 0$, so $\text{supp } \mu = [0, 1]$. Of course also $\text{supp } Leb|_{[0,1]} = [0, 1]$. But of course $\mu \perp Leb$. Also note that we can find Borel measurable sets $A, B \subseteq [0, 1]$ with $\mu(A) = 0$ and $\mu(B) = 1$, and $Leb(B) = 0$ and $Leb(A) = 1$. Of course, A, B cannot be closed!

In general, $\mu_n \rightarrow \mu$ (weakly) does not imply $\text{supp } \mu_n \rightarrow \text{supp } \mu$ (in 2^X).

Example 7. Let $X = [-1, 1]$, and $\mu_n = \frac{1}{2n} Leb|_{[-1,1]} + (1 - \frac{1}{2n})\delta_0$. Then it is easy to check that $\mu_n \rightarrow \delta_0$ but $\mu_n(I) \neq 0$ for every open interval I and every n so

$$\text{supp } \mu_n = [-1, 1] \rightarrow [-1, 1] \neq \{0\} = \text{supp } \delta_0$$

Claim 8. Let $A_n \rightarrow A$ in 2^X and $\mu_n \rightarrow \mu$ weakly. Suppose $\text{supp } \mu_n \subseteq A_n$. Then $\text{supp } \mu \subseteq A$. In particular

$$\text{supp}(\lim \mu_n) \subseteq \lim(\text{supp } \mu_n)$$

assuming both limits exist.

Proof. Let $x \notin A$, we must show that there is a small r such that $\mu(B_r(x)) = 0$. Choose $r = d(x, A)/3$. Then for large enough n we have $B_{2r}(x) \cap A_n = \emptyset$. Choose a continuous function $f : X \rightarrow [0, 1]$ with $f|_{B_r(x)} \equiv 1$ and $f|_{X \setminus B_{2r}(x)} \equiv 0$, for example

$$f(y) = \begin{cases} 1 & x \in B_r(x) \\ 0 & x \in X \setminus B_{2r}(x) \\ \frac{1}{r}d(y, X \setminus B_{2r}(x)) & x \in B_{2r}(x) \setminus B_r(x) \end{cases}$$

Then

$$\mu(B_r(x)) \leq \int f d\mu = \lim \int f d\mu_n = \lim \int_{A_n} f d\mu_n = 0$$

where we have used the definition of weak convergence and in the last equality we used $A_n \cap B_{2r}(x) = \emptyset$ for all large n . \square