Problems in ergodic theory

(1) Let $(X,\mathcal{B},\mu,T)$ be an ergodic measure oreserving system. Let $A\subseteq X$ with $\mu(A)>0$.

   (a) Show that there is a measurable set $A_0\subseteq A$ with $\mu(A_0)>0$ and such that every $x\in A_0$ returns to A_0 infinitely often.

   (b) Let $r(x)=\min\{n>0\,:\,T^nx\in A_0\}$ and define $S:A_0\to A_0$ to be the map $x\mapsto T^{r(x)}x$. Show that S is measure preserving and ergodic

   Note: S is defined almost everywhere on A and is sometimes called the *induced map* of A.

(2) Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system. Let $f:X\to\{0,1,2,3,\ldots\}$ be a measurable function with $\int fd\mu<\infty$. Let

$$X_f=\{(x,n)\,:\,0\leq n\leq f(x)\}$$

This is the "region under the graph of f" . On X_f let μ_f be the "normalized measure under the graph" of f, that is, the unique measure such that $\mu_f(A\times\{n\})=\mu(A)/\int fd\mu$ for $A\subseteq f^{-1}(\{n\})$. Finally let $T_f:X_f\to X_f$ be the map

$$T_f(x,n)=\begin{cases}(x,n+1) & \text{if } n<f(x)\\ (Tx,0) & n=f(x)\end{cases}$$

Show that $(X_f,\mu_f T_f)$ is measure preserving, it is ergodic if and only if (X,μ,T) is, and that $(X,\mathcal{B},\mu,T)$ is the induced map (in the sense of the previous peoblem) on the set $X\times\{0\}\subseteq X_f$.

(3) In a topological system (X,T), show that if x,y are asymptotic and x is generic for a measure μ then so is y.

(4) Construct a point $x\in\{0,1\}^{\mathbb{N}}$ that is not generic for any measure (with respect to the shift).

(5) Let $T:X\to X$ be a continuous map of a compact metric space acting ergodically on a measure μ. Suppose that there is a nonwhere-dense set $X_0\subseteq X$ of positive measure. Show that the ergodic averages of continuous functions cannot converge uniformly (i.e. the system is not uniquely ergodic).

(6) Give an example of a non-compact, locally compact X and continuous $T:X\to X$ without invariant probability measures.

(7) Show that the set of ergodic measures for $\{0,1\}^{\mathbb{N}}$ and the shift is dense in the compact comvex set of invariant measures. (Hint: consider periodic sequences).

(8) Show that if G is a compact group with normalized Haar measure μ, and $g\in G$, then the measure-preserving map $T_g:h\to gh$ is ergodic if and only if it has a dense orbit.

(9) (*) Let $([0,1],Borel,\mu,T)$ be a measure preserving system (you can assume ergodic if you want). Show that

$$\liminf_{n\to\infty} n\cdot|x-T^nx|\leq 1$$

a.e. (this is a quantitative form of the Poincare recurrence theorem).

(10) Let $(X,\mathcal{B},\mu,T)$ be an invertible measure-preserving system. Let $\lambda\in S^1\subseteq\mathbb{C}$ and let $U:L^2\to L^2$ be the unitary operator $U=\lambda T$. Describe the limit of the "ergodic averages" $\frac{1}{N}\sum_{n=0}^{N-1}U^nf$ for $f\in L^2$.

(11) Let $X = \{z \in \mathbb{C} : |z| \leq 1\}$ with normalized area measure μ, and let $T(re^{i\theta}) = re^{i(\theta+r)}$.
  (a) Show that T preserves μ.
  (b) Is μ ergodic? Describe its ergodic components.
  (c) What is the pure-point spectrum of T?

(12) Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic measure preserving system and $1 < n \in \mathbb{N}$. Show that the following are equivalent:
  (a) T^n is ergodic.
  (b) If k divides n then $e^{2\pi ik/n} \notin \mathbb{N}$.
  (c) There is a measurable partition $X = X_1 \cup \ldots \cup X_n$ with $TX_i = X_{i+1} \bmod n$ up to measure 0.

(13) Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system. Let $t_0 > 0$ and define a new map $S : X \times [0,1) \to X \times [0,1)$ by

$$S_{t_0}(x,t) = (T^{[t+t_0]}x, \{t + t_0\})$$

  (a) Show that this map preserves $\nu = \mu \times Leb|_{[0,1)}$. Show that $S_t S_s = S_{t+s}$, so in fact we have an action of $\mathbb{R}$ on $X \times [0,1)$.
  (b) Determine when S_{t_0} is ergodic. Describe its ergodic components and its spectrum.

(14) A joining of topological dynamical systems (X,T) and (Y,S) is a closed and $T \times S$-invariant subset of $X \times Y$ that projects under the coordinate maps to X, Y, respectively. The systems are disjoint if the only joining is the product joining. Show that if X, Y support ergodic invariant measures μ, ν, respectively, with $\operatorname{supp}\mu = X$ and $\operatorname{supp}\nu = Y$, and if $(X,\mu,T) \perp (Y,\nu,S)$, then the topological systems are disjoint.

(15) Show that disjointness of topological dynamical systems $(X,T), (Y,S)$ does not imply disjointness of every pair of invariant measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$. Hint: Consider $X = \{0,1\}^{\mathbb{Z}}$ and Y a periodic orbit.

(16) If $(X,\mathcal{B},\mu,T) \perp (Y,\mathcal{C},\nu,S)$, is $(X,\mathcal{B},\mu,T) \perp (Y,\mathcal{C},\nu,S^{-1})$?

(17) Let $(X,\mathcal{B},\mu,T), (Y,\mathcal{C},\nu,S)$ be ergodic group rotations. Show that $(X,\mathcal{B},\mu,T^m) \perp (Y,\mathcal{C},\nu,S^n)$ for all $m,n \neq 0$.

(18) Let $(X,\mathcal{B},\mu,T)$ and $f \in L^1(\mu)$. Show that for μ-a.e. x the sequence $a_n = f(T^n x)$ has the following property. If $(X,\mathcal{B},\mu,T) \perp (Y,\mathcal{C},\nu,S)$ then for every $g \in L^1(\nu)$,

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} a_n g(S^n y) \to \int f \cdot \int g \qquad \nu\text{-a.e. } y$$

Hint: you may assume X, Y are compact and T, S continuous. First prove it for f, g continuous.

(19) Show that a stationary process $(X_n)_{n=-\infty}^{\infty}$ with trivial tail is strongly mixing.

(20) Construct a probability vector $p = (p_1, p_2, \ldots)$ such that $H(p) = \infty$.

(21) Prove that for any finite-valued random variables X, Y, Z we have $H(X|Y,Z) \geq H(X|Y)$ (we used this repeatedly in class, but did not prove it. Hint: consult the proof that $H(X|Y) \geq H(X)$). Show the same when the conditioning is on σ-algebras.

(22) Prove that the product measure $\mu = (1/2, 1/2)^{\mathbb{Z}}$ is the only measure on $\{0,1\}^{\mathbb{Z}}$ with $h_\mu(S) = 1$ (all others are ≤ 1). Conclude that if $\pi : \{0,1\}^{\mathbb{Z}} \to$

$\{0,1]^{\mathbb{Z}}$ is a continuous map satisfing $\pi S = S\pi$ (S the shift), then if π is 1-1, it must be onto.

(23) Prove that if $\mu_n \in \mathcal{P}(\{0,1\}^{\mathbb{Z}})$ are shift invariant and $\mu = \lim \mu_n$ weak-*, then $h_\mu(S) \geq \limsup h_{\mu_n}(S)$. Show that strict inequality can occur (in fact we can have $h_{\mu_n}(S) = 0$ and $h_\mu(S) = 1$). Show that this lower semi-continuity doesn't hold in general for invariant measures on topological dynamical systems (Hint: construct a topological space in which there is a sequence $X_n \subseteq X$ of invariant subshifts, each of which supports measures of entropy 1, but such that X_n "converge" to a fixed point of the map).

(24) Show that $h_{\mu\times\nu}(T \times S) = h_\mu(T) + h_\nu(S)$.

(25) Prove that if $(X, \mathcal{B}, \mu, T)$ is weak mixing then for any non-trivial partition α of X, there is a sequence $n_k \to \infty$ such that $\frac{1}{N} H(\bigvee_{k=1}^{N} T^{-n_k}\alpha) \to 1$.

(26) Give an example of an ergodic measure preserving system with infinite entropy, and conclude that it does not have a generating partition.

(27) Let $X = \{\pm 1\}^{\mathbb{Z}} \times \{\pm 1\}^{\mathbb{Z}}$, and define the map $X \to X$ by $T(x, y) = (Sx, S^{x(0)}y)$ where $S : \{\pm 1\}^{\mathbb{Z}} \to \{\pm 1\}^{\mathbb{Z}}$ is the shift. Let $\mu = \mu_0 \times \mu_2$ where $\mu_0 = \mu_1 = (1/2, 1/2)^{\mathbb{Z}}$. This system is called the "T, T^{-1}". It is also called "random walk in random scenery", because the first sequence x can be thought of as the increments of a symmetric random walk on the integers; as time goes by we shift the x sequence, producing these increments, and shift the y-sequence either forward or backward according to this increment. Thus, if we view only the$y$ sequence, we see a random sequence of digits being shifted left and right randomly.

(a) Show that (X, T, μ) is measure-preserving (this is a special case of the skew product construction we talked about in class).

(b) Consider the process $(X_n, Y_n)(x, y) = (x_n, y_n)$. Show that given $(X_{-\infty}^{0}, Y_{-\infty}^{0})$ one can determine the entire sequence $Y_{-\infty}^{\infty}$. Hint: Use recurrence of random walk on $\mathbb{Z}$. Conclude that $h((X_n, Y_n)_{-\infty}^{\infty}) = \log 2$.

(c) Show that the process above nevertheless has trivial tail.

Remark: This process turns out to be a process with trivial tail that is not isomorphic to a product measure, although this is not trivial to prove. An earlier example was constructed by Ornstein more explicitly.

(28) A factor map $\pi : X \to Y$ between measure preserving systems is called bounded-to-1 if $|\pi(y)^{-1}| < M$ for some constant M and a.e. $y \in Y$, and finite-to-1 if $\pi^{-1}(y)$ is finite for a.e. $y \in Y$. Show in both cases that $h(X) = h(Y)$ (begin with the bounded-to-1 case, which is easier).