Take-home exam in Ergodic Theory (80859, 80589), Fall 2012

Instructions

Solve 4 of the following 6 problems. Give complete proofs. You may rely on anything proved in class, and use any literature you like as long as it does not solve the specific problem you are working on.

From the moment you read the exam you have one week to complete it. Submit to my mailbox or by email no later than February 7th. If you submit it to my mailbox please email me to let me know.

Problem 1: Uniquely ergodic subshifts and substitution systems

Let A be a finite set, S the shift on $A^{\mathbb{N}}$. Given finite words $a = a_1 \dots a_s \in A^s$ and $b = b_1 \dots b_n \in A^n$, write

$$f(a|b) = \frac{1}{n}\#\{1 \le i \le n-s \,:\, b_i^{i+s-1} = a\}$$

where as usual we write $b_i^j = b_i b_{i+1} \dots b_j$. Thus $f(a|b)$ is the frequency of occurrences of a in b.

1. Let $x \in A^{\mathbb{N}}$ and $X = \overline{\{x, Sx, S^2x, \dots\}}$ its orbit closure, which is S-invariant. Show that X is uniquely ergodic if and only if for every $\varepsilon > 0$ and every finite word $a = a_1 \dots a_s \in A$ there is an N such that for every i, j and every $n > N$,
$$\left| f(a|x_i^{i+n}) - f(a|x_j^{j+n}) \right| < \varepsilon$$
2. A substitution on A is a map $\pi : A \to A^* = \bigcup_{n=0}^{\infty} A^n$, which we think of as a "replacement rule", replacing a with the word $\pi(a)$. We can extend π to finite sequences $x_1 x_2 \dots x_n \in A^n$ by applying it pointwise:
$$\pi(x_1 \dots x_n) \quad = \quad \pi(x_1) \dots \pi(x_n)$$
Suppose that $a \in A$ has the property that $\pi(a)$ begins with an a. Show that the sequence of words $a, \pi(a), \pi^2(a) \dots$ is increasing, that is, $\pi^{n+1}(a) = \pi^n(a)b$ for some word b, and hence converges to a sequence $x \in A^{\mathbb{N}}$ such that $x = \pi(x) = \pi^2(x) = \dots$, where $\pi(x)$ is defined pointwise as above.
As an example, you can consider $A = \{0, 1\}$ with the substitution
$$\begin{aligned} \pi(0) &= 01 \\ \pi(1) &= 10 \end{aligned}$$
and take $a = 0$. Then the resulting sequences x is
$$x = 01\ 10\ 1001\ 10010110 \dots$$
3. Suppose the substitution is **primitive**, i.e. if for every $b \in A$, every letter of A appears in $\pi(b)$ at least once (as is the case in the example above). Let a, x be as in (b). Show that the orbit closure of x is uniquely ergodic. Hint: consider the integer matrix $Q = (q_{a,b})_{a,b \in A}$ where
$$q_{a,b} = \#\{\text{occurrences of } b \text{ in } \pi(a)\}$$
Show that $(Q^n)_{a,b}$ is the number of occurrences of b in $\pi^n a$. By the Perron-Frobenius theorem the eigenvalue λ of Q of largest modulus is real, positive, simple, its eigenvector u_λ has positive coordinates, and for any non-negative vector $u \neq 0$, $Q^n u / \lambda^n \to u_\lambda$. Now use (a) applied first to individual letters, then arbitrary words).

Problem 2: Rigid systems

A measure preserving system $(X, \mathcal{B}, \mu, T)$ is called **rigid** if there is a sequence $n_k \to \infty$ such that $T^{n_k} f \xrightarrow{L^2} f$ for every $f \in L^2(\mu)$.

1. Show that every ergodic group rotation is rigid.

2. Show that rigid systems have entropy 0.

3. Show that the family of rigid automorphisms contains a dense G_δ in the group Aut of automorphisms of $[0,1]$ with Lebesgue measure.

   Remark. Since there is a dense G_δ of weak mixing systems, this shows that there are rigid systems that are not group rotations (in fact, in some sense most systems are of this kind). It also shows that there is a dense G_δ set of zero-entropy systems.

Problem 3: Markov chains and mixing

Let V be a finite set (the state space) and let $P = (p_{u,v})_{u,v\in V}$ be a stochastic matrix, that is, $\sum_{v\in V} p_{u,v} = 1$ for every $u \in V$. We interpret P as giving the transition probabilities of a random walk on V, with $p_{u,v}$ the probability of going from u to v. Given an initial probability measure π on V, we can construct such a random walk V, specifically let X_1 be distributed according to π and given V-valued random variables $X_1, \ldots, X_n$ define X_{n+1} by the condition

$$\mathbb{P}(X_{n+1} = v_{n+1} \,|\, X_1^n = v_1^n) = \mathbb{P}(X_{n+1} = v_{n+1} \,|\, X_n = v_n) = p_{v_n v_{n+1}}$$

π is called a **stationary measure** if the process $X_1, X_2, \ldots$ is stationary. Such a process is called a **stationary Markov chain**.

1. Show that π is stationary if and only if $\pi P = \pi$ (we think of π as a row vector).

2. For π stationary and (X_n) the associated random walk, express the entropy of (X_n) in closed form in terms of π and P.

3. Let Δ denote the set of probability vectors on V, a closed finite-dimensional convex set. Consider the map $P : \Delta \to \Delta$ given by $\pi \mapsto \pi P$.

   Suppose that $P > 0$ (all entries are positive). Show that $P : \Delta \to \Delta$ is a contraction in the $\|\cdot\|_1$-norm on Δ. Using the contraction mapping theorem, conclude that there is unique stationary measure π, and that for any $v \in V$, the random walk $X_1, X_2, \ldots$ started from the initial distribution δ_v has the property that the distribution of X_n (as a measure on V) converges in $\|\cdot\|_1$ to the stationary measure.

4. Prove the conclusion of (3) under the weaker assumption that there is an N with $P^N > 0$.

5. Let μ be the shift-invariant measure on $V^{\mathbb{N}}$ associated to the random walk started from the stationary measure. Assuming $P^N > 0$ for some N, as in (4), show that $\mu([a])\mu(S^{-n}[b]) \to \pi_a \pi_b$ for every $a, b \in A$ (here $[a] = \{a x_2 x_3 \ldots \in A^{\mathbb{Z}} \,:\, x_i \in A\}$). Extend this to all pairs of cylinder sets and deduce that $(A^{\mathbb{Z}}, \mu, S)$ is is strongly mixing.

Problem 4: Group rotations

1. Show that a factor of an ergodic group rotation is also a group rotation (clarification: a measure preserving system is a group rotation if it is isomorphic to a translation on a group with Haar measure).

2. Show that an ergodic joining θ of ergodic group rotations $(X,\mathcal{B},\mu,T),(Y,\mathcal{C},\nu,D)$, is also a group rotation (Suggestion: work with isomorphic copies of X,Y that are actually groups).

3. Describe the ergodic decomposition of $X\times Y$ where X,Y are as in (2).

Problem 5: A skew-product proof of equidistribution of $n^2\alpha$ mod 1

Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Define a transformation transformation S of $\mathbb{T}^2=\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ by $S(x,y)=(x+\alpha,y+2x+\alpha)$. Let ν be Lebesgue measure on $\mathbb{T}^2$.

1. Prove that S is ergodic. Hint: use the fact that the characters $\psi_{m,n}:(x,y)\mapsto e^{2\pi i(mx+ny)}$ are an orthogonal basis for $L^2(\nu)$ to describe the invariant functions in $L^2(\mu)$.

2. Show that S is is uniquely ergodic. (Suggestion: use ergodicity of μ, the fact that the projection of the system to the first coordinate is uniquely ergodic, and the fact that the maps $R_\beta:(x,y)\mapsto(x,y+\beta)$ commute with S for every $\beta\in\mathbb{R}/\mathbb{Z}$).

   Remark: More generally you could prove the following: Suppose G,H are compact abelian groups, T an ergodic group rotation of G, $f:G\to H$ is continuous and $S(g,h)=(Tg,f(g)h)$. Suppose that the product of the Haar measures on $G\times H$ is ergodic under S. Then S is uniquely ergodic.

3. Compute $S^n(0,0)$ and conclude that $n^2\alpha$ mod 1 equidistributes for Lebesgue measure in $[0,1]$.

Problem 6: More about the skew product in Problem 5

Let $(\mathbb{T}^2,\nu,S)$ be the transformation from the previous problem.

1. Show that $(\mathbb{T}^2,\nu,S)$ is not isomorphic to a group rotation. Hint: consider the asymptotic behavior of iterates of $f(x,y)=\frac{1}{2}-1_{[0,1/2)}(y)$, where $[0,1/2)\subseteq\mathbb{R}/\mathbb{Z}$ has the obvious meaning.

   Remark. The topological system S is an example of a **distal** system, that is, it has the property that if $u,v\in\mathbb{T}^2$ and $u\neq v$ then $\inf_{n\in\mathbb{N}}d(S^nu,S^nv)>0$ – that is, distinct orbits remain bounded away from each other for all time. This property is trivial for isometries; the problem above shows that distality is strictly weaker than continuously isomorphic to an isometry.

2. Identifying $\mathbb{T}^2$ with $[0,1)^2$, for (x,y) let $X(x,y)=x_1$, the first binary digit of x, and $Y(x,y)=y_1$, the first binary digit of y. Let $X_n=S^nX$, $Y_n=S^nY$, and let P denote the partition induced by the pair of functions (X,Y).

   Show that P generates for S. (Suggestion: first show that $X_0^\infty(x,y)$ determines x, then use ergodicity (Problem 5(2) above) to show that $(X_0^\infty(x,y),Y_0^\infty(x,y))$ a.s. also determines y).

3. Show that $h_\nu(S)=0$.