Review Problems in Dynamical systems and Entropy

These problems are meant to help review the material.

(1) Let $(X,\mathcal{B},\mu,T)$ be an ergodic measure oreserving system. Let $A\subseteq X$ with $\mu(A)>0$.
  (a) Show that there is a measurable set $A_0\subseteq A$ with $\mu(A_0)>0$ and such that every $x\in A_0$ returns to A_0 infinitely often.
  (b) Let $r(x)=\min\{n>0\,:\,T^n x\in A_0\}$ and define $S:A_0\to A_0$ to be the map $x\mapsto T^{r(x)}x$. Show that S is measure rpreserving and ergodic

  Note: S is defined almost everywhere on A and is sometimes called the *induced map* of A.

(2) Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system. Let $f:X\to\{0,1,2,3,\ldots\}$ be a measurable function with $\int fd\mu<\infty$. Let

$$X_f=\{(x,n)\,:\,0\leq n\leq f(x)\}$$

This is the "region under the graph of f" . On X_f let μ_f be the "normalized measure under the graph" of f, that is, the unique measure such that $\mu_f(A\times\{n\})=\mu(A)/\int fd\mu$ for $A\subseteq f^{-1}(\{n\})$. Finally let $T_f:X_f\to X_f$ be the map

$$T_f(x,n)=\begin{cases}(x,n+1) & \text{if } n<f(x)\\ (Tx,0) & n=f(x)\end{cases}$$

Show that $(X_f,\mu_f T_f)$ is measure preserving, it is ergodic if and only if (X,μ,T) is, and that $(X,\mathcal{B},\mu,T)$ is the induced map (in the sense of the previous peoblem) on the set $X\times\{0\}\subseteq X_f$.

(3) In a topological system (X,T), show that if x,y are asymptotic (i.e. $d(T^n x,T^n y)\to 0$) and x is generic for a measure μ, then so is y.

(4) Construct a point $x\in\{0,1\}^{\mathbb{N}}$ that is not generic for any measure (with respect to the shift).

(5) Show that in $\{0,1\}^{\mathbb{Z}}$ with the shift, the non-generic points form a dense G_δ.

(6) Show that every shift-invariant measure μ on $\{0,1\}^{\mathbb{Z}}$ has a generic point (for ergodic measures this is immediate, since μ-a.e. point is generic for μ. The point is to deal with non-ergodic measures).

(7) Let $T:X\to X$ be a continuous map of a compact metric space acting ergodically on a measure μ. Suppose that there is a nonwhere-dense set $X_0\subseteq X$ of positive measure. Show that the ergodic averages of continuous functions cannot converge uniformly (i.e. the system is not uniquely ergodic).

(8) Give an example of a non-compact, locally compact X and continuous $T:X\to X$ without invariant probability measures.

(9) Show that the set of ergodic measures for $X=\{0,1\}^{\mathbb{N}}$ and the shift is dense in $\mathcal{P}_T(X)$ (Hint: consider periodic sequences).

(10) Let $(X,\mathcal{F},\mu,T)$ be a measure preserving system. Let $\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\ldots\subseteq\mathcal{F}$ be invariant sub-σ-algebras (factors) and suppose that $X,\mathcal{F}_n,\mu,T)$ is ergodic for all n. Suppose $\sigma(\mathcal{F}_0,\mathcal{F}_1,\ldots)=\mathcal{F}$. Show that $(X,\mathcal{F},\mu,T)$ is ergodic. Show the same with the word "ergodic" replaces by "mixing".

(11) (*) Let $([0,1],Borel,\mu,T)$ be a measure preserving system (you can assume ergodic if you want). Show that

$$\liminf_{n\to\infty} n\cdot|x-T^n x|\leq 1$$

a.e. (this is a quantitative form of the Poincare recurrence theorem).

(12) Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure-preserving system. Let $\lambda \in S^1 \subseteq \mathbb{C}$ and let $U : L^2 \to L^2$ be the unitary operator $U = \lambda T$. Describe the limit of the "ergodic averages" $\frac{1}{N}\sum_{n=0}^{N-1} U^n f$ for $f \in L^2$.

(13) Let $X = \{z \in \mathbb{C} : |z| \leq 1\}$ with normalized area measure μ, and let $T(re^{i\theta}) = re^{i(\theta+r)}$.
  (a) Show that T preserves μ.
  (b) Is μ ergodic? Describe its ergodic components.
  (c) What is the pure-point spectrum of T?

(14) Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system. Let $t_0 > 0$ and define a new map $S : X \times [0, 1) \to X \times [0, 1)$ by

$$S_{t_0}(x, t) = (T^{[t+t_0]}x, \{t + t_0\})$$

  (a) Show that this map preserves $\nu = \mu \times Leb|_{[0,1)}$. Show that $S_t S_s = S_{t+s}$, so in fact we have an action of $\mathbb{R}$ on $X \times [0, 1)$.
  (b) Determine when S_{t_0} is ergodic. Describe its ergodic components and its spectrum.

  Remark: compare to problem 2.

(15) Construct a probability vector $p = (p_1, p_2, \dots)$ such that $H(p) = \infty$.

(16) Prove that for any finite-valued random variables X, Y, Z we have $H(X|Y, Z) \geq H(X|Y)$ (we used this repeatedly in class, but I don't think we proved it. Hint: consult the proof that $H(X|Y) \geq H(X)$). Show the same when the conditioning is on σ-algebras.

(17) Prove that the product measure $\mu = (1/2, 1/2)^{\mathbb{Z}}$ is the only measure on $\{0, 1\}^{\mathbb{Z}}$ with $h_\mu(S) = 1$ (all others are ≤ 1).

(18) Use the previous question to conclude that if $\pi : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is a continuous map satisfing $\pi S = S\pi$ (S the shift), then if π is 1-1, it must be onto.

(19) Show that $h_{\mu\times\nu}(T \times S) = h_\mu(T) + h_\nu(S)$.

(20) Prove that if $(X, \mathcal{B}, \mu, T)$ is mixing then for any non-trivial partition α of X, there is a sequence $n_k \to \infty$ such that $\frac{1}{N}H(\bigvee_{k=1}^{N} T^{-n_k}\alpha) \to 1$.

(21) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Define a pseudo-metric between partitions by $d(\mathcal{A}, \mathcal{B}) = H(\mathcal{A}|\mathcal{B}) + H(\mathcal{B}|\mathcal{A})$. Show that the entropy function $\mathcal{A} \mapsto h_\mu(T, \mathcal{A})$ is continuous in the space of finite partitions.

(22) Give an example of an ergodic measure preserving system with infinite entropy, and conclude that it does not have a generating partition.

(23) Show that if (X, T) is a topological system with a unique attracting fixed point (i.e. $T^n x \to x_0 = Tx_0$) for some x_0 and all $x \in X$), then $h_{top}(T) = 0$.

(24) Let $X = \{\pm 1\}^{\mathbb{Z}} \times \{\pm 1\}^{\mathbb{Z}}$, and define the map $X \to X$ by $T(x, y) = (Sx, S^{x(0)}y)$ where $S : \{\pm 1\}^{\mathbb{Z}} \to \{\pm 1\}^{\mathbb{Z}}$ is the shift. Let $\mu = \mu_0 \times \mu_2$ where $\mu_0 = \mu_1 = (1/2, 1/2)^{\mathbb{Z}}$. This system is called the "T, T^{-1}". It is also called "random walk in random scenery", because the first sequence x can be thought of as the increments of a symmetric random walk on the integers; as time goes by we shift the x sequence, producing these increments, and shift the y-sequence either forward or backward according to this increment. Thus, if we view only the$y$ sequence, we see a random sequence of digits being shifted left and right randomly.

(a) Show that (X, T, μ) is measure-preserving (this is an example of a skew product").

(b) Consider the random variables $X_n, Y_n : X \to \{0,1\}$ given by $X_n(x,y) = x_n$ and $Y_n(x,y) = y_n$ and consider the process $(X_n, Y_n)_{n=-\infty}^{\infty}$. Show that a.e., given $(X_{-\infty}^0, Y_{-\infty}^0)$, one can determine the entire sequence $Y_{-\infty}^{\infty}$. Hint: Use recurrence of random walk on $\mathbb{Z}$.

(c) Conclude that $h((X_n, Y_n)_{-\infty}^{\infty}) = \log 2$.

(d) Show that the process above nevertheless has trivial tail.

Remark: This process turns out to be a process with trivial tail that is not isomorphic to a product measure, although this is not trivial to prove. An earlier example was constructed by Ornstein more explicitly.

(25) A factor map $\pi : X \to Y$ between measure preserving systems is called bounded-to-1 if $|\pi(y)^{-1}| < M$ for some constant M and a.e. $y \in Y$, and finite-to-1 if $\pi^{-1}(y)$ is finite for a.e. $y \in Y$. Show in both cases that $h(X) = h(Y)$ (begin with the bounded-to-1 case, which is easier).

(26) Let $x \in A^{\mathbb{N}}$ (A finite). Let

$$L_N(x) = \{w \in A^N \,:\, w = x_i \dots x_{i+N-1} \text{for some } i\}$$

Show that $\log L_N$ is subadditive and that $\lim_{N\to\infty} \frac{1}{N} \log L_N$ exists. Show that this limit is the topological entropy of the orbit closure of x under the shift. Formulate an analogous statement for $x \in A^{\mathbb{Z}}$.

(27) Show that the Morse system has zero topological entropy (the Morse system is defined as follows. Let $\tau : \{0,1\}^* \to \{0,1\}^*$ be defined by $\tau(0) = 01$, $\tau(1) = 10$, and τ is extended pointwise to words, $\tau(w_1 \dots w_k) = \tau(w_1)\tau(w_2)\dots\tau(w_k)$. Then $\tau^{n+1}(0)$ extends $\tau^n(0)$ for all n and thus there is a limiting infinite word $w \in \{0,1\}^{\mathbb{N}}$ such that $w_1 \dots w_{2^n} = \tau^n(0)$. The morse system is the orbit closure of x under the shift.

(28) A topological system (X, T) is called rigid if there is a sequence $n_k \to \infty$ such that $T^{n_k}x \to x$ for every $x \in X$.

(a) Show that if T is a transitive isometry then it is rigid (in fact one can have $T^{n_k}x \to x$ uniformly).

(b) Show that rigid systems have entropy 0.

(29) Let $A \in GL_n(\mathbb{Z})$ be an invertible integer matrix. Let A act on the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ by $T_A x = Ax \bmod 1$. Let $\lambda_1, \dots, \lambda_n$ be the complex eigenvalues of A repeated with multiplicity and ordered such that $|\lambda_1| \geq \dots \geq |\lambda_k| > 1 > |\lambda_{k+1}| \geq \dots \geq |\lambda_n|$. Show that

$$h_{top}(T_A) = \sum_{i=1}^{k} \log|\lambda_i| = -\sum_{i=k+1}^{n} \log|\lambda_i|$$

(30) For a topological system (X, T) and metric d on X, let

$$\overline{d}_n(x,y) = \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y)$$

Let

$$\overline{h}_{top}(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log cov(X, \overline{d}_n, \varepsilon)}{n}$$

Show that $\overline{h}_{top}(X) = h_{top}(X)$.

(31) Prove Abramov's entropy formula for induced maps (see question 1 for definition): If T_A is the induced map on A then $h_{\mu_A}(T_A) = h_\mu(T)/\mu(A)$.