Take-home exam in "Topics in Dynamics" (80940), Spring 2014

Instructions

Solve 6 of the following 8 problems. Give complete proofs. You may rely on anything proved in class and use any literature you like as long as it does not solve the specific problem you are working on. You should solve them yourself.

From the moment you read the exam you have one week to complete it. Submit a hard-copy to me in person or by email no later than August 10.

Problem 1

1. Fix a finite alphabet A. Show that for every $\varepsilon > 0$ there is a $\delta > 0$ with the following property. Let $\xi = (\xi_n), \zeta = (\zeta_n)$ be two stationary processes with values in A, defined on a common probability space, and such that $(\xi_n, \zeta_n)_{n=1}^{\infty}$ is ergodic; if

$$\mathbb{E}(\xi_0 \neq \zeta_0) < \delta$$

then

$$|h(\xi) - h(\zeta)| < \varepsilon$$

where $h(\xi), h(\zeta)$ are the entropies of the corresponding shift-invariant measures.

2. Conclude that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x, y \in \{0,1\}^{\mathbb{N}}$ are generic for ergodic measures μ, ν, and $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} 1_{\{x_i \neq y_i\}} < \delta$, then $h_\mu(\sigma) - h_\nu(\sigma)| < \varepsilon$.

3. Let $x, y \in \{0,1)^{\mathbb{N}}$ and let X, Y be their orbit closures under the shift σ, respectively. Assume also that they are mean-asymptotic , i.e. $\lim_{n\to\infty} \frac{1}{N}\sum_{n=0}^{N} 1_{\{x_i \neq y_i\}} = 0$. Does $h_{top}(X) = h_{top}(Y)$?

Problem 2

Show that for any measure preserving system $(X, \mathcal{F}, \mu, T)$ and every $0 < t < h_\mu(T)$, there is a factor (i.e. invariant sub-σ-algebra of $\mathcal{F}$) with entropy t.

Problem 3

1. A cellular automaton is a map $T : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined by a local rule: $(T(x))_u = f((x_{u+v})_{v\in V}$ for some finite set V and $f : A^V \to A$ (two examples are given below). Show that such a map T is continuous and commutes with the shift.

2. Let $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be given by $(Tx)_i = x_i + x_{i+1} \bmod 1$. Show that T is expansive and compute the topological entropy of T.

3. Let $T : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ be given by $(Tx)_{(i,j)} = x_{(i,j)} + x_{(i+1,j)} + x_{(i,j+1)} \bmod 1$. Compute the topological entropy of T.

Problem 4

For a metric space (X, d), write

$$\begin{aligned} \overline{\dim}_{\mathrm{B}}(X) &= \limsup_{r \to 0} \frac{\log cov(Y, d, r)}{\log(1/r)} \\ \underline{\dim}_{\mathrm{B}}(X) &= \liminf_{r \to 0} \frac{\log cov(Y, d, r)}{\log(1/r)} \end{aligned}$$

These are called the upper and lower box dimension of X, respectively.

Let $T : X \to X$ be a continuous map.

1. Suppose that T is M-Lipschitz, that is, $d(Tx, Ty) \leq M \cdot d(x, y)$ for all $x, y \in X$. Show that

$$h_{top}(T) \leq \overline{\dim}_{\mathrm{B}}(X) \cdot \log(M)$$

2. We say that T expands locally by $m \geq 1$ at x if there is a neighborhood U of x such that $d(Ty, Tz) \geq m \cdot d(y, z)$ for $y, x \in U$. Show that if T expands locally by m at every point, then

$$h_{top}(T) \geq \underline{\dim}_{\mathrm{B}}(X) \cdot \log(m)$$

Problem 5

Let σ be the shift map. Let $\mu \in \mathcal{P}_\sigma(\{0,1\}^{\mathbb{Z}})$ be an ergodic measure. Define a measure on $\{0,1,2\}^{\mathbb{Z}}$ as follows: first form the measure $\mu \times \nu$, where $\nu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{Z}} \in \mathcal{P}_\sigma(\{1,2\}^{\mathbb{Z}}\}$, and let $\tau \in \mathcal{P}(\{0,1,2\}^{\mathbb{Z}})$ be the measure $\tau(A) = (\mu \times \nu)(\pi^{-1}A)$ where π is the map

$$\pi(x, y)_i = x_i \cdot y_i \qquad \text{(multiplication performed in } \mathbb{Z})$$

Verify that τ is stationary and give a short formula for $h_\sigma(\tau)$ in terms of $h_\sigma(\mu)$ and possibly other attributes of μ.

Problem 6

1. Let (X, T) be a topological dynamical system with metric d, and d_n the Bowen metric. Let μ be an ergodic measure for T. Show that for every $\varepsilon > 0$,

$$\limsup_{n \to \infty} -\frac{\log \mu(B_\varepsilon^{d_n}(x))}{n} \leq h_\mu(T) \qquad \mu\text{-a.e. } x$$

   where $B_\varepsilon^{d_n}(x) = \{y : d_n(x, y) < \varepsilon\}$.

2. Let $T : [0,1] \to [0,1]$ be continuously differentiable and $\mu \in \mathcal{P}_T(X)$ an absolutely continuous ergodic measure. Show that

$$h_\mu(T) \geq \int \log |T'(x)| \, d\mu(x)$$

   (Hint: evaluate $\log |(T^n)'(x)|$ using the chain rule, use (1), and the Lebesgue differentiation theorem).

Remark: In fact, the limit in (1) exists and both inequalities are equalities, but this is a little harder (but feel free to prove it).

Problem 7

Let (X, T) be a topological dynamical system. Let Y be a compact metric space and $\Phi : X \to isom(Y)$, $x \mapsto \Phi_x$ a continuous map where $isom(Y)$ is the group of isometries with the topology of uniform convergence. Let $S : X \times Y \to X \times Y$ be the map

$$S(x, y) = (Tx, \Phi_x y)$$

Verify that S is continuous and show that $h_{top}(S) = h_{top}(T)$.

Problem 8

Show that if μ is an invariant ergodic measure on an invertible topological system (X, T), and if $h_\mu(T) > 0$, then for μ-a.e. x there is a y such that $d(T^n x, T^n y) \to 0$ as $n \to \infty$.

Hint: Let $\mathcal{P}_n$ be partitions of X such that every $A \in \mathcal{P}_n$ has diameter $< 1/n$. Observe that if for some $k(n) \to \infty$ we have x, y in the same atom of $\bigvee_{i=k(n)}^{\infty} T^{-i}\mathcal{P}_n$, then $\lim_{n\to\infty} d(T^n x, T^n y) = 0$. Using the fact that $\bigcap_{k=1}^{\infty} \bigvee_{i=k}^{\infty} T^{-i}\mathcal{P}$ is contained in the Pinsker algebra, show that it is possible to choose $n(k) \to \infty$ in such a way that $\bigcap_{n=1}^{\infty} \bigvee_{i=k(n)}^{\infty} T^{-i}\mathcal{P}_n$ is a non-trivial sub-σ-algebra.