**ANALYSIS 1 - SOLUTION FOR EXERCISE 8**

Throughout this solution $E(K)$ will denote the set of extreme points of $K$, where $K$ is a subset of a vector space.

***Question 1:*** The weak topology $\tau$ is stronger than the weak* topology $\tau^*$. It suffice to show that every neighbourhood of 0 in $\tau^*$ contains a neighbourhood of 0 in $\tau$. Let $\Phi : X \to X^{**}$ be the natural embedding. Given $V \in \tau^*$ with $0 \in V$ there exist $x_1, ..., x_n \in X$ and $\epsilon > 0$ with $U(x_1, ..., x_n; \epsilon) \subset V$, where

$$U(x_1, ..., x_n; \epsilon) := \{\Lambda \in X^* \ : \ |\Lambda x_j| < \epsilon \text{ for } 1 \le j \le n\} .$$

Since

$$U(x_1, ..., x_n; \epsilon) = \{\Lambda \in X^* \ : \ |\Phi x_j(\Lambda)| < \epsilon \text{ for } 1 \le j \le n\} \in \tau$$

it follows that $\tau^* \subset \tau$.

***Question 2, part (a):*** Set $S = \{\alpha \in \mathbb{C} \ : \ |\alpha| = 1\}$. We shall show that

$$E(B^*) = \{\alpha \cdot \delta_x \ : \ x \in B, \ \alpha \in S\} .$$

Let $\mu \in E(B^*)$, let $|\mu|$ be the total variation of $\mu$, and let $K \subset B$ be the support of $|\mu|$. If $\mu = 0$ then $\mu = \frac{\delta_x - \delta_x}{2}$ for $x \in B$ which contradicts $\mu \in E(B^*)$, hence $\mu \neq 0$. If $|\mu|(B) < 1$ then $\mu = |\mu|(B) \cdot \frac{\mu}{|\mu|(B)} + (1 - |\mu|(B)) \cdot 0$ which contradicts $\mu \in E(B^*)$, hence $|\mu|(B) = 1$. Assume by contradiction that there exist $x, y \in K$ with $x \neq y$, then there exits a Borel set $E \subset B$ with $|\mu|(E), |\mu|(B \setminus E) > 0$. Let $\mu_1, \mu_2 \in B^*$ be with $\mu_1(F) = \frac{\mu(E \cap F)}{|\mu|(E)}$ and $\mu_2(F) = \frac{\mu((B \setminus E) \cap F)}{|\mu|(B \setminus E)}$ for each Borel set $F \subset B$. Since $\mu = |\mu|(E) \cdot \mu_1 + |\mu|(B \setminus E) \cdot \mu_2$ we get a contradiction to $\mu \in E(B^*)$, and so $K = \{x\}$ for some $x \in B$. From this and from $|\mu|(B) = 1$ it follows that there exists $\alpha \in S$ with $\mu = \alpha \cdot \delta_x$.

Let $x \in B$ and $\alpha \in S$ be given, and set $\mu = \alpha \cdot \delta_x$. Let $\mu_1, \mu_2 \in B^*$ and $0 < t < 1$ be with $\mu = t\mu_1 + (1-t)\mu_2$. Since $|\mu_1\{x\}|, |\mu_2\{x\}| \le 1$ and $\alpha = t\mu_1\{x\} + (1-t)\mu_2\{x\}$, it follows from Lemma 1 below that $\mu_1\{x\} = \mu_2\{x\} = \alpha$. Since $|\mu_1|(B), |\mu_2|(B) \le 1$ and $|\alpha| = 1$ it follows that $\mu = \mu_1 = \mu_2$, and so $\mu \in E(B^*)$. This completes the proof.

**Lemma 1.** *Set $D = \{x \in \mathbb{R}^2 \ : \ |x| \le 1\}$ and $\partial D = \{x \in \mathbb{R}^2 \ : \ |x| = 1\}$, then $E(D) = \partial D$.*

*Proof of Lemma 1:* If $x \in D \setminus \partial D$ it is easy to see that $x \notin E(D)$, hence $E(D) \subset \partial D$. Let $z \in \partial D$ be given, and let $x, y \in D$ and $0 < t < 1$ be with $z = tx + (1-t)y$. Since $|x|, |y| \le 1$ it clearly holds that $|x| = |y| = 1$. We shall now show that $x = y = z$.

Assume first that $x = (1,0)$. Assume by contradiction that $y_1 < 1$, then

$$1 = |z|^2 = |(t + (1-t)y_1, (1-t)y_2)|^2 = t^2 + 2t(1-t)y_1 + (1-t)^2 y_1^2 + (1-t)^2 y_2^2$$
$$= t^2 + 2t(1-t)y_1 + (1-t)^2 < (t + (1-t))^2 = 1\,.$$

This is clearly a contradiction, hence $y_1 = 1$, and so $x = y = z$. Now for the general case, let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal map with $Ux = (1,0)$. Since $Uz = tUx + (1-t)Uy$ and $|Ux| = |Uy| = |Uz| = 1$, it follows from what we have just shown that $Ux = Uy = Uz$, and so $x = y = z$. This shows that $z \in E(D)$, and so $E(D) = \partial D$ as we wanted.

***Part (b):*** Let $\mu \in B^*$ be given. From the Banach-Alauglu theorem it follows that $B^*$ is weak* compact, and it is clear that $B^*$ is convex. From the Krein-Milman theorem it follows that $B^* = \overline{co(E(B^*))}$, where $co(E(B^*))$ stands for the convex hull of $E(B^*)$ and the closure is taken with respect to the weak* topology of $M$. Since $C(B)$ is separable it follows that $B^*$ is metrizable in the weak* topology. From this and from $B^* = \overline{co(E(B^*))}$ it follows that there exists $\{\mu_k\}_{k=1}^{\infty} \subset co(E(B^*))$ with $\mu_k \overset{k}{\to} \mu$ in the weak* topology. From part (a) of this question it follows that for each $k \geq 1$ there exist $N_k \geq 1$, $t_1^k, ..., t_{N_k}^k \in [0,1]$, $\alpha_1^k, ..., \alpha_{N_k}^k \in S$, and $x_1^k, ..., x_{N_k}^k \in B$, with $t_1^k + ...t_{N_k}^k = 1$ and $\mu_k = \sum_{j=1}^{N_k} t_j^k \alpha_j^k \delta_{x_j^k}$. From this and from the definition of the weak* topology it follows that for every $f \in C(B)$

$$\lim_k \sum_{j=1}^{N_k} t_j^k \alpha_j^k f(x_j^k) = \lim_k \int f \, d\mu_k = \int f \, d\mu,$$

which is what we wanted to show.

***Part (c):*** As mentioned above $B^*$ is compact and metrizable in the weak* topology, which implies that it is sequentially compact.

***Part (d):*** Give $f, f_1, f_2, ... \in C(B)$ it holds that $f_k \overset{k}{\to} f$ weakly if and only if $\int f_k \, d\mu \overset{k}{\to} \int f \, d\mu$ for every $\mu \in M$.

***Part (e):*** Let $f, f_1, f_2, ... \in C(B)$ be with $f_k \overset{k}{\to} f$ weakly. Set $H = co\{f_1, f_2, ...\}$, let $\overline{H}$ be the closure of $H$ is the strong topology of $C(B)$, and let $\overline{H}_w$ be the closure of $H$ is the weak topology of $C(B)$, then clearly $f \in \overline{H}_w$. Since $H$ is convex it follows (from a theorem proven in class) that $\overline{H} = \overline{H}_w$, and so $f \in \overline{H}$. Since the strong topology of $C(B)$ is metrizable it follows that there exists $\{g_k\}_{k=1}^{\infty} \subset H$ with $g_k \overset{k}{\to} f$ strongly, hence

$$\lim_k \sup_{x \in B} |f(x) - g_k(x)| = \lim_k \|f - g_k\|_\infty = 0\,.$$

Note that by the definition of $H$ it follows that for every $k \geq 1$ there exist $N_k \geq 1$ and $t_1^k, ..., t_{N_k}^k \in [0,1]$ with $g_k = \sum_{j=1}^{N_k} t_j^k f_j$. This completes the proof.

***Part (f):*** (i) Let $\{P_k\}_{k=1}^{\infty}$ a sequence of partitions of $[0,1]$. As mentioned above $B^*$ is compact and metrizable in the weak* topology. Since $\{\mu_{P_k}\}_{k=1}^{\infty} \subset B^*$ it follows that $\{\mu_{P_k}\}_{k=1}^{\infty}$ has convergent subsequence in the weak* topology.

(ii) Let $\{P_k\}_{k=1}^{\infty}$ a sequence of partitions of $[0,1]$ with $\Delta P_k \xrightarrow{k} 0$. We shall show that $\mu_{P_k} \xrightarrow{k} \mathcal{L}eb$ in the weak* topology, where $\mathcal{L}eb$ is the Lebesgue measure on $[0,1]$. Let $f \in C[0,1]$ and $\epsilon > 0$ be given. There exists $\delta > 0$ with $|f(x) - f(y)| < \epsilon$ for $x, y \in [0,1]$ with $|x - y| < \delta$. There exists $N \geq 1$ with $\Delta P_k < \delta$ for $k \geq N$. Let $k \geq N$ and write $P_k = \{0 = t_0 < ... < t_M = 1\}$, then

$$|\int f\, dx - \int f\, d\mu_{P_k}| \leq \sum_{j=0}^{M-1} \int_{t_j}^{t_{j+1}} |f(x) - f(t_j)|\, dx < \epsilon\,.$$

This shows that $\mu_{P_k} \xrightarrow{k} \mathcal{L}eb$ in the weak* topology, as we wanted.

***Question 3, part (a):*** Set $S = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$ and let $B$ be the closed unit ball of $l^1$. Given $n \geq 1$ let $e^n \in B$ be the $n$'th unit vector of $l^1$, we shall show that

$$E(B) = \{\alpha e^n : n \geq 1 \text{ and } \alpha \in S\}\,.$$

Let $x \in E(B)$, then it is easy to see that $\|x\|_1 = 1$. Assume by contradiction that there exist $1 \leq i < j$ with $x_i \neq 0$ and $x_j \neq 0$. Set $y = \frac{x - x_i e^i}{\|x - x_i e^i\|_1}$, then

$$x = \|x - x_i e^i\|_1 y + |x_i| \frac{x_i e^i}{|x_i|},$$

which is a contradiction to $x \in E(B)$. This shows that there exists a unique $n \geq 1$ with $x_n \neq 0$. Since $\|x\|_1 = 1$ it follows that $x = \alpha e^n$ for some $\alpha \in S$.

Let $n \geq 1$ and $\alpha \in S$ be given and set $x = \alpha e^n$. Let $y, z \in B$ and $0 < t < 1$ be with $x = ty + (1-t)z$. Since $|y_n|, |z_n| \leq 1$ and $\alpha = t y_n + (1-t) z_n$ it follows from Lemma 1 above that $y_n = z_n = \alpha$. Since $\|y\|_1, \|z\|_1 \leq 1$ and $|\alpha| = 1$ it follows that $x = y = z$. This shows that $x \in E(B)$, which completes the proof.

***Part (b):*** Let $B_0$ be the closed unit ball of $c_0$. Let $f \in (c_0)^*$ be with $f(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}$ for $x \in c_0$. It is easy to see that $\sup_{x \in B_0} |f(x)| = \sum_{n=1}^{\infty} n^{-2}$ and $|f(x)| < \sum_{n=1}^{\infty} n^{-2}$ for $x \in B_0$. Since $f$ is continuous with respect to the weak topology of $c_0$ it follows that $B_0$ is not weakly compact, otherwise there would exist $x \in B_0$ with $|f(x)| = \sup_{x \in B_0} |f(x)|$.

***Question 4:*** Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $B_{\mathbb{F}}$ be the closed unit ball of the space of continuous $f : [0,1] \to \mathbb{F}$ equipped with the supremum norm. We shall first show that $E(B_{\mathbb{R}}) = \{\pm 1_{[0,1]}\}$. It is easy to see that $E(B_{\mathbb{R}}) \supset \{\pm 1_{[0,1]}\}$. Let $f \in B_{\mathbb{R}}$ be with $f \notin \{\pm 1_{[0,1]}\}$, then there exists $x \in [0,1]$ with $|f(x)| < 1$. Let $f_1, f_2 \in B_{\mathbb{R}}$ be with $f_1(x) = f(x) + \frac{f(x)^2 - 1}{4}$ and $f_2(x) = f(x) - \frac{f(x)^2 - 1}{4}$ for $x \in [0,1]$. Since $f \neq f_1$ and $f = \frac{f_1 + f_2}{2}$ it follows that $f \notin E(B_{\mathbb{R}})$, which shows $E(B_{\mathbb{R}}) = \{\pm 1_{[0,1]}\}$.

Set $H = \{f \in B_{\mathbb{C}} : |f| = 1\}$. We shall next show that $E(B_{\mathbb{C}}) = H$. Let $f \in B_{\mathbb{C}} \setminus H$ and let $f_1, f_2 \in B_{\mathbb{C}}$ be with $f_1(x) = f(x) + \frac{|f(x)|^2-1}{4}$ and $f_2(x) = f(x) - \frac{|f(x)|^2-1}{4}$ for $x \in [0,1]$. From $f \neq f_1$ and $f = \frac{f_1+f_2}{2}$ it follows that $f \notin E(B_{\mathbb{C}})$, and so $E(B_{\mathbb{C}}) \subset H$. Let $f \in H$ be given, and let $f_1, f_2 \in B_{\mathbb{C}}$ and $0 < t < 1$ be with $f = tf_1 + (1-t)f_2$. For every $x \in [0,1]$ we have $|f_1(x)|, |f_2(x)| \leq 1$, $|f(x)| = 1$, and $f(x) = tf_1(x) + (1-t)f_2(x)$, hence from Lemma 1 we obtain $f(x) = f_1(x) = f_2(x)$. Since this holds for every $x \in [0,1]$ we get $f = f_1 = f_2$, hence $f \in E(B_{\mathbb{C}})$, and so $E(B_{\mathbb{C}}) = H$.

***Question 5:*** We shall prove the claim by induction on $n$. For $n = 1$ the claim is obvious. Let $n > 1$ and assume the claim holds for $n-1$. If $K$ is contained in a hyperplane of $\mathbb{R}^n$ then the claim follows from the induction hypothesis, hence assume that $K$ is not contained in any hyperplane. From Lemma 2 below it follows that $\operatorname{int} K \neq \emptyset$. Fix some $x \in K$, then we need to show that $x$ is the convex combinations of at most $n+1$ points in $E(K)$. From the Krein-Milman theorem it follows that there exists $y \in E(K)$. If $x = y$ then there is nothing to show, hence assume $x \neq y$. Set

$$s_0 = \max\{s \geq 0 \,:\, y + s(x-y) \in K\}$$

and $z = y + s_0(x-y)$, then $s_0 \geq 1$ and $z \in \partial K$. From $\operatorname{int} K \neq \emptyset$ and from question 5 in exercise 1 it follows that there exist $0 \neq f \in (\mathbb{R}^n)^*$ and $c \in \mathbb{R}$ with $f(z) = c$ and $f(w) \leq c$ for all $w \in K$ . Set

$$H = \{w \in \mathbb{R}^n \,:\, f(w) = c\},$$

then $H \cap K$ is convex and compact. Since $H$ is a hyperplane of $\mathbb{R}^n$ and $z \in H \cap K$ it follows from the induction hypothesis that there exist $z_1, ..., z_n \in E(H \cap K)$ and $t_1, ..., t_n \in [0,1]$ with $z = \sum_{j=1}^n t_j z_j$ and $\sum_{j=1}^n t_j = 1$. Since

$$x = \frac{1}{s_0} z + (1 - \frac{1}{s_0})y = \sum_{j=1}^n \frac{t_j}{s_0} z_j + (1 - \frac{1}{s_0})y,$$

$1 = \sum_{j=1}^n \frac{t_j}{s_0} + (1 - \frac{1}{s_0})$, and $y \in E(K)$, the claim will be proven once we show $z_1, ..., z_n \in E(K)$. Let $w \in H \cap K$, $w_1, w_2 \in K$, and $0 < t < 1$, be with $w = tw_1 + (1-t)w_2$. Since $f(w) = tf(w_1) + (1-t)f(w_2)$ and $f(w_i) \leq c = f(w)$ for $i = 1, 2$, it follows that $f(w_i) = c$ for $i = 1, 2$. Hence $w_1, w_2 \in H \cap K$, which shows that $H \cap K$ is an extreme subset of $K$. It was proven in class that extreme points of an extreme subset are extreme points of the original subset, hence $z_1, ..., z_n \in E(H \cap K) \subset E(K)$. This completes the proof of the claim.

**Lemma 2.** *Let $K \subset \mathbb{R}^n$ be a convex set which is not contained in any hyperplane of $\mathbb{R}^n$, then $\operatorname{int} K \neq \emptyset$.*

*Proof of the Lemma:* The proof is by induction on $n$. For $n = 1$ the claim is obvious. Let $n > 1$ and assume the claim holds for $n - 1$. Without loss of generality assume $0 \in K$. From our assumption on $K$ it follows that $span\{x_1, ..., x_n\} = \mathbb{R}^n$ for some $x_1, ..., x_n \in K$. Set $V = span\{x_1, ..., x_{n-1}\}$ and $H = V \cap K$, then since $0, x_1, ..., x_{n-1} \in H$ it follows that $H$ is not contained in a hyperplane of $V$. From this, since $H$ is convex, and from the induction hypothesis, it now follows that there exists an open ball $B \subset \mathbb{R}^n$ with $B \cap V \subset H$. Since the set

$$\{tx_n + (1 - t)y \ : \ 0 < t < 1 \text{ and } y \in B \cap V\}$$

is open in $\mathbb{R}^n$ and contained in $K$ the lemma follows.