ANALYSIS 1 - SOLUTION FOR EXERCISE 7

Question 1: Denote by $\overline{S}_w$ the weak closure of S. Since B is convex and strongly closed it is weakly closed. Since $S \subset B$ it follows that $\overline{S}_w \subset B$. Let $x \in B$ and let $V \subset X$ be a weak neighbourhood of x. There exist $\epsilon > 0$ and $f_1, ..., f_n \in X^*$ with

$$\{y \in X \,:\, |f_i(y - x)| \le \epsilon \text{ for } 1 \le i \le n\} \subset V\,.$$

Since $\dim X = \infty$ there exists $0 \ne y \in X$ with $f_i(y) = 0$ for $1 \le i \le n$. Let $F : [0, \infty) \to [0, \infty)$ be with $F(t) = \|x + ty\|$ for $t \ge 0$. Since $F(0) \le 1$, $\lim_{t\to\infty} F(t) = \infty$, and F is continuous, it follows from the intermediate value theorem that there exists $t_0 \ge 0$ with $F(t_0) = 1$, and then $x + t_0 y \in S$. Since $f_i(x + t_0 y) = f_i(x)$ for every $1 \le i \le n$, it follows that $x + t_0 y \in V$. This shows that $x \in \overline{S}_w$, and so $\overline{S}_w = B$ as we wanted to prove.

Question 2: Let $G_T \subset X \times Y$ be the graph of T. Let $\{(x_n, Tx_n)\}_{n=1}^{\infty} \subset G_T$ and $(x, y) \in X \times Y$ be with $(x_n, Tx_n) \xrightarrow{n} (x, y)$. Given $f \in Y^*$

$$|f(T(x_n - x))| \le C_f \|x_n - x\| \xrightarrow{n} 0,$$

hence $f(Tx) = \lim_n f(Tx_n) = f(y)$. Since Y^* separates points on Y it follows that $y = Tx$, so $(x, y) \in G_T$, and so G_T is closed. It now follows from the closed graph theorem that T is continuous, which is what we wanted to show.

Question 3, part (a): Let $\varphi \in C[0,1]^*$ be with $\varphi(f) = \int_0^1 f\,dx$. From $M = \ker\varphi$ it follows that M is closed.

Part (b): Set $d = dist(h, M)$. From a claim proven in the lecture it follows that there exists $\psi \in C[0,1]^*$ with $\|\psi\| = 1$, $\psi = 0$ on M, and $\psi(h) = d$. From $\ker\varphi \subset \ker\psi$ it follows that there exists $\alpha \in \mathbb{C}$ with $\psi = \alpha\varphi$. It is easy to see that $\|\varphi\| = 1 = \|\psi\|$, hence $|\alpha| = 1$. Since $\varphi(h) > 0$ and $\psi(h) > 0$ it follows that $\psi = \varphi$, and so $d = \varphi(h) = \frac{1}{3}$.

Question 4: Given $g \in L^1(\mathbb{R}^n)$ set $\widetilde{g} = g\frac{\overline{gf}}{|gf|}1_{\{gf\ne 0\}}$, then since $\widetilde{g} \in L^1(\mathbb{R}^n)$

$$\infty > \sup_{k\ge 1} \int_{|x|\le k} f(x)\widetilde{g}(x)\,dx = \int_{\mathbb{R}^n} |f(x)g(x)|\,dx\,. \tag{0.1}$$

For $k \ge 1$ and $g \in L^1(\mathbb{R}^n)$ set $\varphi_k(g) = \int_{\{|f|\le k\}} fg\,dx$ and $\varphi(g) = \int fg\,dx$, then $\varphi_k \in L^1(\mathbb{R}^n)^*$. For every $g \in L^1(\mathbb{R}^n)$ it follows from the dominated convergence theorem and (0.1) that $\lim_k \varphi_k(g) = \varphi(g)$. Hence from a corollary of the uniform boundedness theorem (proven in class) we get $\varphi \in L^1(\mathbb{R}^n)^*$. From Riesz theorem it now follows that there exists $h \in L^\infty(\mathbb{R}^n)$ such that for every $g \in L^1(\mathbb{R}^n)$

$$\int hg\,dx = \varphi(h) = \int fg\,dx\,.$$

This clearly shows that $f(x) = h(x)$ for $\mathcal{L}eb$-a.e. $x \in \mathbb{R}^n$, hence $f \in L^\infty(\mathbb{R}^n)$ which is what we wanted to show.

Question 5, part (a): A proof can be found in page 59 of Rudin's book.

Part (b): (i) Let $\lambda \in \mathbb{C}$, then it is clear that E_λ is convex. Let $f \in X$ and $\epsilon > 0$, then since $C[-1,1]$ is dense in X there exists $g \in C[-1,1]$ with $\|f-g\|_2 < \epsilon$. For $0 < \delta < 1$ let $g_\delta \in C[-1,1]$ be with

$$g_\delta(x) = \begin{cases} \frac{x}{\delta}g(\delta) + \frac{\delta-x}{\delta}\lambda & , \text{if } x \in [0,\delta] \\ \frac{-x}{\delta}g(-\delta) + \frac{\delta+x}{\delta}\lambda & , \text{if } x \in [-\delta,0] \\ g(x) & , \text{if } x \in [-1,1]\setminus[-\delta,\delta] \end{cases},$$

then it is easy to see that $g_\delta \in E_\lambda$ and $\|g-g_\delta\|_2 < \epsilon$ if δ is small enough. It follows that $\|f-g_\delta\|_2 < 2\epsilon$, which shows that E_λ is dense in X.

(ii) Let $\lambda, \mu \in \mathbb{C}$ be with $\lambda \neq \mu$. Assume by contradiction that there exists $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ with

$$\sup_{f \in E_\lambda} \mathrm{Re}\Lambda(f) < \gamma < \inf_{f \in E_\mu} \mathrm{Re}\Lambda(f)\,. \tag{0.2}$$

The set $\{g \in X \,:\, \mathrm{Re}\Lambda(g) > \gamma\}$ is a nonempty open subset of X which is disjoint from E_λ. Since E_λ is dense in X this gives a contradiction, and so there is no $\Lambda \in X^*$ which satisfies (0.2).