ANALYSIS 1 - SOLUTION FOR EXERCISE 5

Question 1, part (a): If $p = \infty$ then $r = q$ and f is essentially bounded, and so the claim is obvious. In a similar manner this is true if $q = \infty$, hence assume $p, q < \infty$. From $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ we get $1 = \frac{r}{p} + \frac{r}{q}$, so from Holder's inequality

$$\int |fg|^r \, d\mu \leq (\int (|f|^r)^{p/r} \, d\mu)^{r/p} \cdot (\int (|g|^r)^{q/r} \, d\mu)^{r/q} = \|f\|_p^r \cdot \|g\|_q^r \, ,$$

which shows that $fg \in L^r(\mu)$ and $\|fg\|_r \leq \|f\|_p \cdot \|g\|_q$.

Part (b): Assume first that $p_1, ..., p_n < \infty$. We shall prove by induction on n that

$$\int |\prod_{k=1}^{n} f_k| \, d\mu \leq \prod_{k=1}^{n} (\int |f_k|^{p_n} \, d\mu)^{1/p_n} \, .$$

The base case $n = 1$ is trivial. Assume $n > 1$ and that the claim holds for $n - 1$. Set $q = (\sum_{k=1}^{n-1} \frac{1}{p_k})^{-1}$, then $\frac{1}{q} + \frac{1}{p_n} = 1$, and so from Holder's inequality

$$\int |\prod_{k=1}^{n} f_k| \, d\mu \leq (\int |\prod_{k=1}^{n-1} f_k|^q \, d\mu)^{1/q} \cdot (\int |f_n|^{p_n} \, d\mu)^{1/p_n} \, . \tag{0.1}$$

Since $\frac{q}{p_1} + ... + \frac{q}{p_{n-1}} = 1$ it follows from the induction hypothesis that

$$\int \prod_{k=1}^{n-1} |f_k|^q \, d\mu \leq \prod_{k=1}^{n-1} (\int |f_k|^{p_k} \, d\mu)^{q/p_k} \, .$$

This together with (0.1) completes the induction. It now follows that

$$\|f\|_1 = \int |f| \, d\mu \leq \prod_{k=1}^{n} (\int |f_k|^{p_n} \, d\mu)^{1/p_n} = \prod_{k=1}^{n} \|f_k\|_{p_k} < \infty,$$

and so $f \in L^1(\mu)$. If $1 \leq k \leq n$ is such that $p_k = \infty$ then $\frac{1}{p_k} = 0$ and f_k is essentially bounded, and so the general case follows easily from what we have just proven.

Question 2: Assume first that $r < \infty$, then for $p \in [s, r]$

$$\int |f|^p \, d\mu \leq \int_{\{|f| \leq 1\}} |f|^s \, d\mu + \int_{\{|f| > 1\}} |f|^r \, d\mu \leq \|f\|_s^s + \|f\|_r^r < \infty,$$

and so $f \in L^p(\mu)$. Assume $r = \infty$, then for $p \in [s, \infty)$

$$\int |f|^p \, d\mu \leq \int_{\{|f| \leq 1\}} |f|^s \, d\mu + \int_{\{|f| > 1\}} |f|^p \, d\mu \leq \|f\|_s^s + \|f\|_\infty^p \cdot \mu\{f > 1\} < \infty,$$

and so $f \in L^p(\mu)$. We shall now show Ψ is convex. Let $s \leq p < q \leq r$ and $t \in (0,1)$ be given, assume $q < \infty$, and set $u = tp + (1-t)q$. Since $t + (1-t) = 1$ it follows from Holder's inequality that

$$\|f\|_u^u = \int |f|^{tp} \cdot |f|^{(1-t)q} \, d\mu \leq (\int |f|^p \, d\mu)^t \cdot (\int |f|^q \, d\mu)^{1-t} = \|f\|_p^{tp} \cdot \|f\|_q^{(1-t)q} \ .$$

From this we get

$$\Psi(u) = \log \|f\|_u^u \leq \log \left(\|f\|_p^{tp} \cdot \|f\|_q^{(1-t)q} \right) = t\Psi(p) + (1-t)\Psi(q),$$

which shows that Ψ is convex.

Question 3: First solution: From $fg \geq 1$ we get $\sqrt{fg} \geq 1$, and so from Holder's inequality with $p = q = 2$ we get

$$1 = \int 1 \, d\mu \leq \int \sqrt{fg} \, d\mu \leq \left(\int f \, d\mu \cdot \int g \, d\mu \right)^{1/2},$$

which gives the desired conclusion.

Remark: when $p = q = 2$ the Holder's inequality is the same as the Cauchy-Schwarz inequality.

Second solution: From $fg \geq 1$ we get $\log f + \log g \geq 0$, hence $\int \log f d\mu + \int \log g d\mu \geq 0$, and so from $\mu(\Omega) = 1$ and Jensen's inequality we get

$$1 \leq \exp(\int \log f \, d\mu) \cdot \exp(\int \log g \, d\mu) \leq \int f \, d\mu \cdot \int g \, d\mu \, .$$

Question 4, part (a): Let $\epsilon > 0$ and let $\delta > 0$ be with $||1 + c|^p - 1| \leq \epsilon$ for all $|c| \leq \delta$. Since

$$\lim_{|c| \to \infty} \frac{||1 + c|^p - 1|}{|c|^p} = 1$$

we have

$$1 \leq C_\epsilon := \sup\{ \frac{||1 + c|^p - 1|}{|c|^p} \ : \ |c| \geq \delta \} < \infty,$$

and so

$$||1 + c|^p - 1| \leq \epsilon + C_\epsilon |c|^p \text{ for all } c \in \mathbb{R} \, .$$

Given $a, b \in \mathbb{R}$ with $a \neq 0$ we can put $c = \frac{b}{a}$ in the last inequality, and then multiply both sides by $|a|^p$. This gives

$$||a + b|^p - |a|^p| \leq \epsilon |a|^p + C_\epsilon |b|^p \text{ for all } a, b \in \mathbb{R} \, . \tag{0.2}$$

For $n \geq 1$ define

$$g_n^\epsilon := \max\{ | \, |f_n|^p - |f_n - f|^p - |f|^p | - \epsilon |f_n - f|^p , \, 0 \} \, .$$

If we apply (0.2) with $a = f_n - f$ and $b = f$ we get

$$| \, |f_n|^p - |f_n - f|^p - |f|^p | \leq | \, |f_n|^p - |f_n - f|^p | + |f|^p \leq \epsilon |f_n - f|^p + (1 + C_\epsilon) |f|^p,$$

hence $g_n^\epsilon \leq (1+C_\epsilon)|f|^p$. From this, from $\|f\|_p < \infty$, from $\lim_n g_n^\epsilon = 0$ pointwise, and from the dominated convergence theorem, we get $\lim_n \|g_n^\epsilon\|_1 = 0$. Given $\delta > 0$ there exists $N \geq 1$ with $\|g_n^\epsilon\|_1 < \delta$ for $n \geq N$, which shows that

$$\delta > \int g_n^\epsilon \, d\mu \geq \int \left| \, |f_n|^p - |f_n - f|^p - |f|^p \right| \, d\mu - \epsilon \int |f_n - f|^p \, d\mu \, .$$

Since this holds for all $\delta > 0$ we obtain

$$\limsup_n \int \left| \, |f_n|^p - |f_n - f|^p - |f|^p \right| \, d\mu \leq \epsilon \cdot \limsup_n \int |f_n - f|^p \, d\mu \, .$$

Since this holds for all $\epsilon > 0$ and since for each $n \geq 1$

$$\int |f_n - f|^p \, d\mu = \|f_n - f\|_p^p \leq (\|f_n\|_p + \|f\|_p)^p \leq (\sup_k \|f_k\|_p + \|f\|_p)^p < \infty,$$

it follows that

$$0 = \lim_n \int \left| \, |f_n|^p - |f_n - f|^p - |f|^p \right| \, d\mu \geq \lim_n \left| \, \|f_n\|_p^p - \|f_n - f\|_p^p - \|f\|_p^p \right| ,$$

which completes the proof.

Part (b): Assume also that $\lim_n \|f_n\|_p = \|f\|_p$, then from part (a)

$$0 = \lim_n \left(\|f_n\|_p^p - \|f_n - f\|_p^p - \|f\|_p^p \right) = \lim_n \|f_n - f\|_p^p \, ,$$

which shows that $f_n \xrightarrow{n} f$ in $L^p(\mu)$.

Question 5: For $1 \leq p \leq \infty$ set $L^p = L^p(0,1)$, and for $n \geq 1$ set

$$E_n = \{ f \in L^2 \ : \ \|f\|_2^2 \leq n \} \, .$$

First we show that E_n is closed in L^1. Let $f_1, f_2, ... \in E_n$ and $f \in L^1$ be with $f_n \xrightarrow{n} f$ in L^1, then there exists a subsequence with $\{f_{n_k}\}_{k=1}^\infty$ with $f_{n_k}(x) \xrightarrow{k} f(x)$ for $\mathcal{L}eb$-a.e. $x \in (0,1)$. It now follows from Fatou's lemma that

$$\int_0^1 |f(x)|^2 \, dx = \int_0^1 \lim_k |f_{n_k}(x)|^2 \, dx \leq \liminf_k \int_0^1 |f_{n_k}(x)|^2 \, dx \leq n,$$

so $f \in E_n$, and so E_n is closed in L^1. We shall now show that E_n is nowhere dense in L^1, since E_n is closed it suffice to show that $L^1 \setminus L^2$ is dense in L^1. Let $f \in L^1$ and $\epsilon > 0$ be given, then since L^∞ is dense in L^1 there exists $g \in L^\infty$ with $\|f - g\|_1 < \epsilon$. Let $\delta > 0$ be with $\int_0^\delta x^{-1/2} \, dx < \epsilon$, and let $h \in L^1$ be such that $h(x) = x^{-1/2} \cdot 1_{(0,\delta)}(x)$ for $x \in (0,1)$. From the Minkowski inequality we get

$$\infty = \|h\|_2 \leq \|h + g\|_2 + \|g\|_2 \, ,$$

so $\|h + g\|_2 = \infty$ (since $\|g\|_2 < \infty$), and so $h + g \in L^1 \setminus L^2$. Since

$$\|f - h - g\|_1 \leq \|f - g\|_1 + \|h\|_1 < 2\epsilon$$

we get that $L^1 \setminus L^2$ is dense in L^1, and so E_n is nowhere dense in L^1. Since $L^2 = \cup_{n=1}^{\infty} E_n$ it follows that L^2 is of the first category in L^1, which is what we wanted to prove.

Question 6: Set

$$l^2 = \{\{x_k\}_{k=1}^{\infty} \subset \mathbb{C} \,:\, \sum_{k=1}^{\infty} |x_k|^2 < \infty\},$$

then l^2 is a Banach space (even a Hilbert space) with the norm $\|x\|_2 = \sum_{k=1}^{\infty} |x_k|^2$ for $x \in l^2$. For every $x \in l^2$ set $f(x) = \sum_{k=1}^{\infty} x_k a_k$, and for each $n \geq 1$ let $f_n \in (l^2)^*$ be with $f_n(x) = \sum_{k=1}^{n} x_k a_k$ for $x \in l^2$. For each $x \in l^2$ we have $f(x) = \lim_n f_n(x)$, hence from a corollary of the uniform boundedness theorem (proven in class) we get $f \in (l^2)^*$. Let μ be the counting measure on $\mathbb{N}$, then $l^2 = L^2(\mu)$. Since the dual space of $L^2(\mu)$ is isomorphic to $L^2(\mu)$ (proven in class), it follows that there exists $y \in l^2$ with $f(x) = \sum_{k=1}^{\infty} y_k x_k$ for $x \in l^2$. Given $k \geq 1$ let $e_k \in l^2$ be the standard k'th unit vector, then $a_k = f(e_k) = y_k$. This holds for all $k \geq 1$, hence $\{a_k\}_{k=1}^{\infty} = y \in l^2$, and so $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, which is what we wanted to prove.

Question 7, part (a): Let $f \in X$ be given, then for $x \in [0,1]$

$$|Gf(x)| \leq \int_0^x |g(t)f(t)|\, dt \leq \int_0^1 |g(t)|\, dt \cdot \|f\|_\infty = \|g\|_1 \|f\|_\infty \,.$$

This shows that $\|Gf\|_\infty \leq \|g\|_1 \|f\|_\infty$, and so G is bounded with $\|G\| \leq \|g\|_1$. Given $\epsilon > 0$ let $f_\epsilon \in X$ be with $f_\epsilon(x) = \frac{\overline{g(x)}}{|g(x)|+\epsilon}$ for $x \in [0,1]$. Since $\|f_\epsilon\|_\infty \leq 1$ we have

$$\|G\| \geq \|Gf_\epsilon\|_\infty \geq |\int_0^1 g(t) f_\epsilon(t)\, dt| = \int_0^1 \frac{|g(t)|^2}{|g(t)|+\epsilon}\, dt \,.$$

This holds for all $\epsilon > 0$, hence from the dominated convergence theorem

$$\|G\| \geq \lim_{\epsilon \downarrow 0} \int_0^1 \frac{|g(t)|^2}{|g(t)|+\epsilon}\, dt = \int_0^1 \lim_{\epsilon \downarrow 0} \frac{|g(t)|^2}{|g(t)|+\epsilon}\, dt = \|g\|_1 \,,$$

and so $\|G\| = \|g\|_1$.

Part (b): Set $K = supp(g)$, we shall show that $\ker G = \{f \in X \,:\, f = 0 \text{ on } K\}$. It is obvious that if $f \in X$ satisfies $f = 0$ on K then $f \in \ker G$. Let $f \in \ker G$ be given, then $Gf(x) = 0$ for all $x \in [0,1]$, and so for all $0 \leq a \leq b \leq 1$

$$0 = Gf(b) - Gf(a) = \int_a^b g(t)f(t)\, dt \,.$$

Since $g \cdot f$ is continuous it follows that we must have $g \cdot f = 0$ on $[0,1]$, so $f(x) = 0$ for all $x \in [0,1]$ with $g(x) \neq 0$, and so $f = 0$ on K which is what we wanted.

Part (c): We shall show that $range(G) = D$, where

$$D := \{f \in C^1([0,1]) \,:\, f(0) = 0\} \,.$$

Let $f \in G$, then $Gf(0) = 0$ and $(Gf)'(x) = g(x)f(x)$ for $x \in [0,1]$, so $Gf \in D$, and so $range(G) \subset D$. Let $h \in D$ and let $f \in X$ be with $f(x) = \frac{h'(x)}{g(x)}$ for $x \in [0,1]$, then for all $x \in [0,1]$

$$Gf(x) = \int_0^x g(t)f(t)\,dt = \int_0^x h'(t)\,dt = h(x) - h(0) = h(x)\,.$$

This shows that $Gf = h$, so $h \in range(G)$, so $D \subset range(G)$, and so $D = range(G)$ as we wanted.