ANALYSIS 1 - SOLUTION FOR EXERCISE 4

Question 1, part (a): It is not necessarily true that $\{k^{-1}U\}_{k=1}^{\infty}$ is a local base at 0 for X. For example take $X = \mathbb{R}^2$ and $U = \{(x, y) \in \mathbb{R}^2 : |x| < 1\}$.

Part (b): If $U = B(0, r)$ it is also not necessarily true that $\{k^{-1}U\}_{k=1}^{\infty}$ is a local base at 0 for X. Example: Let $X = C(\Omega)$ be as in question 2 below. There exist a compact $K \subset \Omega$ and $\epsilon > 0$ with

$$\{f \in X : p_K(f) < \epsilon\} \subset B(0, r) .$$

Let $H \subset \Omega \setminus K$ be compact, let $k \geq 1$, and let $g \in X$ be with $g = 0$ on K and $g \geq 1$ on H. Since $p_K(k \cdot g) = 0$ we have $g \in k^{-1} \cdot B(0, r)$, and so since $p_H(g) \geq 1$

$$k^{-1} \cdot B(0, r) \not\subset \{f \in X : p_H(f) < 1\} .$$

This holds for all $k \geq 1$, hence $\{k^{-1}B(0, r)\}_{k=1}^{\infty}$ is not a local base at 0.

Question 2: Set $X = C(\Omega)$. For $\alpha \in A$ let $\Lambda_\alpha \in X^*$ be with $\Lambda_\alpha f = \int_{K_\alpha} f(x)\, dx$ for $f \in X$. By assumption we have $\sup_{\alpha \in A} |\Lambda_\alpha f| < \infty$ for all $f \in X$. Since X is a Frechet space it follows from the Uniform Boundedness Theorem that there exist a compact $K \subset \Omega$ and $\epsilon > 0$ with

$$\Lambda_\alpha\{f \in X : p_K(f) < \epsilon\} \subset \{z \in \mathbb{C} : |z| < 1\} \text{ for all } \alpha \in A . \tag{0.1}$$

Assume by contradiction that K doesn't contain $\cup_{\alpha \in A} K_\alpha$, then there exist $\alpha \in A$, $x \in \Omega$, and $\delta > 0$ with $\overline{B}(x, \delta) \subset U_\alpha$ and $\overline{B}(x, \delta) \cap K = \emptyset$. Let $f \in X$ be with

$$f(y) \geq \mathcal{L}eb(\overline{B}(x, \delta))^{-1} \cdot 1_{\overline{B}(x,\delta)}(y) \text{ for all } y \in \Omega$$

and $f = 0$ on K, where $\mathcal{L}eb$ is the Lebesgue measure. Since $f = 0$ on K it follows from (0.1) that $|\Lambda_\alpha f| < 1$. On the other hand we have

$$\Lambda_\alpha f = \int_{K_\alpha} f(x)\, dx \geq \int_{\overline{B}(x,\delta)} f(x)\, dx \geq 1 .$$

This contradiction shows that $\cup_{\alpha \in A} K_\alpha \subset K$ as we wanted.

Question 3: Assume there exits $f \in X^*$ with $\|f\| \leq \gamma$ and $f(x_k) = \alpha_k$ for $k \geq 1$. Let $n \geq 1$ and $\beta_1, ..., \beta_n \in \mathbb{C}$ be given, then

$$|\sum_{i=1}^{n} \beta_i \alpha_i| = |f(\sum_{i=1}^{n} \beta_i x_i)| \leq \gamma \left\| \sum_{i=1}^{n} \beta_i x_i \right\| .$$

For the other direction, assume for each $n \geq 1$ and $\beta_1, ..., \beta_n \in \mathbb{C}$

$$(0.2) \qquad |\sum_{i=1}^{n} \beta_i \alpha_i| \leq \gamma \left\| \sum_{i=1}^{n} \beta_i x_i \right\| .$$

Set $Y = span\{x_k : k \geq 1\}$. Given $y \in Y$ there exist $n \geq 1$ and $\beta_1, ..., \beta_n \in \mathbb{C}$ with $y = \sum_{i=1}^{n} \beta_i x_i$, define $f_0(y) = \sum_{i=1}^{n} \beta_i \alpha_i$. We shall now show that $f_0 : Y \to \mathbb{C}$ is well defined. Let $y \in Y$, and let $n, m \geq 1$ and $\beta_1, ..., \beta_n, \eta_1, ..., \eta_m \in \mathbb{C}$ be with

$$\sum_{i=1}^{n} \beta_i x_i = y = \sum_{i=1}^{m} \eta_i x_i .$$

Since

$$|\sum_{i=1}^{n} \beta_i \alpha_i - \sum_{i=1}^{m} \eta_i \alpha_i| \leq \gamma \left\| \sum_{i=1}^{n} \beta_i x_i - \sum_{i=1}^{m} \eta_i x_i \right\| = 0$$

we have $\sum_{i=1}^{n} \beta_i \alpha_i = \sum_{i=1}^{m} \eta_i \alpha_i$, and so f_0 is well defined. It is obvious that f_0 in linear, and from (0.2) it follows that $|f_0(y)| \leq \gamma \|y\|$ for $y \in Y$. From the Hahn-Banach theorem it now follows that there exists $f \in X^*$ with $\|f\| \leq \gamma$ and $f(y) = f_0(y)$ for $y \in Y$. For every $k \geq 1$ we have $x_k \in Y$, hence $f(x_k) = f_0(x_k) = \alpha_k$, and the proof is complete.

Question 4: Assume there exists a signed Radon measure μ on $[0,1]$ with $\int_0^1 x^k d\mu(x) = \alpha_k$ for $k \geq 1$. Set $\gamma := |\mu|([0,1])$, where $|\mu|$ is the total variation measure of μ. Given $\beta_1, ..., \beta_n \in \mathbb{R}$ we have

$$|\sum_{i=1}^{n} \beta_i \alpha_i| = |\sum_{i=1}^{n} \beta_i \int_0^1 x^k \, d\mu(x)| = |\int_0^1 \sum_{i=1}^{n} \beta_i x^k \, d\mu(x)| \leq \gamma \cdot \max_{x \in [0,1]} |\sum_{i=1}^{n} \beta_i x^k| .$$

For the other direction, assume there exists $0 < \gamma < \infty$ such that for all $n \geq 1$ and $\beta_1, ..., \beta_n \in \mathbb{R}$

$$(0.3) \qquad |\sum_{i=1}^{n} \beta_i \alpha_i| \leq \gamma \cdot \max_{x \in [0,1]} |\sum_{i=1}^{n} \beta_i x^k| .$$

Set $X = C[0,1]$, let $\|\cdot\|$ be the supremum norm on X, and for $k \geq 1$ let $p_k \in X$ be with $p_k(x) = x^k$ for $x \in [0,1]$. From (0.3) and question 3, which can also be carried out when the scalar field is $\mathbb{R}$, it follows that there exists $\Lambda \in X^*$ with $\|\Lambda\| \leq \gamma$ and $\Lambda(p_k) = \alpha_k$ for $k \geq 1$. From the Riesz representation theorem it follows that there exists a unique signed Radon measure μ on $[0,1]$ with $\Lambda(f) = \int_0^1 f \, d\mu$ for $f \in X$. In particular for each $k \geq 1$

$$\alpha_k = \Lambda(p_k) = \int_0^1 p_k \, d\mu = \int_0^1 x^k \, d\mu(x),$$

which completes the proof.

Question 5, part (a): The completeness of c_0, c, and l^∞ was established in exercise 2, let us show that l^1 is complete. Let $\{x^n\}_{n=1}^{\infty} \subset l^1$ be a Cauchy sequence.

For each $j, m, n \geq 1$ we have $|x_j^n - x_j^m| \leq \|x^n - x^m\|_1$, where $\|\cdot\|_1$ is the norm of l^1. It follows that $\{x_j^n\}_{n=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence for each $j \geq 1$, hence there exists $x_j \in \mathbb{C}$ with $x_j^n \xrightarrow{n} x_j$. Set $x = \{x_j\}_{j=1}^\infty$. Let $\epsilon > 0$, then there exists $N \geq 1$ with $\|x^n - x^m\|_1 \leq \epsilon$ for $n, m \geq N$. Given $n \geq N$ it holds for all $M \geq 1$ that

$$\sum_{j=1}^{M} |x_j - x_j^n| = \lim_m \sum_{j=1}^{M} |x_j^m - x_j^n| \leq \limsup_m \|x^n - x^m\|_1 \leq \epsilon,$$

and so $\sum_{j=1}^\infty |x_j - x_j^n| \leq \epsilon$. This shows that

$$\sum_{j=1}^{\infty} |x_j| \leq \epsilon + \|x^n\|_1 < \infty,$$

and so $x \in l^1$. It also follows that $\|x - x^n\|_1 \leq \epsilon$ for $n \geq N$, so $x^n \xrightarrow{n} x$, and so l^1 is complete.

part (b): Let $\|\cdot\|_\infty$ be the norm of l^∞, c, and c_0. Given $x \in l^1$ we have for each $y \in c_0$

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|y\|_\infty \|x\|_1 < \infty, \tag{0.4}$$

hence we can define $\Phi x : c_0 \to \mathbb{C}$ by $\Phi x(y) = \sum_{n=1}^\infty x_n y_n$ for $y \in c_0$. It is obvious that Φx is linear. From (0.4) we get $\|\Phi x\| \leq \|x\|_1 < \infty$, where $\|\Phi x\|$ stands for the operator norm of Φx. It follows that $\Phi x \in (c_0)^*$, and so Φ defines a map from l^1 into $(c_0)^*$. It is easy to check that Φ is linear. Let $x \in l^1$ be given, and let $N \geq 1$ and $y \in c_0$ be with

$$y_n = \begin{cases} \frac{\overline{x_n}}{|x_n|} & \text{, if } n \leq N \text{ and } x_n \neq 0 \\ 0 & \text{, else} \end{cases} \quad \text{for all } n \geq 1\,. \tag{0.5}$$

Since $\|y\|_\infty \leq 1$ it follows that

$$\|\Phi x\| \geq |\Phi x(y)| = |\sum_{n=1}^{\infty} x_n y_n| = \sum_{n=1}^{N} |x_n|\,.$$

This holds for all $N \geq 1$, hence $\|\Phi x\| \geq \sum_{n=1}^\infty |x_n| = \|x\|_1$. From this and from $\|\Phi x\| \leq \|x\|_1$ it follows that $\|\Phi x\| = \|x\|_1$, and so Φ is an isometry.
We shall now show that Φ is surjective. Let $f \in (c_0)^*$ be given. For $n \geq 1$ let $\{e_{n,k}\}_{k=1}^\infty = e_n \in c_0$ be such that $e_{n,n} = 1$ and $e_{n,k} = 0$ for $k \neq n$, and set $x_n = f(e_n)$. Let $N \geq 1$ and define $y \in c_0$ as in (0.5), then since $\|y\|_\infty \leq 1$

$$\|f\| \geq |f(y)| = |f(\sum_{n=1}^{N} y_n e_n)| = |\sum_{n=1}^{N} y_n f(e_n)| = \sum_{n=1}^{N} |x_n|\,.$$

Since this holds for all $N \geq 1$ we have $\|f\| \geq \sum_{n=1}^{\infty} |x_n|$, and so $\{x_n\}_{n=1}^{\infty} = x \in l^1$. For each $z \in c_0$ we have $\lim_N \sum_{n=1}^{N} z_n e_n = z$, hence from the continuity of f

$$\Phi x(z) = \sum_{n=1}^{\infty} x_n z_n = \sum_{n=1}^{\infty} z_n f(e_n) = f(\sum_{n=1}^{\infty} z_n e_n) = f(z) .$$

This shows that $\Phi x = f$, and so Φ is surjective. We have thus shown that Φ is an isometry from l^1 onto $(c_0)^*$, and so these two spaces can be identified.

part (c): Let $y \in l^1$, then for $x \in c$

$$|f_y(x)| \leq |y_1 x_\infty| + \sum_{k=2}^{\infty} |y_k x_{k-1}| \leq \|x\|_\infty \sum_{k=1}^{\infty} |y_k| = \|x\|_\infty \|y\|_1 ,$$

and so $\|f_y\| \leq \|y\|_1$. Let $\epsilon > 0$ and let $N \geq 1$ be with $\sum_{k=N+1}^{\infty} |y_k| < \epsilon$. Let $x \in c$ be with

$$(0.6) \qquad x_k = \begin{cases} \frac{\overline{y_{k+1}}}{|y_{k+1}|} & , \text{ if } y_{k+1} \neq 0 \text{ and } k < N \\ 0 & , \text{ if } y_{k+1} = 0 \text{ and } k < N \\ \frac{\overline{y_1}}{|y_1|} & , \text{ if } y_1 \neq 0 \text{ and } k \geq N \\ 0 & , \text{ if } y_1 = 0 \text{ and } k \geq N \end{cases} \qquad \text{for all } k \geq 1,$$

then

$$|f_y(x)| = |y_1 x_\infty + \sum_{k=2}^{\infty} y_k x_{k-1}| \geq \sum_{k=1}^{N} |y_k| - \sum_{k=N+1}^{\infty} |y_k| \geq \|y\|_1 - 2\epsilon .$$

This holds for all $\epsilon > 0$, so $|f_y(x)| \geq \|y\|_1$. Since $\|x\|_\infty \leq 1$ it follows that $\|f_y\| \geq \|y\|_1$, so $\|f_y\| = \|y\|_1$, and so the map $y \to f_y$ is a linear isometry of l^1 into c^*. We shall now show this map is also onto. Let $g \in c^*$ and for each $k \geq 2$ set $y_k = g(e_{k-1})$. Let $N \geq 1$ and let $x \in c$ be as defined in (0.6), then since $|x_k| \leq 1$ for $k \geq 1$

$$\begin{aligned} \sum_{k=2}^{N} |y_k| = |\sum_{k=2}^{N} y_k x_{k-1}| = |\sum_{k=2}^{N} g(e_{k-1}) x_{k-1}| \\ = |g(\sum_{k=1}^{N-1} x_k e_k)| \leq \|g\| \left\| \sum_{k=1}^{N-1} x_k e_k \right\|_\infty \leq \|g\| , \end{aligned}$$

and so $\sum_{k=2}^{\infty} |y_k| < \infty$. Set $y_1 = g(z) - \sum_{k=2}^{\infty} y_k$ where $z \in c$ is such that $z_k = 1$ for all $k \geq 1$, then $\{y_k\}_{k=1}^{\infty} = y \in l^1$. Given $w \in c$ the limit $\lim_N \sum_{k=1}^{N} (w_k - w_\infty) e_k$

exists in c, hence

$$g(w) = g(w_\infty z + \sum_{k=1}^{\infty}(w_k - w_\infty)e_k) = w_\infty \cdot g(z) + \sum_{k=1}^{\infty}(w_k - w_\infty) \cdot g(e_k)$$

$$= w_\infty \cdot y_1 + w_\infty \sum_{k=2}^{\infty} y_k + \sum_{k=1}^{\infty}(w_k - w_\infty) \cdot y_{k+1} = w_\infty \cdot y_1 + \sum_{k=2}^{\infty} w_{k-1} \cdot y_k = f_y(w)\,.$$

It follows that $g = f_y$, and so the map $y \to f_y$ is onto.

Question 6, part (a): c_{00} is not a closed subspace of c_0. For each $n \geq 1$ let $\{x_k^n\}_{k=1}^{\infty} = x^n \in c_{00}$ be with $x_k^n = \frac{1}{k}$ for $1 \leq k \leq n$ and $x_k^n = 0$ for $k > n$. It is clear that $\{x^n\}_{n=1}^{\infty}$ converges to $\{\frac{1}{k}\}_{k=1}^{\infty} \in c_0 \setminus c_{00}$, which shows that c_{00} is not a closed subspace of c_0.

part (b): Fix some $x \in c_{00}$, then the map $Q(x, \cdot) : c_{00} \to \mathbb{C}$ is clearly linear. Set $M := \sum_{i=1}^{\infty} |x_i| < \infty$, then for $y \in c_{00}$

$$|Q(x,y)| = |\sum_{i=1}^{\infty} x_i y_i| \leq M \left\|y\right\|_\infty ,$$

and so the linear functional $Q(x, \cdot)$ is continuous. This shows that Q is continuous in the second argument, in a similar manner it can be shown that Q is continuous in the first argument.
We shall now show that Q is not continuous as a function on $c_{00} \times c_{00}$. For every $n \geq 1$ let $x^n \in c_{00}$ be such that $x_k^n = \frac{1}{n}$ for $1 \leq k \leq n^2$ and $x_k^n = 0$ for $k > n^2$. Clearly $\{x^n\}_{n=1}^{\infty} \subset c_{00}$ converges to 0 in c_{00}, hence $\{(x^n, x^n)\}_{n=1}^{\infty}$ converges to $(0,0)$ in $c_{00} \times c_{00}$. Also, for each $n \geq 1$ we have $Q(x^n, x^n) = 1$. Since $Q(0,0) = 0$ it follows that Q is not continuous on $c_{00} \times c_{00}$.

Question 7: There is no need to assume completeness of the space. Let X be a normed space and let M be a closed subspace of X. It is enough to show that $span\{M, x\}$ is a closed subspace of X for each $x \in X \setminus M$, then by induction on $N \geq 1$ it will follow that $span\{M, \{x_k\}_{k=1}^{N}\}$ is closed for any finite set $\{x_k\}_{k=1}^{N} \subset X$. Let $x \in X \setminus M$ be given, and set

$$d = \inf\{\|x - y\| \,:\, y \in M\}\,.$$

Since M is closed we have $d > 0$. Let $\{z_k\}_{k=1}^{\infty} \subset span\{M, x\}$ and $z \in X$ be with $z_k \xrightarrow{k} z$. For $k \geq 1$ there exist $y_k \in M$ and $\alpha_k \in \mathbb{C}$ with $z_k = y_k + \alpha_k x$. If $\alpha_k \neq 0$ then

$$\|z_k\| = \|y_k + \alpha_k x\| = |\alpha_k| \left\|\alpha_k^{-1} y_k + x\right\| \geq d|\alpha_k|,$$

and so $|\alpha_k| \leq \frac{\|z_k\|}{d}$. It follows that $\{\alpha_k\}_{k=1}^{\infty}$ is a bounded sequence, hence there exists $\alpha \in \mathbb{C}$ with $\alpha_k \xrightarrow{k} \alpha$. From this we get

$$\lim_k y_k = \lim_k (z_k - \alpha_k x) = z - \alpha x,$$

so $z - \alpha x \in M$ since M is closed, and so

$$z = (z - \alpha x) + \alpha x \in span\{M, x\} .$$

This shows that $span\{M, x\}$ is closed and completes the proof.