ANALYSIS 1 - SOLUTION FOR EXERCISE 3

Question 1, part (a): Assume A is invertible, then f is invertible with $f^{-1}(x) = A^{-1}x - A^{-1}b$ for $x \in \mathbb{R}^n$. This shows that f is a homeomorphism and in particular an open map. Assume A is not invertible, then there exists $y \in \mathbb{R}^n$ with $Ax \neq y$ for all $x \in \mathbb{R}^n$. Since $A(\mathbb{R}^n)$ is a subspace for each $\epsilon > 0$ we have $\epsilon y \notin A(\mathbb{R}^n)$, hence $A(\mathbb{R}^n)$ doesn't contain a ball around 0. This shows that A is not an open map, and so f is not open.

Part (b):(i) Let $(x_1, ..., x_n) = x \in \mathbb{R}^n$ be with $\|x\|_\infty \leq 1$, then $|x_j| \leq 1$ for $1 \leq j \leq n$, so

$$\|Ax\|_\infty = \max_{1\leq i\leq n} |\sum_{j=1}^n a_{i,j} \cdot x_j| \leq \max_{1\leq i\leq n} \sum_{j=1}^n |a_{i,j}| =: \alpha,$$

and so $\|A\|_{\infty,\infty} \leq \alpha$. Let $1 \leq i \leq n$ and $x \in \mathbb{R}^n$ be with

$$x_j = \begin{cases} 1 & , \text{ if } a_{i,j} \geq 0 \\ -1 & , \text{ if } a_{i,j} < 0 \end{cases} \quad \text{for } 1 \leq j \leq n,$$

then

$$\sum_{1\leq j\leq n} |a_{i,j}| = \sum_{j=1}^n a_{i,j} \cdot x_j \leq \|Ax\|_\infty \leq \|A\|_{\infty,\infty} .$$

This shows that $\|A\|_{\infty,\infty} \geq \alpha$, and so $\|A\|_{\infty,\infty} = \alpha$.

(ii) Let $x \in \mathbb{R}^n$ be with $\|x\|_1 \leq 1$, then $\sum_{j=1}^n |x_j| \leq 1$, so

$$\|Ax\|_1 = \sum_{1=1}^n |\sum_{j=1}^n a_{i,j} \cdot x_j| \leq \sum_{j=1}^n |x_j| \sum_{i=1}^n |a_{i,j}| \leq \max_{1\leq j\leq n} \sum_{i=1}^n |a_{i,j}| =: \beta,$$

and so $\|A\|_{1,1} \leq \beta$. Given $1 \leq j \leq n$ let e_j be the standard unit vector of $\mathbb{R}^n$, then since $\|e_j\|_1 = 1$

$$\sum_{i=1}^n |a_{i,j}| = \|Ae_j\|_1 \leq \|A\|_{1,1} .$$

This shows that $\|A\|_{1,1} \geq \beta$, and so $\|A\|_{1,1} = \beta$.

(iii) Let $x \in \mathbb{R}^n$ be with $\|x\|_1 \leq 1$, then $\sum_{j=1}^n |x_j| \leq 1$, so

$$\|Ax\|_\infty = \max_{1\leq i\leq n} |\sum_{j=1}^n a_{i,j} \cdot x_j| \leq \max_{1\leq i\leq n} \max_{1\leq j\leq n} |a_{i,j}| =: \gamma,$$

and so $\|A\|_{1,\infty} \leq \gamma$. Given $1 \leq i, j \leq n$ we have

$$|a_{i,j}| \leq \|Ae_j\|_\infty \leq \|A\|_{1,\infty} ,$$

so $\|A\|_{1,\infty} \geq \gamma$, and so $\|A\|_{1,\infty} = \gamma$.

Question 2: Without loss of generality assume $f \geq 0$, otherwise replace f by $|f|$. Assume by contradiction that $f(x_0, y_0) > 0$ for some $(x_0, y_0) \in \mathbb{R}^2$. Since $y \to f(x_0, y)$ is continuous, there exist a closed bounded non-degenerate interval $y_0 \in I \subset \mathbb{R}$ and $\epsilon > 0$ with $f(x_0, y) > \epsilon$ for $y \in I$. For $n \geq 1$ set

$$A_n = \{y \in I \, : \, f(x, y) \geq \frac{\epsilon}{2} \text{ for all } x \in [x_0 - n^{-1}, x_0 + n^{-1}]\},$$

then $I = \cup_{n=1}^{\infty} A_n$ because $f(x_0, y) > \epsilon$ and $x \to f(x, y)$ is continuous for $y \in I$. Since the map $y \to f(x, y)$ is continuous for every $x \in \mathbb{R}$, the set A_n is closed for all $n \geq 1$. From this, from Bair's theorem, and since I is a complete metric space, it follows that there exist $n \geq 1$ and an open interval I_0 with $I_0 \subset A_n$. This shows that

$$f(x, y) \geq \frac{\epsilon}{2} \text{ for all } (x, y) \in [x_0 - n^{-1}, x_0 + n^{-1}] \times I_0,$$

which is a contradiction to the assumption that f vanishes on a dense subset of $\mathbb{R}^2$. Hence we must have $f = 0$ on $\mathbb{R}^2$.

Question 4: Assume Z is open in X, then it is a neighbourhood of 0 in X. Let $x \in X$, then since $0 \cdot x = 0$ and since the map that takes $t \in \mathbb{R}$ to tx is continuous, there exists an open interval $0 \in I \subset \mathbb{R}$ with $tx \in Z$ for $t \in I$. Let $t \in I \setminus \{0\}$, then $x \in t^{-1} Z = Z$, and so $Z = X$.

Question 5: For $n > m \geq 0$ we have

$$\left\| \sum_{k=0}^{n} \frac{T^k}{k!} - \sum_{k=0}^{m} \frac{T^k}{k!} \right\| \leq \sum_{k=m+1}^{n} \frac{\|T\|^k}{k!} \, .$$

Since $\sum_{k=1}^{\infty} \frac{\|T\|^k}{k!} < \infty$ it follows that that $\{\sum_{k=0}^{n} \frac{T^k}{k!}\}_{n=1}^{\infty}$ is a Cauchy sequence in $B(X)$, and so the limit $\sum_{k=0}^{\infty} \frac{T^k}{k!}$ exists in $B(X)$ (since $B(X)$ is complete). This shows that $\exp(T)$ is well defined, and that it is a bounded linear operator on X.

Question 6, part (a): Given a compact $K \subset (0,1)$ it clearly holds that p_K and p'_K are seminorms. Given $f \in C^1(0,1)$ with $f \neq 0$, there exist a non-degenerate closed interval $I \subset (0,1)$ and $\epsilon > 0$ with $|f(x)| > \epsilon$ for $x \in I$. It follows that $p_I(f) > 0$, so the family $\{p_K, p'_K \, : \, K \subset (0,1) \text{ compact}\}$ form a separating family of seminorms on $C^1(0,1)$, and so it makes $C^1(0,1)$ into a locally convex space.

Part (b): Let $\{K_n\}_{n=1}^{\infty}$ be compact subsets of $(0,1)$ with $\cup_n K_n = (0,1)$ and $K_n \subset \text{int } K_{n+1}$ for $n \geq 1$. Given a compact $H \subset (0,1)$ we have $H \subset \cup_n \text{int } K_n$, hence there exists $n \geq 1$ with $H \subset K_n$, and so $p_H \leq p_{K_n}$ and $p'_H \leq p'_{K_n}$. This shows that the family $\{p_{K_n}, p'_{K_n}\}_{n=1}^{\infty}$ also induces the topology of $C^1(0,1)$, and so $C^1(0,1)$ is metrizable.

Part (c): Clearly p is a seminorm on $C_0^1(0,1)$. Let $f \in C_0^1(0,1)$ be with $p(f) = 0$, then $f'(x) = 0$ for all $x \in (0,1)$, and so f is constant on $(0,1)$. Since f is compactly supported it follows that $f = 0$, hence p separates points, and so it makes $C_0^1(0,1)$ into a normed space.

Part (d): We shall show that σ is strictly stronger than τ. Let $K \subset (0,1)$ be compact and let $f \in C_0^1(0,1)$, then clearly $p'_K(f) \leq p(f)$. Let $x \in (0,1)$ be with $|f(x)| = \max_{y\in(0,1)} |f(y)|$, then

$$p_K(f) \leq |f(x)| = |\int_0^x f'(y)\, dy| \leq |\int_0^x |f'(y)|\, dy \leq p(f)\,.$$

This shows that $id : (C_0^1(0,1),\sigma) \to (C_0^1(0,1),\tau)$ is continuous, and so $\sigma \geq \tau$. Assume by contradiction that the set

$$B = \{f \in C_0^1(0,1)\ :\ p(f) < 1\}$$

belongs to τ, then there exist a compact $K \subset (0,1)$ and $\epsilon > 0$ with

$$\{f \in C_0^1(0,1)\ :\ p_K(f) < \epsilon \text{ and } p'_K(f) < \epsilon\} \subset B\,.$$

Let $f \in C_0^1(0,1)$ be such that $f = 0$ on an open neighbourhood of K but $f \neq 0$, then $p(f) > 0$ since p is a norm. For each $n \geq 1$ we have $nf = 0$ on an open neighbourhood of K, hence $p_K(nf) = p'_K(nf) = 0$, and so $p(nf) < 1$. But $\lim_n p(nf) = p(f) \cdot \lim_n n = \infty$, hence we arrived at a contradiction, and so $B \notin \tau$. Since $B \in \sigma$ it follows that $\sigma \gneqq \tau$ as we wanted.

Question 7: If $\Psi \subset \tilde{\Psi}$ then $\sigma \subset \tilde{\sigma}$. This is so since for every $\epsilon > 0$ and $p_1, ..., p_n \in \Psi$ we have

$$\{x \in X\ :\ p_i(x) < \epsilon \text{ for } 1 \leq i \leq n\} =: V(p_1, ..., p_n; \epsilon) \in \tilde{\sigma},$$

and since the sets $V(p_1, ..., p_n; \epsilon)$ form a local base at 0 for σ.

This is related to the previous question as follows: Set $\Psi = \{p_K, p'_K\ :\ K \subset (0,1) \text{ compact}\}$ and $\tilde{\Psi} = \Psi \cup \{p\}$, where p_K, p'_K and p are as defined in the previous question. If σ is the topology (on $C_0^1(0,1)$) induced by Ψ and $\tilde{\sigma}$ is the topology induced by $\tilde{\Psi}$ then $\sigma \subset \tilde{\sigma}$. Note σ is the topology given in **6a** and $\tilde{\sigma}$ is the one given in **6c**.