ANALYSIS 1 - SOLUTION FOR EXERCISE 2

Question 1: Let τ be the topology of X. Assume there exist seminorms $\{p_n\}_{n=1}^{\infty}$ which generate τ. We shall show that X is metrizable, this was done in class but we repeat it here for completeness. For $x, y \in X$ set

$$d(x,y) = \max_{n \geq 1} \frac{n^{-1} \cdot p_n(x-y)}{1 + p_n(x-y)},$$

then d is a metric on X. Denote by τ_d the topology induced by d. Given $V \in \tau$ and $x \in V$ there exist $N \geq 1$ and $0 < \epsilon < 1$ with

$$\{y \in X \,:\, p_n(y-x) < \epsilon \text{ for all } 1 \leq n \leq N\} \subset V\,.$$

Given $y \in X$ with $d(x,y) < \frac{\epsilon}{2N}$ and $1 \leq n \leq N$

$$\frac{p_n(y-x)}{1 + p_n(y-x)} \leq n \cdot d(x,y) < \frac{\epsilon}{2},$$

hence $p_n(y-x) < \epsilon$, and so

$$\{y \in X \,:\, d(x,y) < \frac{\epsilon}{2N}\} \subset V\,.$$

This shows that $V \in \tau_d$ and so $\tau \subset \tau_d$. Given $V \in \tau_d$ and $x \in V$ there exits $\epsilon > 0$ with

$$\{y \in X \,:\, d(x,y) < \epsilon\} \subset V\,.$$

Let $N \geq \frac{1}{\epsilon}$ be an integer and let $y \in X$ be such that $p_n(y-x) < \epsilon$ for $1 \leq n \leq N$, then $d(x,y) < \epsilon$, so

$$\{y \in X \,:\, p_n(y-x) < \epsilon \text{ for all } 1 \leq n \leq N\} \subset V,$$

and so $V \in \tau$. This shows that $\tau_d \subset \tau$, so $\tau = \tau_d$, and so X is metrizable.

Assume X is metrizable, and let d be a metric on X which induces its topology. Since X is locally convex there exist open convex balanced subsets $\{V_n\}_{n=1}^{\infty}$ of X with

$$0 \in V_n \subset B(0, \frac{1}{n}) := \{x \in X \,:\, d(x,0) < \frac{1}{n}\}\,.$$

Since $\{B(0, \frac{1}{n})\}_{n=1}^{\infty}$ is a local base for 0 in X, $\{V_n\}_{n=1}^{\infty}$ is also a local base for 0. For $n \geq 1$ let p_n be the Minkowski functional of V_n. Since V_n is open, convex, and balanced, p_n is a seminorm and

$$V_n = \{x \in X \,:\, p_n(x) < 1\}\,. \tag{0.1}$$

Given $x \in X \setminus \{0\}$ there exits $n \geq 1$ with $x \notin V_n$, hence $p_n(x) \neq 0$, and so $\{p_n\}_{n=1}^{\infty}$ is a separating family of seminorms. Let τ_0 be the topology generated by $\{p_n\}_{n=1}^{\infty}$. From (0.1) it follows that $\{V_n\}_{n=1}^{\infty} \subset \tau_0$, and so $\tau \subset \tau_0$ since $\{V_n\}_{n=1}^{\infty}$ is a local base at 0 for (X, τ). Given $n \geq 1$ and $\epsilon > 0$ it follows from (0.1) that

$$\{x \in X \,:\, p_n(x) < \epsilon\} = \epsilon \cdot V_n \in \tau,$$

hence $\tau_0 \subset \tau$, and so $\tau_0 = \tau$. This shows that the seminorms $\{p_n\}_{n=1}^{\infty}$ generate τ, and completes the proof.

Question 2, part (a): Given $f : X \times X \to \mathbb{R}$ we say that f is continuous if for every $(x_1, x_2) \in X \times X$ and $\epsilon > 0$ there exist neighbourhoods V_1 and V_2 of 0 in X with

$$|f(y_1, y_2) - f(x_1, x_2)| < \epsilon \text{ for every } y_1 \in x_1 + V_1 \text{ and } y_2 \in x_2 + V_2 \,.$$

Part (b): Let $x_1, x_2 \in X$ and $\epsilon > 0$ be given. Let $n \geq 1$ be such that $2^{-n} < \frac{\epsilon}{2}$, and let $y_1, y_2 \in X$ be with

$$p_j(y_i - x_i) < \frac{\epsilon}{4} \text{ for every } i \in \{1, 2\} \text{ and } 1 \leq j \leq n \,.$$

Since

$$|\frac{a}{1+a} - \frac{b}{a+b}| \leq |a - b| \text{ for all } a, b \geq 0,$$

it follows that

$$|d(y_1, y_2) + d(x_1, x_2)| \leq \sum_{j=1}^{n} 2^{-j} \cdot |p(y_1 - y_2) - p(x_1 - x_2)| + \frac{\epsilon}{2}$$

$$\leq \sum_{j=1}^{n} 2^{-j} \cdot (|p(y_1 - y_2) - p(x_1 - y_2)| + |p(x_1 - y_2) - p(x_1 - x_2)|) + \frac{\epsilon}{2}$$

$$\sum_{j=1}^{n} 2^{-j} \cdot (p(y_1 - x_1) + p(y_2 - x_2)) + \frac{\epsilon}{2} < \epsilon,$$

and so $d : X \times X \to \mathbb{R}$ is continuous.

Part (c): Given $x, y \in X$ we have $e(x, y) = 0$ if and only if $p_n(x - y) = 0$ for all $n \geq 1$, which holds if and only if $x = y$. Since e satisfies the triangle inequality and it is symmetric, it follows that e is a metric. Given $\{x_j\}_{j=1}^{\infty} \subset X$ and $x \in X$ it holds that $d(x, x_j) \xrightarrow{j} 0$ if and only if $p_n(x - x_j) \xrightarrow{j} 0$ for all $n \geq 1$, which holds if and only if $e(x, x_j) \xrightarrow{j} 0$. This shows that d and e induce the same topologies.

Question 3: Clearly l^∞ is a vector space, c is a subspace of l^∞, c_0 is a subspace of c, and $\|\cdot\|$ is a norm on l^∞. Let $\{x^m\}_{m=1}^{\infty} \subset l^\infty$ be a Cauchy sequence. For each $n, m, k \geq 1$ we have $|x_n^m - x_n^k| \leq \|x^m - x^k\|$, hence $\{x_n^m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$ for each $n \geq 1$, and so there exists $x_n \in \mathbb{C}$ with $x_n^m \xrightarrow{m} x_n$. There exits $k \geq 1$ with $\|x^m - x^k\| \leq 1$ for all $m \geq k$, hence $|x_n^m| \leq \|x^k\| + 1$ for all $n \geq 1$ and $m \geq k$, and so $|x_n| \leq \|x^k\| + 1$ for $n \geq 1$. This shows that $\{x_n\}_{n=1}^\infty = x \in l^\infty$. Let $\epsilon > 0$, then there exits $M \geq 1$ with $\|x^m - x^k\| < \epsilon$ for $m, k \geq M$. For $n \geq 1$ we have $x_n^m \overset{m}{\to} x_n$, hence for $k \geq M$

$$|x_n - x_n^k| \leq \limsup_m |x_n - x_n^m| + \limsup_m |x_n^m - x_n^k| \leq \|x^m - x^k\| < \epsilon,$$

and so $\|x - x^k\| < \epsilon$. This shows that $\{x^m\}_{m=1}^\infty$ converges to x in l^∞, hence l^∞ is complete, and so it is a Banach space.

Let $\{x^m\}_{m=1}^\infty \subset c$ be a Cauchy sequence, then there exists $x \in l^\infty$ with $x^m \overset{m}{\to} x$ in l^∞. For each $m \geq 1$ there exits $y_m \in \mathbb{C}$ with $\lim_n x_n^m = y_m$. Given $m, k \geq 1$ we have

$$|y_m - y_k| = \lim_n |x_n^m - x_n^k| \leq \|x^m - x^k\|,$$

so $\{y_m\}_{m=1}^\infty$ is a Cauchy sequence, and so there exists $y \in \mathbb{C}$ with $y = \lim_m y_m$. For each $m \geq 1$

$$\limsup_n |y - x_n| \leq |y - y_m| + \limsup_n |y_m - x_n^m| + \limsup_n |x_n^m - x_n| \leq |y - y_m| + \|x^m - x\|,$$

hence

$$\limsup_n |y - x_n| \leq \limsup_m (|y - y_m| + \|x^m - x\|) = 0.$$

This shows that $\lim_n x_n = y$, so $x \in c$, and so c is a closed subspace of l^∞. This also shows that c is complete, and so it is a Banach space.

Let $\{x^m\}_{m=1}^\infty \subset c_0$ be a Cauchy sequence, then there exists $x \in c$ with $x^m \overset{m}{\to} x$ in l^∞. For each $m \geq 1$ we have $\lim_n x_n^m = 0$, hence

$$\limsup_n |x_n| \leq \limsup_n |x_n - x_n^m| + \limsup_n |x_n^m| \leq \|x^m - x\|,$$

and so

$$\limsup_n |x_n| \leq \limsup_m \|x^m - x\| = 0.$$

This shows that $\lim_n x_n = 0$, so $x \in c_0$, and so c_0 is a closed subspace of c. This also shows that c_0 is complete, and so it is a Banach space.

Remark. The fact that c_0 is a closed subspace of c (and so a Banach space) also follows from the following argument: Given $x \in c$ set $f(x) = \lim_n x_n$, then f is a continuous linear functional on c. Now since $c_0 = \ker f$ it follows that c_0 is a closed subspace of c.

Question 4: Let $x, y \in \overline{Y}$ and $\alpha, \beta \in \mathbb{C}$ be given, and let $V \subset X$ be a neighbourhood of 0 in X. Define $f : X \times X \to X$ by

$$f(z, w) = \alpha z + \beta w \text{ for } z, w \in X.$$

Since f is continuous and $f(0,0) = 0$, there exists $W \subset X$ a neighbourhood of 0 with $f(W \times W) \subset V$. Since $x, y \in \overline{Y}$ there exist $x', y' \in Y$ with $x' \in x + W$ and

$y' \in y + W$. Since $\alpha x' + \beta y' \in Y$ and

$$\alpha x' + \beta y' \in \alpha x + \beta y + \alpha W + \beta W = \alpha x + \beta y + f(W \times W) \subset \alpha x + \beta y + V,$$

it follows that $Y \cap (\alpha x + \beta y + V) \neq \emptyset$. This holds for every V which is a neighbourhood of 0, hence $\alpha x + \beta y \in \overline{Y}$, which shows that $\overline{Y}$ is a linear subspace of X.

Question 5, part (a): The space $C(0,1)$ is not normable. Assume by contradiction that there exists a norm $\|\cdot\|$ which induces the topology of $C(0,1)$, then there exist a compact $K \subset (0,1)$ and $\epsilon > 0$ with

$$\{f \in C(0,1) \,:\, p_K(f) < \epsilon\} \subset \{f \in C(0,1) \,:\, \|f\| < 1\}\,.$$

Let $g \in C(0,1)$ be such that $g = 0$ on K but $g(x) \neq 0$ for some $x \in (0,1) \setminus K$. Since $g \neq 0$ we have $\|g\| > 0$, hence $\lim_n \|ng\| = \infty$. But $ng = 0$ on K for all $n \geq 1$, hence $p_K(ng) < \epsilon$, and so $\|ng\| < 1$ for all $n \geq 1$. This contradiction shows that $C(0,1)$ is not normable.

Part (b): $C_0(0,1)$ is not a closed subspace of $C(0,1)$, and $\overline{C_0(0,1)} = C(0,1)$. Fix some $g \in C(0,1)$. For $n \geq 3$ let $f_n \in C_0(0,1)$ be with $f_n = 1$ on $[\frac{1}{n}, 1 - \frac{1}{n}]$. Given a compact $K \subset (0,1)$ there exists $N \geq 3$ with $K \subset [\frac{1}{N}, 1 - \frac{1}{N}]$. For all $n \geq N$ we have $p_K(g - g \cdot f_n) = 0$, and so $g \cdot f_n \xrightarrow{n} g$ in $C(0,1)$. Since $\{g \cdot f_n\}_{n=1}^{\infty} \subset C_0(0,1)$ it follows that $C_0(0,1)$ is dense in $C(0,1)$, and it is not closed since $C(0,1) \setminus C_0(0,1) \neq \emptyset$.

Part (c): It is clear that W is convex and balanced. Let $f \in C_0(0,1)$, then there exists $0 < C < \infty$ with $|f(x)| \leq C$ for $x \in (0,1)$. Given $0 < \delta < (C \cdot \sum_{n=2}^{\infty} \frac{1}{n^2})^{-1}$ we have

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \cdot |\delta \cdot f(\frac{1}{n})| \leq C \cdot \delta \cdot \sum_{n=2}^{\infty} \frac{1}{n^2} < 1,$$

hence $\delta \cdot f \in W$, so $f \in \delta^{-1} W$, and so W is absorbing in $C_0(0,1)$.

Assume by contradiction that $0 \in \operatorname{int} W$ (where $\operatorname{int} W$ is taken in $C_0(0,1)$), then there exist a compact $K \subset (0,1)$ and $\epsilon > 0$ with

$$\{f \in C_0(0,1) \,:\, p_K(f) < \epsilon\} \subset W\,.$$

Let $m \geq 1$ be such that $\frac{1}{m} \notin K$, then there exists $f \in C_0(0,1)$ with $f = 0$ on K and $f(\frac{1}{m}) \geq m^2$. Since $p_K(f) = 0 < \epsilon$ we have $f \in W$, which contradicts

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \cdot |f(\frac{1}{n})| \geq \frac{1}{m^2} \cdot f(\frac{1}{m}) \geq 1\,.$$

This shows that $0 \notin \operatorname{int} W$, and so W is not open in $C_0(0,1)$ since $0 \in W$.

Part (d): Let p be the Minkowski functional of W, then part **c** implies that p is not continuous on $C_0(0,1)$. Assume by contradiction that p is continuous, then

$$\{f \in C_0(0,1) \ : \ p(f) < 1\}$$

is an open set which contains 0 and is contained in W. This shows that $0 \in \operatorname{int} W$ which contradicts part **c**, and so p is not continuous.

Part (e): Let $Q \subset C(0,1)$ be subspace of all polynomials, then Q is dense in $C(0,1)$. Let $V \subset C(0,1)$ be open and let $f \in V$. There exist a compact $K \subset (0,1)$ and $\epsilon > 0$ with

$$\{g \in C(0,1) \ : \ p_K(g-f) < \epsilon\} \subset V \, .$$

From the Weierstrass approximation theorem it follows that there exists $q \in Q$ with $p_K(q-f) < \epsilon$, hence $q \in V$, which shows that Q is dense in $C(0,1)$.

Question 6, part (a): Let $K_1, K_2, ... \subset \Omega$ be compact sets with $\cup_j K_j = \Omega$ and $K_j \subset \operatorname{int} K_{j+1}$ for $j \geq 1$. The family

$$\{p_{K_j,\alpha} \ : \ j \geq 1, \ |\alpha| \leq k\}$$

is countable and induces the topology of $C^k(\Omega)$, hence $C^k(\Omega)$ is metrizable. In order to show that $C^k(\Omega)$ is a Frechet space we need to show that it is complete. Let $\{f_j\} \subset C^k(\Omega)$ be a Cauchy sequence. For each $l \geq 1$ and $|\alpha| \leq k$ the sequence $\{\partial^\alpha f_j\}$ is Cauchy in $C(K_l)$, where $C(K_l)$ is endowed we the supremum norm. Since $C(K_l)$ is complete there exits $g_{\alpha,l} \in C(K_l)$ with $\partial^\alpha f_j \xrightarrow{j} g_{\alpha,l}$ in $C(K_l)$. Given $|\alpha| \leq k$ define $g_\alpha : \Omega \to \mathbb{C}$ by $g_\alpha = g_{\alpha,l}$ on K_l, clearly g_α is well defined and continuous. Set $g = g_0$, we shall show that $g \in C^k(\Omega)$ and $\partial^\alpha g = g_\alpha$ for every multi-index $|\alpha| \leq k$. Let $|\alpha| < k$ and $1 \leq i \leq n$ and set $\beta = \alpha + e_i$, where e_i is the i'th standard unit vector of $\mathbb{R}^n$. Let $x \in \Omega$ and $\epsilon > 0$ be given. Let $1 > \delta_0 > 0$ be such that $\overline{B(x,\delta_0)} =: B \subset \Omega$ and $|g_\beta(y) - g_\beta(x)| < \epsilon$ for $y \in B$. Given $0 < \delta < \delta_0$ there exists $j \geq 1$ with

$$|\partial^\gamma f_j(x) - g_\gamma(x)| \leq \delta\epsilon \text{ for } \gamma \in \{\alpha, \beta\} \text{ and } x \in B \, .$$

From the mean value theorem we get

$$\begin{aligned}
&\left| \frac{g_\alpha(x+\delta e_i) - g_\alpha(x)}{\delta} - g_\beta(x) \right| \\
&\qquad \leq 2\epsilon + \left| \frac{\partial^\alpha f_j(x+\delta e_i) - \partial^\alpha f_j(x)}{\delta} - g_\beta(x) \right| \\
&\qquad \leq 2\epsilon + \sup_{\eta \in [0,\delta]} \left| \partial^\beta f_j(x+\eta e_i) - g_\beta(x) \right| \\
&\qquad \leq 3\epsilon + \sup_{\eta \in [0,\delta]} |g_\beta(x+\eta e_i) - g_\beta(x)| < 4\epsilon,
\end{aligned}$$

which shows that $\partial^{e_i} g_\alpha = g_\beta$. Now by induction on $|\alpha|$ we obtain $\partial^\alpha g = g_\alpha$ for every multi-index α with $|\alpha| \leq k$, which also shows $g \in C^k(\Omega)$. Since $\partial^\alpha f_j \xrightarrow{j} \partial^\alpha g$ in $C(K_l)$ for $l \geq 1$ and $|\alpha| \leq k$, we get $f_j \xrightarrow{j} g$ in $C^k(\Omega)$. This shows that $C^k(\Omega)$ is complete, and finishes the proof.

Part (b): Let β be a multi-index with $|\beta| \leq k$. Since $\partial^\beta : C^k(\Omega) \to C^{k-|\beta|}(\Omega)$ is linear it is enough to show that it is continuous at 0. Let $V \subset C^{k-|\beta|}(\Omega)$ be a neighbourhood of 0, then there exist a compact $K \subset \Omega$ and $\epsilon > 0$ with

$$\{f \in C^{k-|\beta|}(\Omega) \,:\, |\partial^\alpha f(x)| < \epsilon \text{ for } x \in K \text{ and } |\alpha| \leq k - |\beta|\} =: W \subset V\,.$$

Set

$$U = \{f \in C^k(\Omega) \,:\, |\partial^\alpha f(x)| < \epsilon \text{ for } x \in K \text{ and } |\alpha| \leq k\},$$

then U is a neighbourhood of 0 in $C^k(\Omega)$ and $\partial^\beta(U) \subset W \subset V$. This shows that ∂^β is continuous at 0.

Part (c): Let $K \subset \Omega$ be compact and α be a multi-index with $|\alpha| \leq k$. It suffice to show that there exists a continuous seminorm q on $C^k(\Omega)$ with $p_{K.\alpha}(T_\varphi f) \leq q(f)$ for $f \in C^k(\Omega)$. From the Leibniz formula there exist positive integers $\{c_{\alpha,\beta}\}_{\beta \leq \alpha}$ with

$$\partial^\alpha(f \cdot g) = \sum_{\beta \leq \alpha} c_{\alpha,\beta} \cdot \partial^{\alpha-\beta} f \cdot \partial^\beta g \;\text{ for all } f, g \in C^k(\Omega)\,.$$

Hence for $f \in C^k(\Omega)$ and $x \in K$

$$|\partial^\alpha(T_\varphi f)(x)| = |\partial^\alpha(\varphi f)(x)| \leq \sum_{\beta \leq \alpha} c_{\alpha,\beta} \cdot |\partial^{\alpha-\beta}\varphi(x)| \cdot |\partial^\beta f(x)|,$$

and so

$$p_{K.\alpha}(T_\varphi f) \leq \sum_{\beta \leq \alpha} c_{\alpha,\beta} \cdot p_{K,\alpha-\beta}(\varphi) \cdot p_{K,\beta}(f)\,.$$

Since $\sum_{\beta \leq \alpha} c_{\alpha,\beta} \cdot p_{K,\alpha-\beta}(\varphi) \cdot p_{K,\beta}$ is a continuous seminorm on $C^k(\Omega)$ we are done.