FUNDAMENTAL CONCEPTS IN ANALYSIS 1 - EXERCISE 1

Question 1: Assume by contradiction that there exists $x \in \operatorname{int}(\overline{U} \setminus U)$. Since $x \in \overline{U}$ and $\operatorname{int}(\overline{U} \setminus U)$ is an open neighbourhood of x, it follows that there exists $y \in U$ with $y \in \operatorname{int}(\overline{U} \setminus U) \subset \overline{U} \setminus U$. This is clearly a contradiction, hence $\operatorname{int}(\overline{U} \setminus U) = \emptyset$, and so $\overline{U} \setminus U$ is nowhere dense since it is closed.

Since $F \setminus \operatorname{int}(F) \subset F$ it follows that $\operatorname{int}(F \setminus \operatorname{int} F) \subset \operatorname{int} F$. From this and from $\operatorname{int}(F \setminus \operatorname{int} F) \subset F \setminus \operatorname{int} F$ we obtain $\operatorname{int}(F \setminus \operatorname{int} F) = \emptyset$, and so $F \setminus \operatorname{int} F$ is nowhere dense since it is closed.

Question 2a: Let m be the Lebesgue measure on $\mathbb{R}$, let $\epsilon > 0$ be given, let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$, and set

$$F^{\epsilon} = [0,1] \setminus \cup_{n=1}^{\infty} (q_n - \epsilon \cdot 2^{-n-1}, q_n + \epsilon \cdot 2^{-n-1}) \,.$$

Clearly F^{ϵ} is closed, and it is nowhere dense since $F^{\epsilon} \cap \mathbb{Q} = \emptyset$, and so $\operatorname{int} F^{\epsilon} = \emptyset$. In addition we have

$$m(F^{\epsilon}) \geq 1 - \sum_{n=1}^{\infty} m(q_n - \epsilon \cdot 2^{-n-1}, q_n + \epsilon \cdot 2^{-n-1}) = 1 - \epsilon \,.$$

Question 2b: Let $F = \cup_{n=1}^{\infty} F^{1/n}$, then F is of the first category. Also $m(F) \geq m(F^{1/n}) \geq 1 - \frac{1}{n}$ for every $n \geq 1$, so $m(F) = 1$.

Question 3a: Set

$$D_f = \{x \in \mathbb{R} \,:\, f \text{ is not continuous at } x\},$$

and for $n \geq 1$ set

$$E_{f,n} = \{x \in \mathbb{R} \,:\, \limsup f(x) - \liminf f(x) \geq \frac{1}{n}\} \,.$$

Clearly $D = \cup_{n=1}^{\infty} E_{f,n}$, hence it remains to show that the sets $E_{f,n}$ are closed. Let $a, x \in \mathbb{R}$ be such that $\limsup f(x) < a$, then there exists $\epsilon > 0$ with $\limsup f(y) < a$ for $y \in (x - \epsilon, x + \epsilon)$, and so $\limsup f$ is upper semi-continuous. In the same manner $\liminf f$ is lower semi-continuous, and so $\limsup f - \liminf f$ is upper semi-continuous. This shows that the sets $E_{f,n}$ are all closed.

Question 3b: Assume by contradiction that there exists $f : \mathbb{R} \to \mathbb{R}$ with $D_f = \mathbb{R} \setminus \mathbb{Q}$. For each $n \geq 1$ the set $E_{f,n}$ is closed and $E_{f,n} \cap \mathbb{Q} = \emptyset$, hence $E_{f,n}$ is nowhere dense. Now since $\{x\}$ is nowhere dense for $x \in \mathbb{R}$, and since

$$\mathbb{R} = D_f \cup \mathbb{Q} = (\cup_{n=1}^{\infty} E_{f,n}) \cup (\cup_{q \in \mathbb{Q}} \{q\}),$$

it follows that $\mathbb{R}$ is of the first category. Since $\mathbb{R}$ is a complete metric space this contradicts Baire's theorem, and so there is no $f : \mathbb{R} \to \mathbb{R}$ with $D_f = \mathbb{R} \setminus \mathbb{Q}$.

Question 4: Assume by contradiction that there exist open sets $V_1, V_2, ... \subset \mathbb{R}$ with $\cap_{n=1}^{\infty} V_n = \mathbb{Q}$. For $n \geq 1$ set $F_n = \mathbb{R} \setminus V_n$, then since F_n is closed and $F_n \cap \mathbb{Q} = \emptyset$ it follows that F_n is nowhere dense. Now since

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = (\cup_{n=1}^{\infty} F_n) \cup (\cup_{q \in \mathbb{Q}} \{q\})$$

it follows that $\mathbb{R}$ is of the first category, which contradicts Baire's theorem. This proves that $\mathbb{Q}$ is not a countable intersection of open subsets of $\mathbb{R}$.

Question 5: Since int K is non-empty and contains 0, int K is absorbing. Let p be the Minkowski functional corresponding to int K, i.e.

$$p(x) = \inf\{t \in (0, \infty) \ : \ t^{-1} \cdot x \in \text{int } K\} \text{ for } x \in \mathbb{R}^n \ .$$

Since K is convex int K is also convex, hence $p(x + y) \leq p(x) + p(y)$ and $p(t \cdot x) = t \cdot p(x)$ for $x, y \in \mathbb{R}^n$ and $t \in [0, \infty)$ (see Theorem 1.35 in [R]). Let $y \in \partial K$, then $p(y) \geq 1$ since $y \notin \text{int } K$. Set $W = \text{span}\{y\}$ and for $\alpha \in \mathbb{R}$ set $f_0(\alpha \cdot y) = \alpha \cdot p(y)$, then f_0 defines a linear functional on W. If $\alpha < 0$ then

$$f_0(\alpha \cdot y) = \alpha \cdot p(y) < 0 \leq p(\alpha \cdot y),$$

and if $\alpha \geq 0$ then

$$f_0(\alpha \cdot y) = \alpha \cdot p(y) = p(\alpha \cdot y) \ .$$

From this and from the Hahn-Banach theorem it follows that there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = f_0(x)$ for $x \in W$ and $f(x) \leq p(x)$ for $x \in \mathbb{R}^n$. Set

$$H = \{x \in \mathbb{R}^n \ : \ f(x) \leq p(y)\} \ \text{ and } \ \partial H = \{x \in \mathbb{R}^n \ : \ f(x) = p(y)\},$$

then H is a closed half-space and ∂H is a hyperplane. Since $f(y) = f_0(y) = p(y)$, we have $y \in \partial H$. For $x \in \text{int } K$

$$f(x) \leq p(x) \leq 1 \leq p(y),$$

so $f(x) \leq p(y)$ for $x \in K$ since f is continuous. This shows that $K \subset H$ and completes the proof.

Question 6: The function that takes $x \in \mathbb{R}$ to $\{x\}$ is continuous, hence for each $n \geq 0$ the function that takes $x \in \mathbb{R}$ to $s_n(x) = \sum_{j=0}^{n} \frac{\{10^j x\}}{10^j}$ is continuous. From the Weierstrass M-test it follows that the sequence $\{s_n\}_{j=1}^{\infty}$ converges to f uniformly, which implies that f is also continuous.

We shall now show that f is nowhere differentiable. Fix $a \in I := [0, 1)$, then since f is 1-periodic it suffice to show that f is not differentiable at a. Define $\sigma : I \to I$

by

$$\sigma(x) = 10 \cdot x \mod 1 \text{ for } x \in I,$$

set $\sigma^1 = \sigma$, and for each $j \geq 2$ set $\sigma^j = \sigma \circ \sigma^{j-1}$. There exists a sequence $\{a_n\}_{n=1}^{\infty}$, with $a_n \in \{0, ..., 9\}$ for $n \geq 1$, such that $a = \sum_{n=1}^{\infty} a_n \cdot 10^{-n}$. For $n \geq 1$ set

$$\eta_n = \begin{cases} -1 & \text{, if } a_n = 4 \text{ or } 9 \\ 1 & \text{, otherwise} \end{cases}$$

and $h_n = 10^{-n} \cdot \eta_n$, then

$$\frac{f(a+h_n) - f(a)}{h_n} = \eta_n \cdot \sum_{j=0}^{n-1} 10^{n-j} \cdot (\{\sigma^j(a+h_n)\} - \{\sigma^j(a)\}) \,. \tag{0.1}$$

Note that for $x \in [0,1)$

$$\{x\} = \begin{cases} x & \text{, if } x \in [0, \frac{1}{2}) \\ 1-x & \text{, if } x \in [\frac{1}{2}, 1) \end{cases},$$

and that from the way h_n is defined it follows that for $0 \leq j < n$

$$1_{[0,\frac{1}{2})}(\sigma^j(a)) = 1_{[0,\frac{1}{2})}(\sigma^j(a+h_n)) \,.$$

From this and from (0.1) we obtain

$$\frac{f(a+h_n) - f(a)}{h_n} =$$

$$= \eta_n \cdot \sum_{j=0}^{n-1} 10^{n-j} \cdot (1_{[0,\frac{1}{2})}(\sigma^j(a)) - 1_{[\frac{1}{2},1)}(\sigma^j(a))) \cdot (\sigma^j(a+h_n) - \sigma^j(a)) \,.$$

Since

$$\left|10^{n-j} \cdot (\sigma^j(a+h_n) - \sigma^j(a))\right| = 1 \text{ for } 0 \leq j < n,$$

it follows that $\frac{f(a+h_n)-f(a)}{h_n}$ is an integer which is odd if and only if n is odd. This shows that the limit $\lim\limits_{n\to\infty} \frac{f(a+h_n)-f(a)}{h_n}$ does not exist, and in particular f is not differentiable at a.

References

[R] W.Rudin, Functional analysis, second edition.