## ANALYSIS 1 - SOLUTION FOR EXERCISE 12

***Question 1:*** Given $\varphi \in \mathcal{S}(\mathbb{R})$,

$$i\xi\widehat{T_H}(\varphi) = \widehat{\frac{d}{dx}T_H}(\varphi) = \frac{d}{dx}T_H(\hat{\varphi}) = -T_H(\frac{d}{dx}\hat{\varphi})$$
$$= -\int_0^\infty \frac{d}{dx}\hat{\varphi}\,dx = \hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}}\int \varphi(x)\,dx = \frac{1}{\sqrt{2\pi}}T_1(\varphi),$$

which shows $i\xi\widehat{T_H} = \frac{1}{\sqrt{2\pi}}T_1$.

***Question 2, part (a):*** Since

$$\int_0^{2\pi}\sum_{j=-\infty}^{\infty}|f(t+2\pi j)|\,dt = \sum_{j=-\infty}^{\infty}\int_0^{2\pi}|f(t+2\pi j)|\,dt$$
$$= \sum_{j=-\infty}^{\infty}\int_{2\pi j}^{2\pi(j+1)}|f(t)|\,dt = \int_{-\infty}^{\infty}|f|\,dt < \infty,$$

it follows $g$ is well defined and $g \in L^1(0,2\pi)$.

***Part (b):*** For $n \in \mathbb{Z}$

$$\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-int}\,dt = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\int_0^{2\pi} f(t+2\pi j)e^{-int}\,dt$$
$$= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\int_{2\pi j}^{2\pi(j+1)} f(t)e^{-in(t-2\pi j)}\,dt\,.$$

From this and since $e^{-in(t-2\pi j)} = e^{-int}$ for every $t \in \mathbb{R}$ and $j \in \mathbb{Z}$,

$$\hat{g}(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-int}\,dt = \frac{1}{\sqrt{2\pi}}\hat{f}(n)\,.$$

***Part (c):*** *Claim:* If $f \in \mathcal{S}(\mathbb{R})$ then the sum

$$g(t) = \sum_{j=-\infty}^{\infty} f(t+2\pi j)$$

converges uniformly (with all derivatives) to a smooth $2\pi$-periodic function.

*Proof:* Note that, for any integer $l$,

$$\sup_{t\in[0,2\pi]}\sum_{|j|>J}|f(t+2\pi j)| \le C_l J^{-l},$$

with a similar estimate for the series of derivatives (of any order). Thus the convergence is uniform and the Claim follows from the general following calculus proposition:

**Proposition:** Let $I \subset \mathbb{R}$ be a finite open interval and $\{\phi_k(t),\ t \in I\}_{k=1}^{\infty} \subset C^{\infty}(I)$ a sequence of smooth functions such that: (a) $\phi_k \xrightarrow{k} \phi$ uniformly on $I$, (b) for every $j \in \mathbb{N}$ the derivatives of order $j$, $\{\phi_k^{(j)}(t),\ t \in I\}_{k=1}^{\infty}$ converge (uniformly in $t \in I$) to a function $\psi_j(t)$. Then $\phi \in C^{\infty}(I)$ and $\psi_j(t) = (\frac{d}{dt})^j \phi$, $j = 1, 2, \ldots$.

The proof of the proposition is immediate. Pick $t_0 \in I$ and note that

$$\phi(t) - \phi(t_0) = \lim_{k\to\infty} (\phi_k(t) - \phi_k(t_0)) = \lim_{k\to\infty} \int_{t_0}^{t} \phi_k'(s)\, ds = \int_{t_0}^{t} \psi_1(s)\, ds,$$

all limits uniform in $t \in I$. It follows that $\phi'(t) \equiv \psi_1(t)$. This can be continued to all orders, namely, $\psi_1'(t) = \psi_2(t)$...

***Part (d):*** Assume $f \in \mathcal{S}(\mathbb{R})$, then from part (c) we get $g \in C^{\infty}(\mathbb{R})$. From this and since $g$ is $2\pi$-periodic it follows $\sum_{n=-\infty}^{\infty} \hat{g}(n) = g(0)$. Now since $\hat{g}(n) = \frac{1}{\sqrt{2\pi}} \hat{f}(n)$ for $n \in \mathbb{Z}$,

$$\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}(n) = g(0) = \sum_{n=-\infty}^{\infty} f(2\pi n),$$

which is what we wanted to prove.

***Question 3, part (a):*** For $x \in \mathbb{R}$ set $P(x) = x^2$, then since $\widehat{\delta_0} = \frac{T_1}{\sqrt{2\pi}}$

$$\widehat{\frac{d^2}{dx^2}\delta_0} = -P \cdot \widehat{\delta_0} = -\frac{P}{\sqrt{2\pi}}\,.$$

It follows that for each $\varphi \in \mathcal{S}(\mathbb{R}^2)$

$$\widehat{\frac{d^2}{dx^2}\delta_0}(\varphi) = -\frac{1}{\sqrt{2\pi}} \int \xi^2 \cdot \varphi(\xi)\, d\xi\,.$$

***Part (b):*** Given $\varphi \in \mathcal{S}(\mathbb{R}^2)$

$$\frac{\partial}{\partial x} T_{\chi}(\varphi) = -T_{\chi}(\frac{\partial \varphi}{\partial x}) = -\int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy = \int_{-1}^{1} \varphi(-1,y) - \varphi(1,y)\, dy,$$

and similarly

$$\frac{\partial}{\partial y} T_{\chi}(\varphi) = \int_{-1}^{1} \varphi(x,-1) - \varphi(x,1)\, dx\,.$$

***Question 4:*** *Lemma:* If $\psi \in \mathcal{S}(\mathbb{R})$ satisfies $\int_{\mathbb{R}} \psi(x) dx = 0$, then $\psi(x) = \phi'(x)$ for some $\phi \in \mathcal{S}(\mathbb{R})$.

*Proof of the Lemma:* Define $\phi(x) = \int_{-\infty}^{x} \psi(t) dt$. Clearly the function is well-defined (because of the decay of $\psi$) and infinitely differentiable. Furthermore, for any integer $l \in \mathbb{N}$, the decay $|\psi(t)| \leq C_l |t|^{-(l+1)}$, $t < -1$, implies by integration

$$|\phi(x)| \leq C_l |x|^{-l}, \quad x < -1.$$

Since $\phi'(x) = \psi(x)$, the rapid decay of derivatives follows from that of $\psi$ (and its derivatives).

Now, to obtain a similar decay of $\phi(x)$ as $x \to \infty$, use the vanishing of the integral (over the line) of $\psi$ to get

$$\phi(x) = -\int_x^\infty \psi(t)dt,$$

and repeat the above argument.

***Part (a):*** Let $\omega \in \mathcal{S}(\mathbb{R})$ be with $\int \omega \, dx = 1$ and set $c = v(\omega)$. Fix some $\psi \in \mathcal{S}(\mathbb{R})$ and set $a = \int \psi \, dx$. Since

$$\int_{-\infty}^{\infty} \psi(y) - a \cdot \omega(y) \, dy = 0,$$

it follows from the lemma above that there exists $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi' = \psi - a \cdot \omega$. It now holds that

$$0 = \frac{d}{dx} v(\varphi) = -v(\frac{d}{dx}\varphi) = -v(\psi - a\omega) = -v(\psi) + c \int \psi \, dx,$$

and so $v(\psi) = c \int \psi \, dx$. This shows $v = T_c$, which is what we wanted to prove.

***Part (b):*** For $k \geq 0$ and $x \in \mathbb{R}$ set $P_k(x) = x^k$. Let $u \in \mathcal{S}'(\mathbb{R})$ be with $\frac{d^2}{dx^2} u = T_{x^2}$, then $\frac{d}{dx}(\frac{d}{dx}u - \frac{P_3}{3}) = 0$, and so from part (a) we get $\frac{d}{dx}u = \frac{P_3}{3} + T_{c_1}$ for some $c_1 \in \mathbb{C}$. In the same manner since $\frac{d}{dx}(u - \frac{P_4}{12} - c_1 P_1) = 0$ it follows $u = \frac{P_4}{12} + c_1 P_1 + T_{c_2}$ for some $c_2 \in \mathbb{C}$. We have thus found all solutions for the equation $\frac{d^2}{dx^2} u = T_{x^2}$.