ANALYSIS 1 - SOLUTION FOR EXERCISE 11

Question 1: Let $x \in \mathbb{R}$, then from the inversion theorem and the definition of $\hat{f}$

$$f(x) = (2\pi)^{-\frac{1}{2}} \lim_{a\to\infty} \int_{-a}^{a} \hat{f}(\xi) e^{ix\xi}\, d\xi = \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} \int_{\mathbb{R}} f(t) e^{i\xi(x-t)}\, dt\, d\xi\,. \tag{0.1}$$

Since for each $a > 0$

$$\int_{-a}^{a} \int_{\mathbb{R}} |f(t) e^{i\xi(x-t)}|\, dt\, d\xi < \infty$$

we can use Fubini's theorem on (0.1), and so

$$f(x) = \frac{1}{2\pi} \lim_{a\to\infty} \int_{\mathbb{R}} f(t) \cdot \int_{-a}^{a} e^{i\xi(x-t)}\, d\xi\, dt = \frac{1}{2\pi} \lim_{a\to\infty} \int_{\mathbb{R}} f(x+t) \cdot \int_{-a}^{a} e^{-i\xi t}\, d\xi\, dt\,.$$

Now since for $a > 0$

$$\int_{-a}^{a} e^{-i\xi t}\, d\xi = \frac{2\sin(at)}{t}$$

it follows that

$$f(x) = \frac{1}{\pi} \lim_{a\to\infty} \int_{\mathbb{R}} f(x+t) \cdot \frac{\sin(at)}{t}\, dt,$$

which is what we wanted.

Question 2, part (a): For every $x \in \mathbb{R}$ we have $\frac{d}{dx}\varphi(x) = -x \cdot e^{-\frac{1}{2}x^2}$, so

$$\frac{d^2}{dx^2}\varphi(x) = -e^{-\frac{1}{2}x^2} + x^2 \cdot e^{-\frac{1}{2}x^2},$$

and so

$$-\frac{d^2}{dx^2}\varphi(x) + x^2\varphi(x) = e^{-\frac{1}{2}x^2} = \varphi(x),$$

which is what we wanted.

Part (b): For $x \in \mathbb{R}$ set $P(x) = x^2$. From claims stated in class it follows $\widehat{P\varphi} = -\frac{d^2}{dx^2}\hat{\varphi}$ and $\widehat{\frac{d^2}{dx^2}\varphi} = -P\hat{\varphi}$. From this and part (a) we get

$$\hat{\varphi} = \mathcal{F}(-\frac{d^2}{dx^2}\varphi + P\varphi) = P\hat{\varphi} - \frac{d^2}{dx^2}\hat{\varphi},$$

i.e. φ also satisfies the differential equation given in part (a).

Part (c): For $x \in \mathbb{R}$ set $Q(x) = x$, then

$$\frac{d}{dx}\hat{\varphi}(0) = -i\widehat{Q\varphi}(0) = \frac{1}{\sqrt{2\pi}i} \int_{\mathbb{R}} x \cdot e^{-\frac{1}{2}x^2}\, dx = 0,$$

where the last equality holds since $x \to x \cdot e^{-\frac{1}{2}x^2}$ is an odd function. Let $c = \hat{\varphi}(0) > 0$, then

$$(c^{-1} \cdot \hat{\varphi}(0), c^{-1} \cdot \frac{d}{dx}\hat{\varphi}(0)) = (1, 0) = (\varphi(0), \frac{d}{dx}\varphi(0)) .$$

From this, since φ and $c^{-1} \cdot \hat{\varphi}$ both satisfy the differential equation given in (a), and since this equation has order 2, it follows $\hat{\varphi} = c\varphi$. In addition we have $c = 1$ since

$$c = \hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\, dx = 1 .$$

Question 3, part (a): For $x \in \mathbb{R}$ set $P(x) = x^2$ and let φ be as in the previous question. From parts (c) and (a) of the last question we obtain

$$\begin{aligned}\int_{\mathbb{R}} x^2 \cdot e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi} \cdot \widehat{P\varphi}(0) = -\sqrt{2\pi} \cdot \frac{d^2\hat{\varphi}}{dx^2}(0) \\ = -\sqrt{2\pi} \cdot \frac{d^2\varphi}{dx^2}(0) = -\sqrt{2\pi} \cdot (P(0)\varphi(0) - \varphi(0)) = \sqrt{2\pi} .\end{aligned}$$

Part (b): Let $a > 0$ and set $b = \sqrt{\frac{a}{2}}$, then

$$\begin{aligned}\int_0^\infty e^{-ax^2} \cos(ax)\, dx = \frac{1}{2} \int_{\mathbb{R}} e^{-ax^2} \frac{e^{iax} + e^{-iax}}{2}\, dx \\ = \frac{\sqrt{2\pi}}{4} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} (e^{ibx} + e^{-ibx}) \cdot (2a)^{-\frac{1}{2}}\, dx = \sqrt{\frac{\pi}{16a}} (\hat{\varphi}(-b) + \hat{\varphi}(b)) \\ = \sqrt{\frac{\pi}{16a}} (\varphi(-b) + \varphi(b)) = \sqrt{\frac{\pi}{4a}} \cdot \exp(-\frac{a}{4}) .\end{aligned}$$

Question 4, part (a): For $x \in \mathbb{R}$ set $Q(x) = x$, then

$$\begin{aligned}\mathcal{F}(\sigma) = \mathcal{F}(Q\psi) - \mathcal{F}(\frac{d\psi}{dx}) = i\frac{d}{dx}\mathcal{F}(\psi) - iQ\mathcal{F}(\psi) \\ = i\lambda \cdot \frac{d}{dx}\psi - i\lambda \cdot Q\psi = -i\lambda\sigma,\end{aligned}$$

which is what we wanted.

Part (b): Since $\mathcal{F}\varphi = \varphi$ we have

$$\mathcal{F}(\theta) = 2\mathcal{F}(Q\varphi) = 2i\frac{d}{dx}\mathcal{F}(\varphi) = 2i\frac{d}{dx}\varphi = -2iQ\varphi = -i\theta .$$

Part (c): We shall prove the claim by induction on k. For $k = 1$ the claim follows from part (b). Let $k \geq 1$ and assume there exists a real polynomial H_k with $\deg H_k = k$ and $\mathcal{F}(H_k\varphi) = (-i)^k H_k\varphi$. Let H_{k+1} be the polynomial with $H_{k+1}\varphi = (Q - \frac{d}{dx})(H_k\varphi)$, then $\deg H_{K+1} = k + 1$ and from part (a)

$$\mathcal{F}(H_{k+1}\varphi) = (-i)^{k+1} H_{k+1}\varphi .$$

This completes the induction and the proof.