ANALYSIS 1 - SOLUTION FOR EXERCISE 10

Question 1: We first show T is well defined. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be given, and set $K = supp(\varphi)$. There exists $N \geq 1$ with $y + ka \notin K$ for every $y \in B(x,1)$ and $k > N$, hence

$$T\varphi(y) = \sum_{k=1}^{N} \varphi(y + ka) \text{ for } y \in B(x,1),$$

which shows $T\varphi|_{B(x,1)}$ is well defined and smooth. This holds for all x hence $T\varphi \in C^\infty(\mathbb{R}^n)$.

Given $K \Subset \mathbb{R}^n$ and $N \geq 0$ set

$$p_{K,N}(f) = \sup\{|\partial^\alpha f(x)| \,:\, x \in K,\ |\alpha| \leq N\} \text{ for } f \in C^\infty(\mathbb{R}^n)\,.$$

Equip $C^\infty(\mathbb{R}^n)$ with the topology induced by the seminorms $\{p_{K,N}\}$. We shall now show T is continuous. It is clear that T is linear. Fix $K \Subset \mathbb{R}^n$, then since $C^\infty(\mathbb{R}^n)$ is locally convex it follows from a claim proven in class that it suffice to show $T|_{\mathcal{D}_K(\mathbb{R}^n)}$ is continuous. Let $H \Subset \mathbb{R}^n$ and $N \geq 1$ be given. There exists an integer $M \geq 1$ with $x + ka \notin K$ for all $x \in H + B(0,1)$ and $k > M$, hence

$$T\varphi(x) = \sum_{k=1}^{M} \varphi(x + ka) \text{ for } \varphi \in \mathcal{D}_K(\mathbb{R}^n) \text{ and } x \in H + B(0,1)\,.$$

It follows that for $\varphi \in \mathcal{D}_K(\mathbb{R}^n)$, a multi-index α with $|\alpha| \leq N$, and $x \in H$,

$$|\partial^\alpha T\varphi(x)| \leq \sum_{k=1}^{M} |\partial^\alpha \varphi(x + ka)| \leq M \cdot p_{K,N}(\varphi),$$

and so $p_{H,N}(T\varphi) \leq M \cdot p_{K,N}(\varphi)$. Since T is linear it follows $T|_{\mathcal{D}_K(\mathbb{R}^n)}$ is continuous, which completes the proof.

First solution for question 2. Part (a): Let $1 \leq k \leq n$, let $e_k \in \mathbb{R}^n$ be the k'th standard unit vector, and set $\partial_k = \partial^{e_k}$. Given $\phi \in \mathcal{D}(\Omega)$

$$\begin{aligned}\partial_k(\psi \cdot u)(\phi) &= -\psi \cdot u(\partial_k \phi) = -u(\psi \cdot \partial_k \phi) = u(\partial_k \psi \cdot \phi) - u(\partial_k \psi \cdot \phi) - u(\psi \cdot \partial_k \phi)\\ &= ((\partial_k \psi) \cdot u)(\phi) - u(\partial_k(\psi \cdot \phi)) = ((\partial_k \psi) \cdot u)(\phi) + (\psi \cdot \partial_k u)(\phi),\end{aligned}$$

which shows $\partial_k(\psi \cdot u) = (\partial_k \psi) \cdot u + \psi \cdot \partial_k u$ as we wanted.

Part (b): Given multi-indexes $\beta \leq \alpha$ set $\binom{\alpha}{\beta} = \Pi_{i=1}^{n}\binom{\alpha_i}{\beta_i}$. We shall prove by induction on $|\alpha|$ that for every multi-index α

$$\partial^\alpha(\psi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta \psi \cdot \partial^{\alpha-\beta} u \,. \tag{0.1}$$

For $\alpha = 0$ the claim is obvious since $\binom{0}{0} = 1$. Let α be a multi-index for which (0.1) holds, let $1 \leq k \leq n$, and set $\gamma = \alpha + e_k$. From part (a) of this question we obtain

$$\begin{aligned} \partial^\gamma(\psi u) = \partial_k(\partial^\alpha(\psi u)) &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial_k(\partial^\beta \psi \cdot \partial^{\alpha-\beta} u) \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\beta+e_k} \psi \cdot \partial^{\alpha-\beta} u + \partial^\beta \psi \cdot \partial^{\alpha-\beta+e_k} u) \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\beta+e_k} \psi \cdot \partial^{\gamma-(\beta+e_k)} u + \partial^\beta \psi \cdot \partial^{\gamma-\beta} u) \\ &= \sum_{\eta \leq \gamma} (\binom{\alpha}{\eta - e_k} \cdot 1_{\{\eta_k > 0\}} + \binom{\alpha}{\eta} \cdot 1_{\{\eta_k \leq \alpha_k\}}) \partial^\eta \psi \cdot \partial^{\gamma-\eta} u \,. \end{aligned} \tag{0.2}$$

Let $\eta \leq \gamma$ be a multi-index. If $\eta_k > 0$ and $\eta_k \leq \alpha_k$ then from Pascal's rule

$$\begin{aligned} \binom{\alpha}{\eta - e_k} + \binom{\alpha}{\eta} &= (\binom{\alpha_k}{\eta_i - 1} + \binom{\alpha_k}{\eta_k}) \cdot \Pi_{i \neq k} \binom{\alpha_i}{\eta_i} \\ &= \binom{\alpha_k + 1}{\eta_k} \cdot \Pi_{i \neq k} \binom{\gamma_i}{\eta_i} = \binom{\gamma}{\eta} \,. \end{aligned} \tag{0.3}$$

If $\eta_k = 0$ then

$$\binom{\alpha}{\eta} = \binom{\alpha_k}{0} \cdot \Pi_{i \neq k} \binom{\alpha_i}{\eta_i} = \binom{\gamma_k}{0} \cdot \Pi_{i \neq k} \binom{\gamma_i}{\eta_i} = \binom{\gamma}{\eta} \,. \tag{0.4}$$

If $\eta_k > \alpha_k$ then clearly $\eta_k = \alpha_k + 1 = \gamma_k$, hence

$$\binom{\alpha}{\eta - e_k} = \binom{\alpha_k}{\alpha_k} \cdot \Pi_{i \neq k} \binom{\alpha_i}{\eta_i} = \binom{\gamma_k}{\eta_k} \cdot \Pi_{i \neq k} \binom{\gamma_i}{\eta_i} = \binom{\gamma}{\eta} \,. \tag{0.5}$$

From (0.2), (0.3), (0.4), and (0.5) it follows (0.1) holds for γ in place of α. This completes the induction and the proof of the claim.

Secon solution for question 2: Let $D_j = i^{-1}\frac{\partial}{\partial x_j}$ and $D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$. Then for $\phi, \psi \in C^\infty$,

$$D^\alpha(\phi\psi) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta \phi \cdot D^{\alpha-\beta} \psi.$$

Proof: Write

$$D^\alpha(\phi\psi) = \sum_{\beta \leq \alpha} D^\beta \phi \cdot P_\beta(D)\psi,$$

where clearly P_β is a monomial of order $\alpha-\beta$. Now take $\phi(x)=e^{ix\xi}$ and $\psi(x)=e^{ix\eta}$ to get

$$(\xi+\eta)^\alpha = \sum_{\beta\le\alpha}\xi^\beta P_\beta(\eta).$$

On the other hand the binomial formula reads

$$(\xi+\eta)^\alpha = \sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\xi^\beta\eta^{\alpha-\beta},$$

and we conclude

$$P_\beta(D)=\frac{\alpha!}{\beta!(\alpha-\beta)!}D^{\alpha-\beta}.$$

Question 3: Set $K_0=supp(\varphi)$, let $H\Subset\mathbb{R}^n$, and set $K=K_0+H$. It is clear that T_φ is linear. In order to show T_φ is continuous it suffice to show it is continuous as a map from $\mathcal{D}_H(\mathbb{R}^n)$ into $\mathcal{D}_K(\mathbb{R}^n)$ (clearly we have $T_\varphi(\mathcal{D}_H(\mathbb{R}^n))\subset\mathcal{D}_K(\mathbb{R}^n)$). Let $\{p_{K,N}\}$ be the seminorms defined in question 1. Given $N\ge 1$, a multi-index α with $|\alpha|\le N$, $\psi\in\mathcal{D}_H(\mathbb{R}^n)$, and $x\in K$,

$$\begin{aligned}|\partial^\alpha T_\varphi\psi(x)| = |\varphi*\partial^\alpha\psi(x)| &\le \int_{K_0}|\varphi(y)\partial^\alpha\psi(x-y)|\,dy\\ &\le \mathcal{L}eb(K_0)\cdot p_{K_0,0}(\varphi)\cdot p_{H,N}(\psi),\end{aligned}$$

which shows

$$p_{K,N}(T_\varphi\psi)\le\mathcal{L}eb(K_0)\cdot p_{K_0,0}(\varphi)\cdot p_{H,N}(\psi)\,.$$

It follows $T_\varphi:\mathcal{D}_H(\mathbb{R}^n)\to\mathcal{D}_K(\mathbb{R}^n)$ is continuous, which completes the proof.

Question 4, part (a): $u\in\mathcal{D}'(\mathbb{R}^n)$ since $e^{|x|^2}$ is locally integrable.

Part (b): By using polar coordinates it is easy to see $\frac{1}{|x|^m}$ is locally integrable if and only if $m<n$. It follows $u_m\in\mathcal{D}'(\mathbb{R}^n)$ if and only if $m<n$. Note that when $m\ge n$ there exists $\varphi\in\mathcal{D}(\mathbb{R}^n)$ with $\int\frac{1}{|x|^m}\varphi(x)\,dx=\infty$.

First solution for part (c): We shall show $u\notin\mathcal{D}'(\mathbb{R}^n)$. For simplicity assume $n=1$, the general case follows by using polar coordinates. Let $0\le\varphi\in\mathcal{D}(\mathbb{R})$ be with $\varphi = 1$ on $[-1,1]$, then for $j \geq 1$

$$\int_0^\infty e^{-x/j!}\varphi(\cos\left(\frac{x}{j}\right) \cdot x)\,dx \geq e^{-\pi}\int_0^{j!\pi} 1_{[-1,1]}(\cos\left(\frac{x}{j}\right) \cdot x)\,dx$$

$$= e^{-\pi} j \int_0^{(j-1)!\pi} 1_{[-1,1]}(jx \cdot \cos x)\,dx$$

$$= e^{-\pi} j \sum_{l=0}^{(j-1)!-1} \int_{l\pi}^{(l+1)\pi} 1_{[-1,1]}(jx \cdot \cos x)\,dx$$

$$\geq e^{-\pi} j \sum_{l=1}^{(j-1)!} \int_0^{\pi} 1_{[-1,1]}(jl\pi \cdot \cos x)\,dx$$

$$= e^{-\pi} j \sum_{l=1}^{(j-1)!} \int_{-1}^{1} 1_{[-1,1]}(jl\pi \cdot x) \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\geq e^{-\pi} j \sum_{l=1}^{(j-1)!} \frac{2}{jl\pi} = \frac{2}{\pi} e^{-\pi} \sum_{l=1}^{(j-1)!} l^{-1}\,.$$

Since $\sum_{l=1}^{\infty} l^{-1} = \infty$ it follows $u(\varphi)$ is not well defined, and so $u \notin \mathcal{D}'(\mathbb{R}^n)$.

Second solution for part (c): Define the functional

$$I_j(\phi) = \int_{\mathbb{R}^n} e^{-\frac{|x|}{j!}} \phi(\cos(\frac{|x|}{j}) \cdot x)dx \;\text{ for } \phi \in \mathcal{D}(\mathbb{R}^n) \text{ and } j \in \mathbb{N}.$$

Given $\phi \in \mathcal{D}(\mathbb{R}^n)$ set $I(\phi) = \lim_{j\to\infty} I_j(\phi)$, then $I \notin D'(\mathbb{R}^n)$.

Proof: Use the fact that $|\cos(\frac{\pi}{2} + \epsilon)| \leq |\epsilon|$ if $|\epsilon| \leq \frac{\pi}{4}$. Take $0 \leq \phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\phi(y) = 1$ for $|y| < \pi$.

Now consider spherical shell $S_j = \{j(\frac{\pi}{2} - \frac{1}{j}) \leq |x| \leq j(\frac{\pi}{2} + \frac{1}{j})\}$. The volume (Lebesgue measure) of this shell satisfies $\nu(S_j) > cj^{n-1}$ and

$$|\cos(\frac{|x|}{j}) \cdot x| \leq \frac{1}{j}(\frac{\pi}{2} + 1)j < \pi, \;\; x \in S_j.$$

Thus

$$|I_j(\phi)| \geq \int_{S_j} e^{-1}\phi(\cos(\frac{|x|}{j}) \cdot x)dx \geq cj^{n-1},$$

hence $\lim_{j\to\infty} I_j(\phi) = \infty$.

Question 5, part (a): Given $\varphi \in \mathcal{D}(\mathbb{R})$ it follows from the dominated convergence theorem

$$\lim_k u_k(\varphi) = \int \lim_k \frac{\sin(kx)}{\sqrt{k}}\varphi(x)\,dx = 0,$$

which shows $u_k \xrightarrow{k} 0$ in $\mathcal{D}'(\mathbb{R})$.

Part (b): Let $j \in \mathbb{N}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ be given, then from part (a)

$$\lim_k \partial^j u_k(\varphi) = (-1)^j \cdot \lim_k u_k(\partial^j \varphi) = 0,$$

which shows $\partial^j u_k \xrightarrow{k} 0$ in $\mathcal{D}'(\mathbb{R})$.

Part (c): $\frac{du_k}{dx}$ does not tend to 0 pointwise, since for $k \geq 1$ and $y \in \mathbb{R}$

$$\frac{du_k}{dx}(y) = \sqrt{k} \cdot \cos(ky) .$$

Question 6: Given $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$|u_k(\varphi) - \delta_0(\varphi)| = |\int k^n u(kx)\varphi(x)\, dx - \varphi(0)| \leq \int |u(x)(\varphi(\frac{x}{k}) - \varphi(0))|\, dx .$$

For $x \in \mathbb{R}^n$ we have $|u(x)(\varphi(\frac{x}{k}) - \varphi(0))| \leq 2\|\varphi\|_\infty |u(x)|$ and $\varphi(\frac{x}{k}) \xrightarrow{k} \varphi(0)$, hence from the dominated convergence theorem

$$\int |u(x)(\varphi(\frac{x}{k}) - \varphi(0))|\, dx \xrightarrow{k} 0 .$$

It follows $u_k(\varphi) \xrightarrow{k} \delta_0(\varphi)$, which shows $u_k \xrightarrow{k} \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.