AUTONOMOUS SYSTEMS, HYPERBOLIC CASE: STABILITY, LYAPUNOV FUNCTIONS, STABLE MANIFOLD THEOREM

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Notation

- The scalar product in $\mathbb{R}^m$ is denoted by $(\cdot, \cdot)$.
- Euclidean norm $|x|^2 = \sum_{i=1}^{m} x_i^2$ in $\mathbb{R}^m$.
- For every $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ we denote by $\|A\|$ its (operator) norm with respect to $|\cdot|$.
- Notation: $B(x, r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x, r)$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.

  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.

  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m)$ which are continuously differentiable up to (including) order k.

  (d) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.

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- **BASIC DEFINITIONS**
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- We consider (**AN AUTONOMOUS SYSTEM**)

$$(A) \qquad y'(t) = f(y(t)), \quad t \in I \subseteq \mathbb{R} \text{ open interval, } 0 \in I,$$

  $f(y) \in C^1(D, \mathbb{R}^m)$, where $D \subseteq \mathbb{R}^m$ is an open set.
- We denote by $y(t; P)$ the (unique) solution such that $y(0; P) = P$, $P \in D$.

  Note that the solution does not necessarily exist for all $t \in I$.
- DEFINITION (**critical point**): A point $Q \in D$ is said to be **critical** (also **equilibrium**) for (A) if $f(Q) = 0$.

  NOTE: The unique solution passing through Q is $y(t; Q) \equiv Q$.
- DEFINITION (**stable point**): A critical point $Q \in D$ is **stable** if , for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$P \in B(Q, \delta) \Rightarrow y(t; P) \in B(Q, \varepsilon) \quad \text{for all} \quad t \geq 0.$$

  REMARK: We assume in particular that $y(t; P)$ exists for all $t \geq 0$.
- DEFINITION (**asymptotically stable point**): A critical point $Q \in D$ is **asymptotically stable** if :

  (a) It is stable.

  (b) There exists a $\delta_0 > 0$ such that

$$P \in B(Q, \delta_0) \Rightarrow y(t; P) \to Q \quad \text{as} \quad t \to \infty.$$

- DEFINITION (**Lyapunov function**): Let $U \subseteq D$ be open. A function $V \in C^1(U)$ is said to be **Lyapunov function** (for (A) in U) if

$$(\nabla V, f) \leq 0 \quad \text{in} \quad U.$$

It is called a **strong Lyapunov function** if

$$(\nabla V, f) < 0 \quad \text{in} \quad U.$$

- **BASIC STABILITY / INSTABILITY THEOREMS**
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- LEMMA: If V is a Lyapunov function for (A) in U and $P \in U$ then $v(t) = V(y(t;P))$ is a *nonincreasing function* of t for every solution $y(t;P)$, as long as $y(t;P)$ is in U.
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Theorem (stability)**.** *Let V be a Lyapunov function for (A) in U. Let $Q \in U$ be a critical point and assume that for some $\varepsilon_0 > 0$*

$$V(Q) < V(P), \quad P \in B(Q, \varepsilon_0) \setminus \{Q\}.$$

Then Q is a stable point.

Furthermore, if V is a ***strict Lyapunov function*** *in $B(Q, \varepsilon_0) \setminus \{Q\}$, then Q is asymptotically stable.*

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Theorem (instability)**.** *Let V be a Lyapunov function for (A) (in U). Let $Q \in U$ be a critical point and let*

$$U_- = \{P \in U, \quad V(P) < V(Q)\}.$$

assume that

$$(i) \quad Q \in \overline{U_-}.$$

(ii) V is a strict Lyapunov function in U_-.

Then Q is an unstable equilibrium.

Proof. Intersecting U with a ball centered at Q, we can assume that $\overline{U} \subseteq D$.

Take any $P \in U_-$ and let $y(t;P)$ be the solution to (A), $y(0;P) = P$. The function $v(t)$ is strictly decreasing (as long as trajectory stays in U_-). Let

$$\tau = \sup\{t > 0, \quad y(t;P) \in U_-\}.$$

If $\tau < \infty \Rightarrow y(\tau;P) \in \partial U$ and $|y(\tau;P) - Q| > \frac{1}{2} dist(Q, \partial U)$.

If $\tau = \infty$ let $E = \lim\limits_{t\to\infty} v(t)$. If $\{t_k\} \uparrow \infty$ is such that $y(t_k;P) \to S \in U$ then $S \in U_-$ and $V(S) = E$. For $\theta > 0$ sufficiently small $y(\theta;S) \in U_-$ and $y(\theta;S) = \lim\limits_{k\to\infty} y(t_k + \theta;P) \Rightarrow V(y(\theta;S)) = E$. This contradicts the fact that V is a strict Lyapunov function. Hence, $\overline{\{y(t;P),\, t > 0\}} \cap \partial U \neq \emptyset$.

We conclude that for every $P \in U_-$ there is a point on the trajectory $y(t;P)$ which is outside the ball of radius $\frac{1}{2} dist(Q, \partial U)$ centered at Q. □

- **LINEAR CONSTANT-COEFFICIENT CASE**
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- $$(LC) \qquad y'(t) = Ay(t), \quad t \in \mathbb{R},$$
  where $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$. We assume that A is **nonsingular**.
- **CLAIM:** Suppose that $\mu \in \mathbb{R}$ is such that $\Re\lambda < \mu$ for every eigenvalue λ of A. Then there exists $K > 0$ such that
  $$\|e^{At}\| \leq Ke^{\mu t}, \quad t \geq 0.$$
  Proof. A Jordan block (of order $k \leq m$) of A is
  $$B = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda \end{pmatrix} = \lambda I + N,$$
  and
  $$e^{Bt} = e^{\lambda t}(I + \sum_{j=1}^{k-1} \frac{N^j t^j}{j!}),$$
  from which the claim follows immediately. □
- **CLAIM:** Suppose that $\nu \in \mathbb{R}$ is such that $\Re\lambda > \nu$ for every eigenvalue λ of A. Then there exists $K > 0$ such that
  $$\|e^{At}\| \leq Ke^{\nu t}, \quad t \leq 0.$$
  Proof. Apply preceding claim to $-A$, so that
  $$\|e^{-At}\| \leq Ke^{-\nu t}, \quad t \geq 0.$$
  Now change t to $-t$. □
- **Theorem.** *Consider the linear system (LC). Then the (only) equilibrium at $y = 0$ is:*
  (a) Asymptotically stable if $\Re\lambda < 0$ for every eigenvalue of A.
  (b) Unstable if for some eigenvalue λ we have $\Re\lambda > 0$.
- **A LYAPUNOV APPROACH TO THIS THEOREM.**
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- **Lemma** Suppose all eigenvalues of A have *negative* real parts. Then there exists a positive definite matrix S such that
  $$A^T S + SA = -I.$$
  Proof. Take $S = \int_0^\infty (e^{At})^T e^{At} dt$. □
- To prove (a) of the theorem, we define, with S as in the Lemma,
  $$V(x) = (x, Sx).$$
  Then $(\nabla V(x), Ax) = (Ax, Sx) + (x, SAx) = -|x|^2$, so V is a strict Lyapunov function.

  To prove (b), note that there exists a real nonsingular matrix Z, which transforms A to a block diagonal matrix
  $$(BD) \qquad Z^{-1}AZ = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 has eigenvalues with real part ≤ 0, and A_2 has eigenvalues with positive real part (at least one by assumption). Assume A is already of this form and let $V(x) = -(x, S_2 x)$, where by the Lemma (applied to $-A_2$),

$$A_2^T S_2 + S_2 A_2 = I.$$

It is easy to see that we can apply the instability theorem (with $U_- = \{V < 0\}$).

- **COROLLARY TO THE PROOF:** Assume that A is already in the block-diagonal form (BD), with a corresponding decomposition $\mathbb{R}^m = \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2}$, with A_i acting in $\mathbb{R}^{m_i}$. Then

$$y(t;P) \xrightarrow[t\to\infty]{} 0 \quad \text{iff} \quad P \in \mathbb{R}^{m_1}.$$

- **SMALL NONLINEAR PERTURBATIONS.**
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$$(NL) \qquad y'(t) = Ay(t) + g(y(t)), \quad t \in \mathbb{R}.$$

We assume:
 (i) $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ is nonsingular.
 (ii) $g \in C^1(U, \mathbb{R}^m)$, where U is a neighborhood of $0 \in \mathbb{R}^m$, and $g(0) = g'(0) = 0$.
 ($g'(0)$ is the Jacobian matrix of g at 0).

- **REMARK:** We could write (NL) in the form

$$y'(t) = f(y(t)),$$

where $f(0) = 0, \quad f'(0) = A$.

- **GRONWALL'S LEMMA-A BASIC TOOL**.
  (This is Problem 1 of Chapter 1 in the book by Coddington-Levinson).
- **Lemma:** Assume $\psi, \gamma, \alpha \in C[t_0, q]$ where $q > t_0$. Assume in addition that $\alpha(t) \geq 0$ and that

$$\psi(t) \leq \gamma(t) + \int_{t_0}^{t} \alpha(s)\psi(s)ds, \quad t_0 \leq t \leq q.$$

Then

$$(*)\, \psi(t) \leq \gamma(t) + \int_{t_0}^{t} \alpha(s)\gamma(s) \exp\left(\int_{s}^{t} \alpha(\tau)d\tau\right)ds, \quad t_0 \leq t \leq q.$$

If moreover $\gamma(t) \in C^1[t_0, q]$ and $\gamma'(t) \geq 0$ then

$$(**)\, \psi(t) \leq \gamma(t) \exp\left(\int_{t_0}^{t} \alpha(s)ds\right).$$

Proof. Define $\Psi(t) = \int_{t_0}^{t} \alpha(s)\psi(s)ds$, so that multiplying the inequality by $\alpha(t)$ yields

$$\Psi'(t) \leq \alpha(t)\gamma(t) + \alpha(t)\Psi(t),$$

so that

$$\frac{d}{dt}\{\Psi(t)\exp(-\int_{t_0}^{t}\alpha(s)ds)\} \leq \alpha(t)\gamma(t)\exp(-\int_{t_0}^{t}\alpha(s)ds).$$

We get

$$\Psi(t) \leq \int_{t_0}^{t}\alpha(s)\gamma(s)\exp(\int_{s}^{t}\alpha(\tau)d\tau)ds,$$

and (*) follows since $\psi(t) \leq \gamma(t) + \Psi(t)$.

Now, to prove (**),

$$\alpha(s)\exp(\int_{s}^{t}\alpha(\tau)d\tau) = -\frac{d}{ds}\exp(\int_{s}^{t}\alpha(\tau)d\tau),$$

so that if $\gamma(t)$ is differentiable, we integrate by parts to get

$$\int_{t_0}^{t}\alpha(s)\gamma(s)\exp(\int_{s}^{t}\alpha(\tau)d\tau)ds$$

$$= \int_{t_0}^{t}\gamma'(s)\exp(\int_{s}^{t}\alpha(\tau)d\tau)ds - \gamma(t) + \gamma(t_0)\exp(\int_{t_0}^{t}\alpha(\tau)d\tau),$$

so that

$$\psi(t) \leq \gamma(t) + \Psi(t) \leq (\gamma(t_0) + \int_{t_0}^{t}\gamma'(s)ds)\exp(\int_{t_0}^{t}\alpha(\tau)d\tau) = \gamma(t)\exp(\int_{t_0}^{t}\alpha(\tau)d\tau).$$

□

- **BACK TO SMALL NONLINEAR PERTURBATIONS.**
- **ASYMPTOTICALLY STABLE CASE**
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- **consider (NL) again**
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Theorem. *Assume that $\Re\lambda < -\nu$, $\nu > 0$, for every eigenvalue λ of A. Then the origin is an asymptotically stable equilibrium of (NL).*

In fact, there exist $K, \delta > 0$ such that, if $|y_0| < \delta \Rightarrow |y(t; y_0)| < Ke^{-\nu t}|y_0|$ for all $t > 0$.

Proof. Take $\varepsilon > 0$, to be determined later. There exists $\eta > 0$ such that $\|g'(x)\| < \varepsilon$ if $|x| < \eta$.

Let $\mu > \nu$ be such that still $\Re\lambda < -\mu$ for all eigenvalues. Then $\|e^{tA}\| < Ke^{-\mu t}$ for $t \geq 0$.

For small $t \geq 0$ the solution is given by

$$(*) \qquad y(t; y_0) = e^{tA}y_0 + \int_{0}^{t}e^{(t-s)A}g(y(s; y_0)ds.$$

Let $\tau > 0$ be such that $|y(s;y_0)| < \eta$ if $|y_0| < \frac{1}{2}\eta$ and $s \in [0,\tau]$. This is possible by the local existence and continuous dependence on initial data theorems.

We obtain from (*),

$$|y(t;y_0)| < Ke^{-\mu t}|y_0| + \int_0^t Ke^{-\mu(t-s)}\varepsilon|y(s;y_0)|ds, \quad t \in [0,\tau],$$

so that,

$$e^{\mu t}|y(t;y_0)| < K|y_0| + \int_0^t Ke^{\mu s}\varepsilon|y(s;y_0)|ds, \quad t \in [0,\tau],$$

which by Gronwall's Lemma (part (**) there) yields

$$(**) \qquad |y(t;y_0)| \le K|y_0|e^{-(\mu - K\varepsilon)t}, \quad t \in [0,\tau].$$

Now take $\varepsilon > 0$ so small that $\mu - K\varepsilon = \nu$ and then take $0 < \delta < \frac{1}{2}\eta$ so that $K\delta < \frac{1}{2}\eta$. Then if $|y_0| < \delta$ we have $|y(t;y_0)| < K\delta < \frac{1}{2}\eta$ if $t \in [0,\tau]$. The point $y(\tau;y_0)$ can therefore be used as a new initial point and the solution $y(t;y_0)$ exists for $t \in [\tau, 2\tau]$ with $|y(t;y_0)| < \eta$, $t \in [0,2\tau]$. In particular the estimate (**) now holds in $[0,2\tau]$, yielding $|y(t;y_0)| < \frac{1}{2}\eta$, $t \in [0,2\tau]$. In this way the solution can be continued to all $t \ge 0$, and satisfies the estimate (**) for all $t \ge 0$.. □

- REMARK **(LYAPUNOV APPROACH TO THIS THEOREM):** The same approach, with the same function, can be applied here as in the linear case ($g \equiv 0$).
- **THE HYPERBOLIC CASE**
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- **consider (NL) again**

  We now assume that *not all eigenvalues* of A have negative real parts. However, we assume that *none of them is purely imaginary.*
- DEFINITION**(hyperbolic critical point)**: We say that the critical point $y = 0$ is **hyperbolic** if $\Re\lambda \ne 0$ for every eigenvalue λ of A.
- DEFINITION: If $\Re\lambda \ne 0$ for every eigenvalue λ of A we say that A is **infinitesimally hyperbolic**.
- ASSUMPTION: In what follows we assume that $y = 0$ is hyperbolic.

  We assume that $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ where $m_1, m_2 > 0$ and A is in block diagonal form as before

$$(BD) \qquad Z^{-1}AZ = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad A_i \in Hom(\mathbb{R}^{m_i}, \mathbb{R}^{m_i}).$$

  There are $\mu, \nu > 0$ such that:

  $\Re\lambda < -\nu$ for every eigenvalue of A_1.

  $\Re\lambda > \mu$ for every eigenvalue of A_2.
- Using the norm estimates above, we have $K > 0$ such that

$$\|e^{tA_1}\| \le Ke^{-\nu t},\ t \ge 0, \quad \|e^{tA_2}\| \le Ke^{\mu t},\ t \le 0,$$

  in the respective spaces.
- Notation: For $x \in \mathbb{R}^m$ we decompose $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$.
- Consider the nonlinear equation (NL) with g as above, in particular $g(0) = g'(0) = 0$. We denote $g = (g^{(1)}, g^{(2)}) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$.

- We write (NL) using the (BD) form:

$$(NLD) \qquad \begin{aligned} {y^{(1)}}'(t) &= A_1 y^{(1)} + g^{(1)}(y^{(1)}, y^{(2)}) \\ {y^{(2)}}'(t) &= A_2 y^{(2)} + g^{(2)}(y^{(1)}, y^{(2)}). \end{aligned}$$

- We define the set of all initial data of bounded trajectories (defined for all $t \geq 0$) by

$$W^S = \{y_0 \in U, \quad y(t; y_0) \in C_b([0,\infty), U)\}.$$

(see Notation above for $C_b([0,\infty), U)$ and its norm $\|\cdot\|$).

- **REMARK:** The set W^S is *invariant under the flow*, i.e., $y_0 \in W^S \Rightarrow y(t; y_0) \in W^S$, $t \geq 0$. In particular, it is actually the union of all bounded trajectories (in U).
- Given $\delta > 0$ the set of initial data of trajectories contained in the ball $B(0,\delta) \subseteq U$ is denoted by

$$W^S_\delta = \{y_0 \in U, \quad y(t; y_0) \in C_b([0,\infty), B(0,\delta))\}.$$

As before, W^S_δ is invariant under the flow and is the union of all trajectories contained in $B(0,\delta)$.

- The bounded trajectories have the following property.

LEMMA: Let $y_0 = (y_0^{(1)}, y_0^{(2)}) \in W^S$. Then the solution $y(t; y_0) = (y^{(1)}(t; y_0), y^{(2)}(t; y_0))$ satisfies

$$(NLB) \qquad \begin{aligned} y^{(1)}(t; y_0) &= e^{A_1 t} y_0^{(1)} + \int_0^t e^{A_1(t-s)} g^{(1)}(y(s; y_0)) ds, \\ y^{(2)}(t; y_0) &= -\int_t^\infty e^{A_2(t-s)} g^{(2)}(y(s; y_0)) ds. \end{aligned}$$

<u>PROOF</u>: Since the solution exists for all $t \geq 0$ we have by the variation-of-constants formula:

$$y^{(1)}(t; y_0) = e^{A_1 t} y_0^{(1)} + \int_0^t e^{A_1(t-s)} g^{(1)}(y(s; y_0)) ds,$$

$$y^{(2)}(t; y_0) = e^{A_2(t-\tau)} y^{(2)}(\tau; y_0) + \int_\tau^t e^{A_2(t-s)} g^{(2)}(y(s; y_0)) ds,$$

for every $0 \leq t, \tau < \infty$.

Letting $\tau \to \infty$ and taking into account the assumed boundedness of the solution and the norm estimates for $e^{A_i t}$ the claim follows.

- **THE STABLE MANIFOLD THEOREM**

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- The theorem says that for sufficiently small δ, i.e., trajectories close to the origin, W^S_δ is a graph over a small ball in $\mathbb{R}^{m_1}$.
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Theorem. *Let $y = 0$ be a hyperbolic critical point for (NL).*

There exist $0 < \rho < \delta$ *such that the intersection*

$$W^S_{\delta,\rho} = W^S_\delta \cap \{\{B(0,\rho) \subseteq \mathbb{R}^{m_1}\} \times \mathbb{R}^{m_2}\}$$
$$= \left\{ y_0, \quad \|y(\cdot; y_0)\| < \delta,\ |y_0^{(1)}| < \rho \right\},$$

has the form

$$W^S_{\delta,\rho} = \left\{ (y_0^{(1)}, h(y_0^{(1)})), \quad y_0^{(1)} \in B(0,\rho) \right\},$$

where

$$h : B(0,\rho) \subseteq \mathbb{R}^{m_1} \hookrightarrow \mathbb{R}^{m_2}.$$

Furthermore, $W^S_{\delta,\rho}$ *is* globally invariant *with respect to the flow for small initial data, i.e., there exists* $0 < \tilde{\rho} < \rho$*, such that if* $y_0 \in W^S_{\delta,\rho}$ *and* $|y_0^{(1)}| < \tilde{\rho}$*, then* $y(t; y_0) \in W^S_{\delta,\rho}$ *for all* $t \geq 0$.

Proof. (1) We assume that $U \subseteq \mathbb{R}^m$ is a neighborhood of 0, $\overline{U}$ bounded and $g \in C^1$ in a neighborhood of $\overline{U}$. Also $U_i \subseteq \mathbb{R}^{m_i},\ i = 1, 2$ open neighborhoods of 0, and $\overline{U_1} \times \overline{U_2} \subseteq U$.

(2) Let $X_i = C_b([0,\infty), \overline{U_i})$, $i = 1, 2$, (see Notation above).

(3) We equip X_i with norm $\|\phi\|_i = \sup\limits_{0 \leq t < \infty} |\phi(t)|$. Now define

$$X = X_1 \times X_2, \quad \|\phi\| = \|\phi^{(1)}\|_1 + \|\phi^{(2)}\|_2, \quad \phi = (\phi^{(1)}, \phi^{(2)}) \in X.$$

(4) Given $y_0^{(1)} \in \mathbb{R}^{m_1}$ we define a nonlinear map $\mathcal{T} : X \hookrightarrow C_b([0,\infty), \mathbb{R}^m)$ by

$$\psi(t) = (\mathcal{T}\phi)(t) = (\psi^{(1)}, \psi^{(2)}), \quad \text{where} \begin{cases} \psi^{(1)}(t) = e^{A_1 t} y_0^{(1)} + \int\limits_0^t e^{A_1(t-s)} g^{(1)}(\phi(s)) ds, \\ \psi^{(2)}(t) = -\int\limits_t^\infty e^{A_2(t-s)} g^{(2)}(\phi(s)) ds. \end{cases}$$

(Check that $\mathcal{T}$ is well defined and bounded).

(5) Let $\varepsilon > 0$ (to be chosen later) and let $\delta > 0$ be such that $x \in B(0,\delta) \Rightarrow \|g'(x)\| < \varepsilon$. Suppose that $\|\phi\| < \delta$. Then $|g^{(i)}(\phi(s))| \leq \varepsilon \|\phi\|$, $i = 1, 2,\ s \geq 0$.

(6) It follows that $\|\phi\| < \delta$ implies

$$|\psi^{(1)}(t)| \leq K e^{-\nu t} |y_0^{(1)}| + \int_0^t K e^{-\nu(t-s)} \varepsilon \|\phi\| ds \leq K |y_0^{(1)}| + K\varepsilon\nu^{-1}\|\phi\|,$$

$$|\psi^{(2)}(t)| \leq \int_t^\infty K e^{\mu(t-s)} \varepsilon \|\phi\| ds = K\varepsilon\mu^{-1}\|\phi\|.$$

(7) Choose now $\varepsilon > 0$ so small that $K\varepsilon(\mu^{-1} + \nu^{-1}) < \frac{1}{2}$ and determine the corresponding δ.

Conclusion: Let $\Gamma_\delta \subseteq X$ be the ball of radius δ centered at 0. Then for every $y_0^{(1)} \in \mathbb{R}^{m_1}$ such that $K|y_0^{(1)}| < \frac{1}{2}\delta$ the map $\mathcal{T} : \Gamma_\delta \hookrightarrow \Gamma_\delta$.

(8) Let now $\phi, \tilde{\phi} \in \Gamma_\delta$. Let $\psi = \mathcal{T}\phi, \quad \tilde{\psi} = \mathcal{T}\tilde{\phi}$. Then

$$|\psi^{(1)}(t) - \tilde{\psi}^{(1)}(t)| \leq \int_0^t Ke^{-\nu(t-s)}\varepsilon\|\phi - \tilde{\phi}\|ds \leq K\varepsilon\nu^{-1}\|\phi - \tilde{\phi}\|,$$

$$|\psi^{(2)}(t) - \tilde{\psi}^{(2)}(t)| \leq \int_t^\infty Ke^{\mu(t-s)}\varepsilon\|\phi - \tilde{\phi}\|ds = K\varepsilon\mu^{-1}\|\phi - \tilde{\phi}\|.$$

$$\|\mathcal{T}\phi - \mathcal{T}\tilde{\phi}\| \leq \frac{1}{2}\|\phi - \tilde{\phi}\|.$$

Conclusion: $\mathcal{T}$ is a contraction on Γ_δ.

(9) For every $y_0^{(1)} \in \mathbb{R}^{m_1}$ such that $K|y_0^{(1)}| < \frac{1}{2}\delta$ the map $\mathcal{T} : \Gamma_\delta \hookrightarrow \Gamma_\delta$ has a *unique fixed point* $\psi(t; y_0^{(1)}) \in \Gamma_\delta \subseteq X$.
Let

$$y_0^{(2)} = h(y_0^{(1)}) = -\int_0^\infty e^{A_2(t-s)}g^{(2)}(\psi(s; y_0^{(1)}))ds,$$

and set $y_0 = (y_0^{(1)}, y_0^{(2)})$.
Conclusion: The function $y(t; y_0) = \psi(t; y_0^{(1)})$ is a solution to (NL) which satisfies $|y(t; y_0)| < \delta$, $t \geq 0$.

(10) We pick $\rho > 0$ such that $\rho < min\left\{\delta, \frac{\delta}{2K}\right\}$.

(11) Conversely, suppose that $y(t; y_0)$ is a solution to (NL) such that

$$|y_0^{(1)}| < \rho, \quad |y(t; y_0)| < \delta, \ t \geq 0.$$

By the Lemma above, $y(t; y_0)$ must satisfy (NLB) and is therefore a fixed point of $\mathcal{T}$ in Γ_δ with $|y_0^{(1)}| < \rho$. Since such a fixed point is unique, it must coincide with the solution constructed above and we must have $y_0^{(2)} = h(y_0^{(1)})$.
This concludes the proof of the statement concerning the structure of $W^S_{\delta,\rho}$.

(12) Finally, to prove the global invariance of $W^S_{\delta,\rho}$ we simply replace in the proof above δ by the smaller number ρ. The proof gives the existence of some $0 < \tilde{\rho} < \rho$ such that, if $|y_0^{(1)}| < \tilde{\rho}$ and $y_0^{(2)} = h(y_0^{(1)})$, then $|y(t; y_0)| < \rho$ for all $t \geq 0$. But then, by the above arguments, $y(t; y_0) \in W^S_{\delta,\rho}$ for all $t \geq 0$. In particular,

$$y^{(2)}(t; y_0) = h(y^{(1)}(t; y_0)), \quad t \geq 0.$$

Note that we have used that the "h-function" for the smaller ball coincides (in that ball) with the "h-function" of the larger ball. This is obvious since a trajectory contained in the smaller ball is necessarily contained in the larger one.

□

- DEFINITION (**Stable Manifold**): The surface $W^S_{\delta,\rho}$ is called the **Stable Manifold** of (NL) (locally at 0).

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- **PROPERTIES OF THE STABLE MANIFOLD.**

Claim. *(*Lipschitz continuity*) The map $h : B(0,\rho) \subseteq \mathbb{R}^{m_1} \hookrightarrow \mathbb{R}^{m_2}$ is Lipschitz. In fact, for every $\eta > 0$ there exists a $0 < \theta < \rho$ such that if $x^{(1)}, x^{(2)} \in B(0,\theta) \subseteq \mathbb{R}^{m_1}$ then*

$$|h(x^{(2)}) - h(x^{(1)})| < \eta |x^{(2)} - x^{(1)}|.$$

PROOF. Let $\varepsilon, \delta, \rho > 0$ be as in the proof of the theorem. If $y_0 \in W^S_{\delta,\rho}$ then the solution $y(t;y_0)$, satisfies in view of the Lemma,

$$y^{(1)}(t;y_0) = e^{A_1 t} y_0^{(1)} + \int_0^t e^{A_1(t-s)} g^{(1)}(y(s;y_0))ds,$$

$$y^{(2)}(t;y_0) = -\int_t^\infty e^{A_2(t-s)} g^{(2)}(y(s;y_0))ds.$$

If also $z_0 \in W^S_{\delta,\rho}$ then similar expressions apply with y_0 replaced by z_0. We take the difference of the two solutions in the above expressions and use the norm estimates for e^{tA} as well as $\|g'\| < \varepsilon$ in the ball to get

$$|y^{(1)}(t;y_0) - y^{(1)}(t;z_0)| \le K e^{-\nu t} |y_0^{(1)} - z_0^{(1)}| + \int_0^t K e^{-\nu(t-s)} \varepsilon |y(s;y_0) - y(s;z_0)| ds$$

$$|y^{(2)}(t;y_0) - y^{(2)}(t;z_0)| \le \int_t^\infty K e^{\mu(t-s)} \varepsilon |y(s;y_0) - y(s;z_0)| ds.$$

Adding the two inequalities and taking $\sup\limits_{0 \le t < \infty}$ we obtain

$$\|y(\cdot;y_0) - y(\cdot;z_0)\| \le K|y_0 - z_0| + K\varepsilon(\mu^{-1} + \nu^{-1}) \|y(\cdot;y_0) - y(\cdot;z_0)\|.$$

Taking $\varepsilon > 0$ such that $K\varepsilon(\mu^{-1} + \nu^{-1}) < \frac{1}{2}$ (which of course forces suitable choices for δ, ρ), we get

$$\|y(\cdot;y_0) - y(\cdot;z_0)\| \le 2K|y_0 - z_0|.$$

This estimate can now be inserted into the inequality above for $y^{(2)}$ to get

$$|y^{(2)}(t;y_0) - y^{(2)}(t;z_0)| \le 2K^2\varepsilon\mu^{-1}|y_0 - z_0| \le 2K^2\varepsilon\mu^{-1}(|y_0^{(1)} - z_0^{(1)}| + |y_0^{(2)} - z_0^{(2)}|).$$

Taking $t = 0$ and requiring further that $2K^2\varepsilon\mu^{-1} < \frac{1}{2}$ we conclude

$$|y_0^{(2)} - z_0^{(2)}| \le 4K^2\varepsilon\mu^{-1}|y_0^{(1)} - z_0^{(1)}|.$$

The claim is proved in view of the fact that $\varepsilon > 0$ can be taken arbitrarily small(with δ, ρ determined accordingly). □

- **Corollary.** The stable manifold is tangent to $\mathbb{R}^{m_1}$ at the origin.

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  The trajectories starting on the stable manifold sufficiently close to the origin stay on the manifold (by the last part of the Theorem). Furthermore, the next claim shows that they approach the equilibrium at an exponential rate, similar to the corresponding linear case.

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Claim. *(*Exponential decay of stable trajectories*) Let $y_0 \in W^S_{\delta,\rho}$ where $y_0^{(1)}$ is small. Then $y(t;y_0) \in W^S_{\delta,\rho}$ for all $t \geq 0$ and there exists a constant $C > 0$ such that*

$$|y(t;y_0)| \leq Ce^{-\kappa t}, \quad t \geq 0,$$

where $\kappa > 0$ is such that $\Re\lambda < -\kappa$ for every eigenvalue of A_1.

PROOF. The fact that the full trajectory lies on the manifold is given by the last part of the Theorem.

We now use the estimate for $y^{(1))}$ in the proof of the previous Claim, with $z_0 = 0$, to get

$$|y^{(1)}(t;y_0)| \leq Ke^{-\nu t}|y_0^{(1)}| + \int_0^t Ke^{-\nu(t-s)}\varepsilon|y(s;y_0)|ds.$$

Since the trajectory lies on the manifold, the Lipschitz estimate for h obtained in the previous proof yields

$$|y^{(2)}(t;y_0)| \leq 4K^2\varepsilon\mu^{-1}|y^{(1)}(t;y_0)|,$$

so that we can rewrite the estimate for $y^{(1)}$ above as

$$|y^{(1)}(t;y_0)| \leq Ke^{-\nu t}|y_0^{(1)}| + CK\varepsilon\int_0^t e^{-\nu(t-s)}|y^{(1)}(s;y_0)|ds,$$

where $C > 0$ is a constant. It follows that the function $r(t) = |y^{(1)}(t;y_0)|e^{\nu t}$ satisfies the inequality

$$r(t) \leq K|y_0^{(1)}| + CK\varepsilon\int_0^t r(s)ds, \quad t \geq 0,$$

so that by Gronwall's Lemma (see (**) there),

$$r(t) \leq K|y_0^{(1)}|e^{CK\varepsilon t}, \quad t \geq 0,$$

hence

$$|y^{(1)}(t;y_0)| \leq K|y_0^{(1)}|e^{(CK\varepsilon-\nu)t}, \quad t \geq 0.$$

Given κ as in the claim we can find $\nu > \kappa$ such that still $\Re\lambda < -\nu$ for every eigenvalue of A_1.

We conclude the proof by taking $\varepsilon = \frac{\nu-\kappa}{CK}$ and noting that $|y^{(2)}(t;y_0)|$ can be estimated in terms of $|y^{(1)}(t;y_0)|$ in view of the the Lipschitz continuity of h. □

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THE UNSTABLE MANIFOLD

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- Changing t to $-t$, the matrix A is replaced by $-A$, so that the stable behavior is determined by $-A_2$, acting in $\mathbb{R}^{m_2}$. Switching back to the original variable t, we define,

$$W^U = \{y_0 \in U, \quad y(t;y_0) \in C_b((-\infty,0],U)\}.$$

- Given $\delta > 0$ the set of initial data of "back" trajectories contained in the ball $B(0,\delta) \subseteq U$ is denoted by

$$W_\delta^U = \{y_0 \in U, \quad y(t;y_0) \in C_b((-\infty,0], B(0,\delta))\}.$$

  We get the following analogous theorem.
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Theorem. *There exist* $0 < \rho < \delta$ *such that the intersection*

$$W_{\delta,\rho}^U = W_\delta^U \cap \mathbb{R}^{m_1} \times \{\{B(0,\rho) \subseteq \mathbb{R}^{m_2}\}\}$$
$$= \left\{y_0, \quad \|y(\cdot;y_0)\| < \delta, \, |y_0^{(1)}| < \rho\right\},$$

has the form

$$W_{\delta,\rho}^U = \left\{(k(y_0^{(2)}), y_0^{(2)}), \quad y_0^{(2)} \in B(0,\rho)\right\},$$

where

$$k : B(0,\rho) \subseteq \mathbb{R}^{m_2} \hookrightarrow \mathbb{R}^{m_1}.$$

Furthermore, $W_{\delta,\rho}^U$ *is* globally invariant *with respect to the flow for small initial data, i.e., there exists* $0 < \tilde{\rho} < \rho$, *such that if* $y_0 \in W_{\delta,\rho}^U$ *and* $|y_0^{(2)}| < \tilde{\rho}$, *then* $y(t;y_0) \in W_{\delta,\rho}^U$ *for all* $t \le 0$.

- DEFINITION (**Unstable Manifold**): The surface $W_{\delta,\rho}^U$ is called the **Unstable Manifold** of (NL) (locally at 0).
- REMARK. The claims concerning the Lipschitz continuity of h and the exponential decay of trajectories can directly be translated to the unstable manifold. In particular, note that $W_{\delta,\rho}^U$is tangent to $\mathbb{R}^{m_2}$ at the origin and the trajectories approach the origin at an exponential rate as $t \to \infty$.

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