AUTONOMOUS SYSTEMS IN THE PLANE: PHASE PLANE ANALYSIS

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May 2011

Notation

- The scalar product in $\mathbb{R}^m$ is denoted by $(\cdot,\cdot)$.
- Euclidean norm $|x|^2 = \sum\limits_{i=1}^{m} x_i^2$ in $\mathbb{R}^m$.
- For every $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ we denote by $\|A\|$ its (operator) norm with respect to $|\cdot|$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.

  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.

  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m)$ which are continuously differentiable up to (including) order k.

  (d) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.

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THE CASE $y' = Ay$, WHERE $A \in Hom(\mathbb{R}^2, \mathbb{R}^2)$ IS REAL, NONSINGULAR, CONSTANT MATRIX

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All solutions can be classified as **nodes, spirals, saddle points or centers** (with respect to the origin–the only critical point).

READ: Coddington-Levinson, Ch. 15, Sec. 1, or Boyce-DiPrima, Ch. 9, Secs. 1-3.

- **BASIC DEFINITIONS AND FACTS**
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- DEFINITION (**AN AUTONOMOUS SYSTEM**): Given a continuous function $f(y) \in C(\mathbb{R}^m, \mathbb{R}^m)$, the system

  $$(A) \qquad\qquad y'(t) = f(y(t)), \quad t \in I \subseteq \mathbb{R},$$

  is called **autonomous.**
- REMARKS: (a) In general we do not assume f to be Lipschitz, so there is no guarantee of uniqueness of the solutions.

(b) For simplicity we assume f to be defined in all of $\mathbb{R}^m$. If it is defined in an open set $U \subseteq \mathbb{R}^m$ then the solution $y(t)$ exists as long as $y(t) \in U$.

(c) Even if f is "nice" the solutions are not necessarily defined for all $t \in \mathbb{R}$ (think of $m = 1$ and $f(y) = y^2$).

(d)**The role of the t parameter:** If $y(t), \quad t \in I$ is a solution then for every "shift" t_0 also $\tilde{y}(t) = y(t - t_0), \quad t - t_0 \in I$ is a solution of (A).

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- **IN THIS SUMMARY WE DISCUSS THE CASE $m = 2$.**
- **THIS CONTAINS THE SECOND ORDER SCALAR EQUATION.**

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- **A NONLINEAR PENDULUM**

$$Lz''(t) = -g \sin z(t),$$

where $z(t)$ is the angle of deviation from (the vertical) equilibrium, $L > 0$ is the length of the pendulum and $g > 0$ is the gravitation constant.

- Multiply by $z'(t)$ and integrate to obtain

$$\frac{L}{2}\left(\frac{dz}{dt}\right)^2 = g(\cos z(t) - 1) + E,$$

where $E \geq 0$ is the energy (=nonnegative constant). The zero level of the potential (gravitational) energy is set to zero at the lowest (vertical) position $z = 0$.

- If $E = 0$ we have only the *trivial solution* $z(t) \equiv 0$.
- *The case* $0 < E < 2g$.

Then we must have $|z(t)| \leq z^{max} = \arccos(1 - \frac{E}{g})$ if $z(t) \in (-\pi, \pi)$.

The trajectories in the *phase plane* z, z' diagram (for these values of E) are closed curves centered at the equilibrium (critical) points $\{(2k\pi, 0), \quad k = 0, \pm 1, \pm 2...\}$

CONCLUSION: In this case all trajectories are **periodic**.

COMPUTATION OF THE PERIOD:

$$\frac{dt}{dz} = \sqrt{\frac{L}{2}\frac{1}{g(\cos z - 1) + E}} \quad .$$

A quarter of a period is given by the time of motion from the bottom $z = 0$ to the maximal angle z^{max}, so

$$T = 4\sqrt{\frac{L}{2}} \int_0^{z^{max}} \frac{dz}{\sqrt{g(\cos z - 1) + E}}.$$

Note that T depends on E. In fact, with the change of variable

$$u = 1 - \frac{g}{E}(1 - \cos z),$$

we get

$$T(E) = 2\sqrt{\frac{L}{g}} \int_0^1 \frac{du}{u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}[1 - \frac{E}{2g}(1-u)]^{\frac{1}{2}}}.$$

Note that

$$T(0) = 2\sqrt{\frac{L}{g}} \int_0^1 \frac{du}{u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}} = 2\pi\sqrt{\frac{L}{g}},$$

that is, in the limit of "infinitesimal amplitude", the period is equal to that of the *linear pendulum*.

- *The case $E = 2g$.*

  In this case we have the set of equilibria $\{(2k + 1\pi, 0), \quad k = 0, \pm 1, \pm 2...\}$ These are the *unstable equilibria* where the pendulum is at rest at the "top". The trajectories "connect" one such equilibrium to the next one ("full swing" of the pendulum, clockwise or counterclockwise).

  QUESTION: The pendulum starts from its lowest position $z(0) = 0$. It reaches the highest point $z(t_0) = \pi$ and then stops there?

  ANSWER: **Yes and No.** In fact, it will approach the highest point only in infinite time!

  To see this, write

$$\frac{dt}{dz} = \sqrt{\frac{L}{2}\frac{1}{g(\cos z - 1) + 2g}},$$

then

$$t_0 = \int_0^{\pi} \sqrt{\frac{L}{2}\frac{1}{g(\cos z + 1)}} dz = \infty.$$

- *The case $E > 2g$.*

  In this case the derivative $z'(t)$ of a trajectory never vanishes. The trajectories in the phase plane are continuous graphs over the $z-$axis, oscillating between minimal and maximal values of $z'(t)$, both positive (or both negative). The actual motion of the pendulum is a continuous rotation (clockwise or counter-clockwise).
- **FRICTION IS ADDED...**

  The equation is now

$$Lz''(t) + kz'(t) = -g \sin z(t), \quad k > 0.$$

  The term $kz'(t)$ is proportional to the (angular) velocity; when $z'(t) > 0$, i.e., the pendulum is moving *away* from equilibrium (at $z = 0$) it serves to *decrease* the acceleration $z''(t)$, i.e., to *slow down* the motion.
- SOLUTION.
- Multiply by $z'(t)$ and integrate to obtain

$$\frac{L}{2}(\frac{dz}{dt})^2 + k\int_{t_0}^{t} z'(s)^2 ds = g(\cos z(t) - 1) + E,$$

  where we assume that $z(t_0) = 0$ and $E = \frac{L}{2}z'(t_0)^2$ is the initial energy.
- COROLLARY: The function $f(t) = \frac{L}{2}(\frac{dz}{dt})^2 + g(1 - \cos z(t))$ is a **decreasing function** of t.

  Since it is positive, it must converge to a limit

$$f(t) \to \xi \geq 0, \quad \text{as} \quad t \to \infty.$$

  From the equation

$$f'(t) = -kz'(t)^2 \leq -\frac{2k}{L}f(t),$$

  so that

$$f(t) \leq f(t_0)\exp(-\frac{2k}{L}(t - t_0)).$$

  Thus $\xi = 0$ and

$$z'(t) \to 0, \quad z(t) \to 2\pi j, \quad j \in \mathbb{Z}, \quad \text{as} \quad t \to \infty.$$

  Note that the pendulum always approaches a *stable* equilibrium (at the bottom position).

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- The **LOTKA-VOLTERRA EQUATION**

- $$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} ay_1(t) - by_1(t)y_2(t) \\ -cy_2(t) + dy_1(t)y_2(t) \end{pmatrix}, \quad a, b, c, d > 0.$$
- **This is a model of "predator"(y_2)-"prey"(y_1) interaction.**

  (i) The prey population grows at a rate proportional to its size but decreases in proportionality to the size of the predator population.

  (ii) The opposite statement applies to the predator population.
- The critical (or *equilibrium*) point (other than $(0,0)$) is
$$\overline{y} = (\frac{c}{d}, \frac{a}{b}).$$
- Take a trajectory $y(t)$ through a point $y(0) = y^0$, so that $y_1^0, y_2^0 > 0$.
- $$dy_1'(t) + by_2'(t) = ady_1(t) - bcy_2(t),$$
$$c\frac{y_1'(t)}{y_1(t)} + a\frac{y_2'(t)}{y_2(t)} = ady_1(t) - bcy_2(t).$$
- $$c\frac{y_1'(t)}{y_1(t)} + a\frac{y_2'(t)}{y_2(t)} = dy_1'(t) + by_2'(t),$$
  so that
$$c \log y_1(t) + b \log y_2(t) = dy_1(t) + by_2(t) + \log K,$$
$$y_1(t)^c e^{-dy_1(t)} y_2(t)^a e^{-by_2(t)} = K.$$

  Let
$$F = \max_{0 \le x < \infty} x^c e^{-dx}, \quad G = \max_{0 \le x < \infty} x^a e^{-bx}.$$
  These maximal values are attained respectively at the critical coordinates $x = \overline{y}_1$ and $x = \overline{y}_2$.

  A trajectory (in the first quadrant $y_1, y_2 > 0$) exists if and only if $0 < K \le FG$.

  If $K = FG$ the solution is *stationary*, $y(t) \equiv \overline{y}$.

  If $K < FG$ we have:
- **CLAIM**: If $0 < K < FG$ the trajectory is **closed**, in other words, the solution is **periodic** in time.

  PROOF: (i) Because $\lim_{x \to \infty} (x^c e^{-dx} + x^a e^{-bx}) = 0$, every trajectory is *bounded*. Since it is *a closed set*, and never touches the axes (why?), it is *compact*, hence by the general theory the solution exists *for all* $t \in \mathbb{R}$.

(ii) In the first quadrant of the (y_1, y_2) plane $(y_1, y_2 > 0)$ consider the implicit equation

$$g(y_1, y_2) \equiv y_1^c e^{-dy_1} y_2^a e^{-by_2} = K.$$

Clearly on this set $\nabla_{y_1,y_2} g(y_1, y_2) \neq 0$, since the gradient vanishes only at y^c and we assume $K < FG$.

By the Implicit Function Theorem it follows that the equation determines a *closed curve* Λ.

The solution $y(t)$ stays on Λ (why?) and its tangent $y'(t)$ does not vanish (why?) and is perpendicular to $\nabla_{y_1,y_2} g(y_1, y_2)$ at $y(t)$ (why?), hence tangent to Λ. It follows that indeed the solution "goes around" Λ, which completes the proof.

(iii)Here is also a geometric picture of the situation:

It is easy to see from the equation that if $y_1(t) > \frac{c}{d} = y_1^c$ then $y_2'(t) > 0$ so that $y_2(t)$ increases from a minimal value $y_{2,min}$ (satisfying $y_{2,min}^a e^{-by_{2,min}} F = K$) till it reaches the maximal value $y_{2,max}$, satisfying the same equation, where $y_1(t) = y_1^c$. At this point it begins to decrease, down to $y_{2,min}$.

Q.E.D.

- THE PERIOD AND THE EQUILIBRIUM.

  The above claim gives that, for some $T > 0$, possibly depending on the trajectory (i.e., on K), such that $y(t+T) = y(t)$, $t \in \mathbb{R}$. It is interesting to note the following:

- **CLAIM**: The equilibrium value $\overline{y}$ is the *average over a period* of the populations.

  PROOF: Integrate $\frac{y_1'(t)}{y_1(t)} = a - by_2(t)$ to get

$$0 = aT - b \int_{t_0}^{t_0+T} y_2(t)dt,$$

for any fixed $t_0 \in \mathbb{R}$. Hence indeed

$$\frac{a}{b} = \overline{y}_2 = \frac{1}{T} \int_{t_0}^{t_0+T} y_2(t)dt,$$

and similarly,

$$\frac{c}{d} = \overline{y}_1 = \frac{1}{T} \int_{t_0}^{t_0+T} y_1(t)dt.$$

Q.E.D.

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- **DEFINITIONS IN THE GENERAL CASE OF** $\mathbb{R}^m$.
- We consider the autonomous system

$$(A) \qquad y'(t) = f(y(t)), \quad t \in I \subseteq \mathbb{R},$$

where now we assume that $f(y) \in C^1(D, \mathbb{R}^m)$ where $D \subseteq \mathbb{R}^m$ is an open set.
- DEFINITION (**critical point**): A point $Q \in D$ is said to be **critical** (for (A)) if $f(Q) = 0$.
- NOTATION: For every $P \in D$ we denote by $y(t; P)$ the (unique) solution of (A) such that $y(0; P) = P$.
- DEFINITION (**periodic solution**): A solution $y(t; P)$ is called **periodic** if for some $T > 0$ we have $y(T; P) = y(0; P) = P$.
- REMARK: Such a periodic solution exists for all $t \in \mathbb{R}$ and satisfies $y(t; P) = y(t + T; P)$ for all $t \in \mathbb{R}$.

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- The **POINCARÉ-BENDIXSON THEOREM**
- It is a **theorem in the plane**—$m = 2$.

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Theorem. *Let $K \subseteq D \subseteq \mathbb{R}^2$ be compact. Assume that:*
(i) There are no critical points in K.
(ii) For some $\overline{P} \in K$ the solution $y(t; \overline{P})$ exists for all $t \geq 0$ and is contained in K. Then either
(a) $y(t; \overline{P})$ is periodic,
or
(b) The set

$$\omega^+ = \left\{ z \in K, \quad z = \lim_{n \to \infty} y(t_n; \overline{P}), \quad t_n \uparrow \infty \right\},$$

is a periodic solution.

Below is an outline of the proof. For the full proof, see

E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, Ch. 16.

- REMARK: In the case of the Lotka-Volterra system considered above, the theorem implies that ALL trajectories in the quarter-plane $y_1 > 0, y_2 > 0$, are periodic (except for the one at the critical point).
- **Outline of the proof**.

(a) Suppose $y(t; \overline{P})$ is not periodic.

(b) A line segment $l \subseteq D$ is called *transversal* if for every $y \in l$ the vector $f(y)$ is not parallel to l.

Note: If $Q \in K$ then it is a center of a transversal segment (because $f(Q) \neq 0$).

(c) LEMMA: Let $Q \in K$ and let l be a transversal segment centered at Q. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $P \in B(Q, \delta)$ then the trajectory $y(t; P)$ intersects l for some $|t| < \varepsilon$.

Proof: Use the implicit function theorem.

(d)CLAIM (**heart of the proof**): Let $y(t; P)$ be a trajectory defined for $t \in I \subseteq \mathbb{R}$. Let $l \subseteq D$ be transversal. Then, if $[a, b] \subseteq I$, there is at most a *finite* number of values $a \leq t_1 < t_2 < ... < t_n \leq b$, such that $y(t_j; P) \in l$.

Furthermore, the points $S_1 = y(t_1; P), ..., S_n = y(t_n; P)$ are **monotonically ordered** on l.

Proof: Using the **Jordan curve theorem**.

(e)COROLLARY: The set of intersection points of the trajectory $y(t; P)$ with l is *at most* a monotonically ordered sequence.

(f)BACK TO $y(t; \overline{P})$: Let $t_j \uparrow \infty$ be a sequence such that the points $S_j = y(t_j; \overline{P})$ converge to some $Q \in K$ (exists because K is compact and the trajectory is assumed to be contained in it).

We can assume further that $t_{j+1} - t_j > 2$.

(g) Let l be transversal, centered at Q (from (f)). Take $\varepsilon = 1$ in (c) and let $\delta > 0$ be the radius given by the Lemma. By discarding a finite number of points, we can assume that $\{S_j\}_{j=1}^{\infty} \subseteq B(Q, \delta)$. The trajectory $y(t; S_j)$ intersects l for some $|\tau_j| < 1$. Thus the points $S_j^* = y(t_j + \tau_j; \overline{P})$ are on l and are monotonically ordered.

Note: The sequence $\{t_j + \tau_j\}$ is increasing.

(h) If the sequence $\left\{S_j^*\right\}$ is finite then $y(t; \overline{P})$ is periodic (points for two different t's coincide)–contrary to hypothesis.

Thus, the sequence $\left\{S_j^*\right\}$ is infinite, converging to Q.

(i) Consider the trajectory $y(t; Q)$. By the maximal interval theorem (and continuity wrt initial data) we have $y(t; Q) \subseteq K, \quad t \in [0, \infty)$.

FURTHERMORE, all points of $y(t; Q)$ are limit points of $y(t; \overline{P})$(as in (f)). This also follows from continuity wrt initial data (for any point $y(s; Q)$, take $\left\{y(t_j + s; \overline{P})\right\}$).

(j) Let $s_j \uparrow \infty$ be a sequence such that the points $W_j = y(s_j; Q)$ converge to some $W \in K$ (exists because K is compact and the trajectory was shown to be contained in it).

(k) Let l be transversal centered at W. As in (g), there exists a monotonically ordered set of points $W_j^* = y(s_j^*; Q) \in l$.

(l) If the sequence is finite, then $y(t; Q)$ is periodic, concluding the proof.

(m) Otherwise, The sequence $\left\{W_j^* = y(s_j^*; Q)\right\}_{j=1}^{\infty}$ is infinite, contained in l, and each point is (by (h)) a *limit point* of an *infinite* sequence of points on $y(t; \overline{P})$ (intersection with l).

BUT THIS IS A CONTRADICTION TO THE MONOTONICITY (e).

(n) CLAIM: The set of limit points of $y(t; \overline{P})$ (in the sense of (f)) is **connected**.

(o) CONCLUSION: The periodic trajectory $y(t; Q)$ is the set of *all* limit points of $y(t; \overline{P})$.

Q.E.D.

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