PROPERTIES OF THE SOLUTIONS

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Notation

- Euclidean norm $|x|^2 = \sum_{i=1}^{n} x_i^2$ in $\mathbb{R}^n$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- An open BOX in $\mathbb{R}^n$ is $Q = \prod_{i=1}^{n}(a_i, b_i)$. The corresponding closed box is $\overline{Q} = \prod_{i=1}^{n}[a_i, b_i]$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.

  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.

  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m)$ which are continuously differentiable up to (including) order k.

  (d) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.

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- **REMINDER: BASIC ASSUMPTIONS IN WHAT FOLLOWS**
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- Here $n = m + 1$ and $D \subseteq \mathbb{R}^n$ is OPEN.
- A point in D is denoted by $(t, x) \in \mathbb{R} \times \mathbb{R}^m$.
- **ASSUMPTIONS ON** f: $f \in C(D, \mathbb{R}^m)$ and satisfies:
- (**Lipschitz continuity**): Let $K \subset D$ be compact. Then there exists a constant $L_K > 0$ such that for any two points $(t, \xi^1), (t, \xi^2) \in K$,

$$|f(t, \xi^1) - f(t, \xi^2)| < L_K |\xi^1 - \xi^2|.$$

- Note: We take the *same* t.
- **ALL FUNCTIONS BELOW ARE ASSUMED TO SATISFY THESE ASSUMPTIONS.**
- **THE EQUATION:**

$$(*) \quad y'(t) = f(t, y(t)), \quad t \in I, \quad (t, y(t)) \in D.$$

- **INITIAL VALUE PROBLEM=IVP**: Suppose that $(t_0, y^0) \in D$.

  Find a solution of (*), in some open interval $I \subseteq \mathbb{R}$, such that

$$(**) \qquad t_0 \in I, \quad y(t_0) = y^0.$$

- **FUNDAMENTAL NEW FORMULATION**
- $y(t) \in C(I, \mathbb{R}^m)$ is a solution of the initial value problem if and only if $(t, y(t)) \in D$ for all $t \in I$ and

$$(***) \qquad y(t) = y^0 + \int_0^t f(s, y(s))ds, \quad t \in I.$$

- **IMPORTANT OBSERVATION**: Equation (***) implies that $y(t) \in C^1(I, \mathbb{R}^m)$.
- **MOST GENERAL EXISTENCE THEOREM** (Euler 1768, Cauchy 1820-1830).
- **Theorem.** *Let the closed set ("cylinder")* $\Gamma = [t_0 - \eta, t_0 + \eta] \times \overline{B(y^0, \alpha)} \subseteq D$.

  Let $M = \max_{(t,x)\in\Gamma} |f(t, x)|$ *and assume that* $M\eta < \alpha$.

  Then the initial value problem (*)-(**) has a solution in $I = (t_0 - \eta, t_0 + \eta)$.
- "DEFINITION" (just for this summary): The set Γ will be called the "local existence cylinder" (centered at (t_0, y^0)).

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- **MAXIMAL INTERVAL OF EXISTENCE**
- **Theorem.** *Consider the IVP (*)-(**). Then there exists an open interval(finite or infinite)* $I^{max} = (t_{min}, t_{max}) \subseteq \mathbb{R}$ *such that*

  (i) $t_0 \in I^{max}$ *and the IVP (*)-(**) has a solution* $y(t)$ *in* I^{max}.

  (ii) If $z(t)$ *is any solution to the IVP in some interval* $J \subseteq \mathbb{R}$ *(containing* t_0*) then* $J \subseteq I^{max}$ *and* $z(t) \equiv y(t), \quad t \in J$.

  (iii) If $t_{max} < \infty$ *then the solution* $y(t)$ *"escapes to the boundary" of* D *as* $t \to t_{max}$ *in the following sense:*

  For every compact $K \subseteq D$ *there exists an* $\varepsilon > 0$ *such that*

$$t > t_{max} - \varepsilon \Rightarrow (t, y(t)) \notin K.$$

A similar statement holds for t_{min}.

Proof. (a) Uniqueness is always guaranteed by the assumed Lipschitz continuity.

(b) Let Φ be the collection of all *finite* intervals $I = (a_I, b_I) \ni t_0$ such that a solution exists in I.

Define:

$$t_{min} = \inf_{I \in \Phi} a_I, \quad t_{max} = \sup_{I \in \Phi} b_I.$$

(c) By uniqueness the solution exists in $I^{max} = (t_{min}, t_{max})$. However, if $t_{max} < \infty$ and $y(t)$ can be extended continuously to t_{max} then we can construct a local existence cylinder centered at $(\tau, y(\tau)) \in D$, with $\tau < t_{max}$ but sufficiently close, so that the cylinder contains $(t_{max}, y(t_{max})) \in D$, and therefore the solution can be extended beyond $(t_{max}, y(t_{max})) \in D$, contradicting the maximality of I^{max}.

(d) To prove (iii) of the theorem suppose that $t_{max} < \infty$ and that there exists a compact $K \subseteq D$ such that for some sequence $\{t_j\} \uparrow t_{max}$ we have $\{(t_j, y(t_j))\} \subseteq K$. By restricting to a subsequence (and not changing notation) we can assume that $(t_j, y(t_j)) \to (t_{max}, y^*) \in K$.

Let $U \subseteq D$ be open such that $K \subseteq U$ and $\overline{U} \subseteq D$ is compact. Let $M = \max_{(s,x) \in \overline{U}} |f(s,x)|$. There exists a local existence cylinder

$$\Gamma_j = [t_j - \eta, t_j + \eta] \times \overline{B(y(t_j), \alpha)} \subseteq \overline{U},$$

centered at $(t_j, y(t_j))$ such that $M\eta < \alpha$ (see the above notation– note that η, α are independent of j), and such that, for sufficiently large j, $(t_{max}, y^*) \in \Gamma_j$ (and, in fact, is an interior point of the cylinder).

Clearly the whole solution segment

$$\{(t, y(t)), \quad t_j \leq t < t_{max}\} \subseteq \Gamma_j,$$

and it can be extended further (centered at (t_{max}, y^*)), which is a contradiction. □

- **REMARK**: In the special case that $D = \mathbb{R} \times \mathbb{R}^m$, i.e., that $f(t, y)$ is everywhere defined (and satisfies the Lipschitz condition), the fact that $t_{max} < \infty$ implies $|y(t)| \to \infty$ as $t \uparrow t_{max}$.
- **EXAMPLE:** Consider the scalar equation

$$y'(t) = (t^2 + y(t)^2) \sin(y(t)e^t), \quad y(0) = 1.$$

The function $z(t) = y(t)e^t$ satisfies

$$z'(t) = z(t) + y'(t)e^t = z(t) + (t^2 e^t + z(t)^2 e^{-t}) \sin(z(t)), \quad z(0) = 1.$$

We show that $t_{max} = +\infty$ for this equation (hence also for the y equation).

If not, then $0 < t_{max} < \infty$. Take a positive integer l so large that $\pi(2l - \frac{1}{2})e^{-t_{max}} > 1$. Suppose that, for some $0 < t_1 < t_{max}$, we have $z(t_1) = \pi(2l - \frac{1}{2})$. Then from the equation, since $sin(z(t_1) = -1$,

$$z'(t_1) < z(t_1) - z(t_1)^2 e^{-t_1} \leq z(t_1) - z(t_1)^2 e^{-t_{max}} < 0,$$

so $z(t) \leq \pi(2l - \frac{1}{2}),\ t < t_{max}$. Similarly we show that $z(t)$ is bounded from below, so that $t_{max} = +\infty$, and similarly $t_{min} = -\infty$.

- **COROLLARY-THE LINEAR CASE.**

  Take $D = I \times \mathbb{R}^m$, where $I = (\alpha, \beta) \subseteq \mathbb{R}$ is a finite or infinite open interval.

  Let $f(t, y) = A(t)y + b(t)$, where $A(t) \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ is a continuous $m \times m$ matrix function and $b(t) \in C(I, \mathbb{R}^m)$ is a continuous vector function.

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Theorem. *Under these assumption, for every $(t_0, y^0) \in D$ there exists a unique solution of the IVP (*)-(**), which exists for all $t \in I$ (i.e., its maximal interval is I).*

Proof. Suppose to the contrary that $t_{max} < \beta$. By the preceding remark, we reach a contradiction by showing that

$$\limsup_{t \uparrow t_{max}} |y(t)| < \infty.$$

Let

$$M = \max_{t_0 \leq \tau \leq t_{max}} \|A(\tau)\|, \quad \|A(\tau)\| \text{ is matrix norm.}$$

The equation yields

$$\frac{d}{dt}|y(t)|^2 = 2(y(t), y'(t)) = 2(y(t), A(t)y(t)) + 2(y(t), b(t)),$$

so that the (scalar) nonnegative function $\xi(t) = |y(t)|^2$ satisfies

$$\xi'(t) \leq (2M + 1)\xi(t) + N, \quad N = \max_{t_0 \leq \tau \leq t_{max}} |b(\tau)|^2.$$

Hence

$$\frac{d}{dt}(\xi(t)e^{-(2M+1)t}) \leq Ne^{-(2M+1)t},$$

from which it follows that

$$\limsup_{t \uparrow t_{max}} |y(t)|^2 = \limsup_{t \uparrow t_{max}} \xi(t) < \infty.$$

(complete the details).

□

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- **STABILITY WITH RESPECT TO CHANGE OF f.**
- We study the dependence of the IVP (*)-(**) on a *variation* of the function f. We use the above notation I^{max} for the maximal interval related to the IVP (*)-(**).
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Theorem. *Let $\{f_j\}_{j=1}^{\infty} \subseteq C(D, \mathbb{R}^m)$ be a sequence satisfying the Lipschitz condition (possibly with constants depending on j!). Suppose that $f_j \to f$, uniformly in every compact $K \subseteq D$. Let $D \ni (t_j, x^j) \to (t_0, y^0)$ and let $y^j(t)$ be the solution to the IVP:*

$$(y^j)'(t) = f_j(t, y^j(t)), \quad (t, y^j(t)) \in D,$$

$$y^j(t_j) = x^j.$$

Denote by $I^{j,max}$ the maximal interval of existence for this IVP.

Let the finite closed interval $[p, q] \subseteq I^{max}$. Then $[p, q] \subseteq I^{j,max}$ for sufficiently large j and

$$y^j(t) \to y(t), \quad \textit{uniformly in} \quad t \in [p, q].$$

Remark: Recall that we always assume also that f satisfies the Lipschitz condition.

Proof. – (a) Without loss of generality we can assume that $p \le t_0$ and that $t_j \in [p, q]$, since otherwise we can "extend" slightly the interval $[p, q]$ (for sufficiently large j, of course).

In the following arguments, we will use (sometimes implicitly) the index j in the sense of "sufficiently large" j.

– (b) Let $\delta > 0$ be sufficiently small, so that the set

$$K = \{(t, x), \quad t \in [p, q], \quad |x - y(t)| \le \delta\}$$

is compact and contained in D.

– Let $K \subseteq U \subseteq D$ where U is open and $\overline{U} \subseteq D$ is compact.

– Let $M = \sup_{1 \le j < \infty} \max_{(t,x) \in \overline{U}} |f_j(t, x)| < \infty$.

(why is it finite?)

– Let $\eta, \alpha > 0$ with $2M\eta < \alpha$ so that the local existence cylinder $\Gamma_j = [t_j - 2\eta, t_j + 2\eta] \times \overline{B(x^j, \alpha)} \subseteq U$ for every j (again, we start with sufficiently large j).

– Assume further that
$$\alpha < \frac{1}{6}\delta.$$
For j sufficiently large we have
$$t_j \in [t_0 - \eta, t_0 + \eta], \quad |x^j - y^0| < \frac{1}{12}\delta.$$
– Since
$$|x^j - y^j(t_0)| = |y^j(t_j) - y^j(t_0)| \le M\eta < \frac{1}{12}\delta$$
we have $|y^j(t_0) - y(t_0)| < \frac{1}{6}\delta$ so that
$$|y^j(t) - y(t)| \le |y^j(t) - y^j(t_0)| + |y^j(t_0) - y(t_0)| + |y(t_0) - y(t)|,$$
$$< 2M\eta + \frac{1}{6}\delta + 2M\eta < \frac{1}{2}\delta, \quad t \in [t_0, t_0 + 2\eta], \quad j \quad \text{sufficiently large.}$$
– (c) Suppose that (passing to a subsequence, if needed, without changing index) for every j there exists a point $\tau_j \in [t_0, q]$ such that $|y^j(\tau_j) - y(\tau_j)| = \delta$. We can assume that τ_j is the *first* such point and by the above $\tau_j > t_0 + 2\eta$. We can further assume that $\tau_j \to \tau^* \in [t_0 + 2\eta, q]$.
– (d) Denote $\bar\tau = \tau^* - \eta \ge t_0 + \eta$. Since $\bar\tau < \tau_j$ (for j sufficiently large) we have $[t_0, \bar\tau] \subseteq I^{j,max}$.
The solutions $\{y^j\}$ are uniformly bounded and equicontinuous on $[t_0, \bar\tau]$ (why? contained in K) and
$$|y(\bar\tau) - y^j(\bar\tau)| \ge |y(\tau^*) - y^j(\tau^*)| - 2M\eta \ge \frac{1}{2}\delta.$$
– (e) Using the Arzela-Ascoli theorem there is a subsequence (we again do not change index) $\{y^j\}$ which converges uniformly to some function $z(t) \in C([t_0, \bar\tau], \mathbb{R}^m)$.
– (f) Since
$$y^j(t) = x^j + \int_{t_j}^t f_j(s, y^j(s))ds, \quad t \in [t_0, \bar\tau],$$
we have in the limit
$$z(t) = y^0 + \int_{t_0}^t f(s, z(s))ds.$$
– (g) We conclude by uniqueness that $z(t) \equiv y(t) \quad t \in [t_0, \bar\tau]$ which contradicts the fact (see (d)) that $|y(\bar\tau) - z(\bar\tau)| \ge \frac{1}{2}\delta$.

– (h) We conclude that for the given $\delta > 0$, for all sufficiently large j,

$$|y^j(t) - y(t)| \le \delta, \quad t \in [t_0, q],$$

and a similar argument for the interval $[p, t_0]$.

– (i) In particular, $[p, q] \subseteq I^{j,max}$ for sufficiently large j.

□

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- **STABILITY AND CONTINUITY WITH RESPECT TO THE INITIAL POINT**
- SPECIAL CASE: In the above treatment, take $f_j \equiv f$ for all j.
- Let $(\tau, x) \in D$ and let $y(t; \tau, x)$ be the solution of (*) satisfying $y(\tau) = x$.
- We now regard the maximal interval of existence as *a function of the initial data*:

$$I^{max}_{\tau,x} = (t_{min}(\tau, x), t_{max}(\tau, x)).$$

As a direct corollary of the above stability theorem we have:

- **Theorem.** *(i)*

$$\liminf_{(\tau,x)\to(t_0,y^0)} t_{max}(\tau, x) \ge t_{max}(t_0, y^0),$$

$$\limsup_{(\tau,x)\to(t_0,y^0)} t_{min}(\tau, x) \le t_{min}(t_0, y^0).$$

(ii) Let $[a, b] \subseteq I^{max}_{t_0,y^0}$. The solution $y(t; \tau, x)$ converges uniformly in $[a, b]$ to $y(t; t_0, y^0)$ as $(\tau, x) \to (t_0, y^0)$.

- REMARK: It follows that $t_{max}(\tau, x)$ is *lower semicontinuous* as a function of (τ, x). Similarly, $t_{min}(\tau, x)$ is *upper semicontinuous* as a function of (τ, x).
- Let $E \subseteq \mathbb{R} \times D$ be the existence set of the solution $y(t; \tau, x)$:

$$E = \left\{(t, \tau, x) \in \mathbb{R} \times D, \quad t \in I^{max}_{\tau,x}\right\}.$$

- **CLAIM**: E is open in $\mathbb{R} \times D$.
- EXERCISE: Prove this!
- **THE FLOW MAP**

Take $(t_0, y^0) \in D$. Let $W \subseteq \mathbb{R}^m$ be an open neighborhood of y^0, so that

$$\{t_0\} \times W \subseteq D.$$

The following theorem says that if we take a closed time interval contained in $I^{max}_{t_0,y^0}$, we can choose W sufficiently small so that*the* *solution beginning* at (t_0, x), $x \in W$ exists in this interval and it maps W, one-to-one, onto open neighborhoods.

- **Theorem.** *given* $[t_0 - T, t_0 + T] \subseteq I^{max}_{t_0,y^0}$, *we can choose an open neighborhood* $W \subseteq \mathbb{R}^m$ *of* y^0, *so that* $\{t_0\} \times W \subseteq D$, *and such that*
  (1) $[t_0 - T, t_0 + T] \subseteq I^{max}_{t_0,x}, \quad x \in W$.
  Definition: *The map* $\Phi_\tau(x) = y(\tau; t_0, x)$, $x \in W$, *is called the flow map.*
  (2) *For every* $\tau \in [t_0 - T, t_0 + T]$ *the flow map is one-to-one and its image* $W_\tau = \{\Phi_\tau(x), \ x \in W\} \subseteq \mathbb{R}^m$ *is an open neighborood of* $y(t_0; \tau, y^0)$.
  (3) *The map* Φ_τ *is open; an open subset of* W *is mapped onto an open subset of* W_τ.
  (4) *There exists* $r > 0$ *such that for every* $\tau \in [t_0 - T, t_0 + T]$ *the ball* $B(y(\tau; t_0, y^0), r) \subseteq W_\tau$.

  Proof. (1) If there is no such W, we can find a sequence $x^j \to y^0$, such that $I^{max}_{t_0,x^j}$ does not contain $[t_0 - T, t_0 + T]$, contrary to the previous theorem.
  (2) Φ_τ is one-to-one by uniqueness of the solution; if $\Phi_\tau(\xi) = \Phi_\tau(\zeta)$ then the solution beginning at this point arrives, at $t = t_0$, to the points ξ, ζ, which must be the same.
  If W_τ is not open, there exists $\Phi_\tau(\xi) \in W_\tau$ which is not interior. But we can take a small ball $B_\delta = B(\Phi_\tau(\xi), \delta)$ so that all solutions $y(t; \tau, z)$, $z \in B$, exist at $t = t_0$, and by the previous theorem $y(t_0; \tau, z) \in W$ if $\delta > 0$ is small. Then B is in the image W_τ of W under Φ_τ, a contradiction.
  (3) This is already contained in the previous argument of the proof, as W can be replaced by any open subset.
  (4) Otherwise there will be a sequence $\{\tau_j\}_{j=1}^\infty \subseteq [t_0 - T, t_0 + T]$ and $\{z^j\}_{j=1}^\infty$ such that $|z^j - y(\tau_j; t_0, y^0)| \to 0$ but $z^j \notin W_{\tau_j}$. Taking a subsequence (without changing index) we have $\tau_j \to \tau \in [t_0 - T, t_0 + T]$ so $z^j \to y(\tau; t_0, y^0)$. But then by the previous theorem the interval of existence of $y(t; \tau_j, z^j)$ contains $t = t_0$ (for sufficiently large j) and $y(t_0; \tau_j, z^j) \to y(t_0; \tau, y(\tau; t_0, y^0)) = y^0$. In particular $y(t_0; \tau_j, z^j) \in W$, so $z^j = \Phi_{\tau_j}(y(t_0; \tau_j, z^j)) \in W_{\tau_j}$, a contradiction.

  □

- EXAMPLE: Consider the scalar equation $(m = 1)$

$$y'(t) = y(t)^2 - \varepsilon y(t)^4, \quad \varepsilon \geq 0.$$

For $y(0) = 1$ we have $t_{max} = 1$ if $\varepsilon = 0$, but $t_{max} = +\infty$ if $\varepsilon > 0$. (Prove this!).

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- **DEPENDENCE ON PARAMETERS**
- Let $\mathcal{P} \subseteq \mathbb{R}^p$ be open and suppose that

$$f(t, y; \mu) : D \times \mathcal{P} \to \mathbb{R}^m.$$

- Consider the IVP

$$y' = f(t, y; \mu), \quad y(t_0) = y^0, \quad (t_0, y^0) \in D, \quad \mu \in \mathcal{P}.$$

- DEFINITION: We say that the IVP *depends* on a parameter μ in the *parameter domain* $\mathcal{P}$.
- REMARK: We assume that $f(t, y; \mu)$ satisfies the (local) Lipschitz condition with respect to y, possibly with constants depending on μ. Let $(\tau, x, \mu) \in D \times \mathcal{P}$. Then the IVP has a unique solution $y(t; \tau, x, \mu)$, such that $y(\tau; \tau, x, \mu) = x$.

  The maximal interval of existence depends of course on μ. We denote it by $I^{max}_{\tau,x,\mu} = (t_{min}(\tau, x, \mu), t_{max}(\tau, x, \mu))$.

Theorem. *Let $f(t, y; \mu) : D \times \mathcal{P} \to \mathbb{R}^m$ be continuous in all variables and satisfy a (local) Lipschitz condition in y for fixed $(t, \mu) \in D \times \mathcal{P}$.*

Fix $(t_0, y^0, \mu^0) \in D \times \mathcal{P}$.

Then:

(i)

$$\liminf_{(\tau,x,\mu)\to(t_0,y^0,\mu^0)} t_{max}(\tau, x, \mu) \geq t_{max}(t_0, y^0, \mu^0),$$

$$\limsup_{(\tau,x,\mu)\to(t_0,y^0,\mu^0)} t_{min}(\tau, x, \mu) \leq t_{min}(t_0, y^0, \mu^0).$$

(ii) Let $[a, b] \subseteq I^{max}_{t_0,y^0,\mu^0}$. The solution $y(t; \tau, x, \mu)$ converges uniformly in $[a, b]$ to $y(t; t_0, y^0, \mu^0)$ as $(\tau, x, \mu) \to (t_0, y^0, \mu^0)$.

Proof. Nice trick: Define a continuous function $g : D \times \mathcal{P} \to \mathbb{R}^{m+p}$ by

$$g(t, y, \mu) = (f(t, y; \mu), 0) \in \mathbb{R}^m \times \mathbb{R}^p.$$

Clearly g is continuous in its variables and satisfies a (local) Lipschitz condition in (y, μ).

For $z(t) \in \mathbb{R}^{m+p}$ solve the IVP

$$z'(t) = g(t, z(t)), \quad z(\tau) = (x, \mu), \quad (t, z(t)) \in D \times \mathcal{P}.$$

By uniqueness

$$z(t) = (y(t;\tau,x,\mu),\mu), \quad t \in I^{max}_{\tau,x,\mu},$$

and all the assertions follow from the previous theorem (dependence on initial data). □

- REMARK: Thus, the parameter μ "has the status" of the initial data y^0 in what concerns the dependence of the solution on these data. In what follows we therefore omit the parameter.

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- **REGULARITY OF THE SOLUTION**
- We return to the solution $y(t;\tau,x)$ of the IVP (*), with $y(\tau;\tau,x) = x$. . We know it is continuous (as a function of t,τ,x) on

$$E = \left\{(t,\tau,x) \in \mathbb{R} \times D, \quad t \in I^{max}_{\tau,x}\right\}.$$

- In addition, by the equation, it is *continuously differentiable* with respect to t.
- We show that if f is more regular, then so is the solution.

Theorem. *Suppose that for some integer $l \geq 1$, the function $f(t,y) \in C^l(D,\mathbb{R}^m)$. Fix $\tau = t_0$. Then the solution $y(t;t_0,x)$ is* $l+1$ times continuously differentiable *with respect to t and l* times continuously differentiable *with respect to x.*

Proof. Take first $l = 1$.
For notational simplicity we assume $m = 1$, so that the unknown y is a scalar (there is no loss of generality, otherwise we consider components of y).
The fact that y is twice continuously differentiable with respect to t follows from the equation (*) and the assumed differentiability of f, since

$$y''(t;t_0,x) = \frac{\partial f}{\partial t}(t,y(t;t_0,x)) + \frac{\partial f}{\partial y}(t,y(t;t_0,x))f(t,y(t;t_0,x)).$$

- To prove differentiability with respect to x, we fix $b > t_0$ such that $[t_0,b] \subseteq I^{max}_{t_0,\tilde{x}}$, the maximal interval for $y(t;t_0,\tilde{x})$, where $\tilde{x} \in (x-h_0,x+h_0)$ for some $h_0 > 0$.
  Let $K = \{(t,y(t;t_0,x)),\ t \in [t_0,b]\}$ be the (compact) graph of y on $[t_0,b]$ and let $U \subseteq D$ be open , such that $K \subseteq U$ and $\bar{U} \subseteq D$ is compact.
  We denote $N = \max_{\bar{U}} |\frac{\partial f(t,y)}{\partial y}|$.
- By the theorem on continuous dependence

$$\|y(t;t_0,x+h) - y(t;t_0,x)\| = \max_{t\in[t_0,b]} |y(t;t_0,x+h) - y(t;t_0,x)| \xrightarrow[h\to 0]{} 0,$$

so that, for $h_0 > 0$ sufficiently small,

$$\|y(t;t_0,x+h) - y(t;t_0,x)\| < \frac{1}{2} dist(K, D \setminus U), \quad |h| < h_0.$$

In particular, the union of all graphs

$$\{(t, y(t;t_0,x+h)),\ t \in [t_0,b],\ |h| < h_0\} \subseteq U.$$

We have

$$y(t;t_0,x) = x + \int_{t_0}^{t} f(s, y(s;t_0,x))ds, \quad t \in [t_0,b].$$

Subtracting it from the same equation with x replaced by $x+h$ we get

$$\frac{y(t;t_0,x+h) - y(t;t_0,x)}{h}$$

$$= 1 + \int_{t_0}^{t} h^{-1}(f(s, y(s;t_0,x+h)) - f(s, y(s;t_0,x)))ds, \quad 0 < |h| < h_0.$$

Denote for simplicity $z^h(t) = \frac{y(t;t_0,x+h)-y(t;t_0,x)}{h}$. By the mean value theorem the last equality gives

$$|z^h(t)| \le 1 + \int_{t_0}^{t} N|z^h(s)|ds, \quad t \in [t_0,b].$$

CLAIM: For every integer r,

$$|z^h(t)| \le \sum_{j=0}^{r} \frac{(N(t-t_0))^r}{r!} + \frac{(N(t-t_0))^{r+1}}{(r+1)!}\|z^h\|.$$

The case $r = 0$ is obtained from the above inequality:

$$|z^h(t)| \le 1 + N(t-t_0) \max_{s\in[t_0,b]} |z^h(s)| = 1 + N(t-t_0)\|z^h\|, \quad t \in [t_0,b],$$

and the general case is obtained by induction (as in the proof of Picard's theorem).
Letting $r \to \infty$ we obtain

$$|z^h(t)| \le e^{N(t-t_0)}, \quad t \in [t_0,b].$$

(Remark: This is in fact a simple consequence of Gronwall's inequality.)

- COROLLARY: The family $\left\{z^h(t)\right\}_{|h|<h_0}$ is uniformly bounded in $t \in [t_0, b]$.
- From

$$z^h(t) = 1 + \int_{t_0}^{t} h^{-1}(f(s, y(s; t_0, x+h)) - f(s, y(s; t_0, x)))ds, \quad 0 < |h| < h_0,$$

it now follows that

$$|z^h(t+\delta) - z^h(t)|$$

$$\leq \int_{t}^{t+\delta} h^{-1}|(f(s, y(s; t_0, x+h)) - f(s, y(s; t_0, x)))|ds \leq N|\delta|\|z^h\|, \quad 0 < |h| < h_0.$$

- CONCLUSION: The family $\left\{z^h(t)\right\}_{|h|<h_0}$ is equicontinuous in $t \in [t_0, b]$.
- By the Arzela-Ascoli theorem there exists a subsequence $\left\{z^{h_j}(t)\right\}$ with $h_j \to 0$ that converges uniformly (on $[t_0, b]$) to a function $w(t; t_0, x)$.
  From the equation and the continuity of $\frac{\partial f}{\partial y}$ and $y(s; t_0, x+h)$ it follows that

$$w(t; t_0, x) = 1 + \int_{t_0}^{t} \frac{\partial f}{\partial y}(s, y(s; t_0, x))w(s; t_0, x)ds.$$

- It follows that the limit function w satisfies the LINEAR IVP:

$$\frac{d}{dt}w(t; t_0, x) = \frac{\partial f}{\partial y}(t, y(t; t_0, x))w(t; t_0, x), \quad t \in [t_0, b], \quad w(t_0; t_0, x) = 1.$$

By uniqueness (of solutions to linear equations) the whole family $\left\{z^h(t)\right\}_{|h|<h_0}$ converges to the same limit w as $h \to 0$, so that , by definition, it is the derivative with respect to x:

$$w(t; t_0, x) = \frac{\partial y(t; t_0, x)}{\partial x}, \quad t \in [t_0, b], \quad w(t_0; t_0, x) = 1.$$

- NOTE in particular that ,as a solution of a linear equation, $\frac{\partial y(t;t_0,x)}{\partial x}$ is defined for $t \in I^{max}_{t_0,x}$.
- Suppose $l > 1$.

We refer to the equation

$$\frac{\partial y(t;t_0,x)}{\partial x} = 1 + \int_{t_0}^{t} \frac{\partial f}{\partial y}(s, y(s;t_0,x))\frac{\partial y(s;t_0,x)}{\partial x} ds,$$

as we referred before to y. Repeating the same reasoning (based on the fact that f is at least twice continuously differentiable), we get

$$\frac{\partial^2 y(t;t_0,x)}{\partial x^2}$$

$$= \int_{t_0}^{t} [\frac{\partial^2 f}{\partial y^2}(s, y(s;t_0,x))\big(\frac{\partial y(s;t_0,x)}{\partial x}\big)^2 + \frac{\partial f}{\partial y}(s, y(s;t_0,x))\frac{\partial^2 y(s;t_0,x)}{\partial x^2}]ds,$$

or, in other words, that $\frac{\partial^2 y(t;t_0,x)}{\partial x^2}$ satisfies a LINEAR IVP (as function of t)

$$\begin{aligned}\frac{d}{dt}\frac{\partial^2 y(t;t_0,x)}{\partial x^2} &= \frac{\partial^2 f}{\partial y^2}(t, y(t;t_0,x))(\frac{\partial y(t;t_0,x)}{\partial x})^2 \\ &\quad + \frac{\partial f}{\partial y}(t, y(t;t_0,x))\frac{\partial^2 y(t;t_0,x)}{\partial x^2}, \\ \frac{\partial^2 y(t_0;t_0,x)}{\partial x^2} &= 0 \quad .\end{aligned}$$

– It is clear how to do higher order pure $x-$ derivatives; at each level the highest order derivative satisfies a LINEAR equation.
– The pure $t-$ derivatives are simpler–just differentiate the equation (*) with respect to t. Justification is simple;the right hand side at each step is differentiable with respect to t by the chain rule.
For mixed derivatives, we do first the $t-$derivatives and then proceed with the $x-$ derivatives as above.

□

- REMARK(Differentiability with respect to parameter): recall that parameters have the "status" of the initial data x, so if $f(t, y; \mu)$ is $l-$times continuously differentiable with respect to (t, y, μ) then the solution is $l-$times continuously differentiable with respect to μ.

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