BASIC THEOREMS ON EXISTENCE AND UNIQUENESS

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Notation

- Euclidean norm $|x|^2 = \sum_{i=1}^{n} x_i^2$ in $\mathbb{R}^n$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- An open BOX in $\mathbb{R}^n$ is $Q = \prod_{i=1}^{n}(a_i, b_i)$. The corresponding closed box is $\overline{Q} = \prod_{i=1}^{n}[a_i, b_i]$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.
  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.
  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m)$ which are continuously differentiable up to (including) order k.
  (d) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.

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- **A "DIFFERENTIAL EQUATION" MEANS FINDING AN "UNKNOWN FUNCTION"**
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- Here $n = m + 1$ and $D \subseteq \mathbb{R}^n$ is OPEN.
- A point in D is denoted by $(t, x) \in \mathbb{R} \times \mathbb{R}^m$.
- **PROBLEM**: We are given a function $f \in C(D, \mathbb{R}^m)$.
  Let $I \subseteq \mathbb{R}$ be an open interval.
  Find a function $y(t) \in C^1(I, \mathbb{R}^m)$ such that:
  (a) The point $(t, y(t)) \in D$ for every $t \in I$.
  (b) $\quad (*) \quad y'(t) = f(t, y(t)), \quad t \in I.$
- This problem covers **ALL POSSIBLE** solutions of the equation (*).
- **INITIAL VALUE PROBLEM**: Suppose that $(t_0, y_0) \in D$.
  Find a solution of (*), in some open interval $I \subseteq \mathbb{R}$, such that

$$(**) \qquad t_0 \in I, \quad y(t_0) = y_0.$$

- **FUNDAMENTAL NEW FORMULATION**
- $y(t) \in C(I, \mathbb{R}^m)$ is a solution of the initial value problem if and only if $(t, y(t)) \in D$ for all $t \in I$ and

$$(***) \qquad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds, \quad t \in I.$$

- **IMPORTANT OBSERVATION**: Equation (***) implies that $y(t) \in C^1(I, \mathbb{R}^m)$.
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- **MOST GENERAL EXISTENCE THEOREM** (Euler 1768, Cauchy 1820-1830).
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Theorem. *Let the closed set ("cylinder")* $\Gamma = [t_0 - \eta, t_0 + \eta] \times \overline{B(y_0, \alpha)} \subseteq D$.

Let $M = \max\limits_{(t,x)\in\Gamma} |f(t,x)|$ *and assume that* $M\eta < \alpha$.

Then the initial value problem (*)-(**) has a solution in $I = (t_0 - \eta, t_0 + \eta)$.

- *Proof.* (a) For any integer $p > 2$ define a (finite) set of points in $\bar{I}$ by:

$$t_k^p = t_0 + \frac{k}{p}\eta, \quad k = -p, ..., -1, 0, 1, ..., p.$$

(b) Define (for the same p) points in $B(y_0, \alpha)$ by $z^{0,p} = y_0$ and

$$z^{k,p} = z^{\tilde{k},p} + sgn(k)\frac{\eta}{p} f(t_{\tilde{k}}^p, z^{\tilde{k},p}), \quad k = \pm 1, ..., \pm p.$$

where $\tilde{k} = sgn(k)(|k| - 1)$.

(c) Define a corresponding "piecewise linear" function $z^p(t), \quad t \in I$, by

$$z^p(t) = \frac{p}{\eta}[(t_{k+1}^p - t)z^{k,p} + (t - t_k^p)z^{k+1,p}], \quad t \in [t_k^p, t_{k+1}^p],$$
$$k = -p, -(p-1), ..., -1, 0, 1, ..., (p-1).$$

(d) The functions $\{z^p(t)\}_{p=2}^{\infty}$ are uniformly bounded and equicontinuous (Lipschitz constant $\leq M$) on $\bar{I}$.

(e) By the Arzela-Ascoli theorem there is a (uniformly) convergent subsequence $\{z^{p_j}\}$. Call the limit function $y(t), \quad t \in I$.

(f) Let $\varepsilon > 0$. By uniform continuity of f in Γ there exists $\delta > 0$ such that if $\xi^1, \xi^2 \in \Gamma$ and $|\xi^1 - \xi^2| < \delta$ then $|f(\xi^1) - f(\xi^2)| < \varepsilon$.

(g) In particular, if $(\frac{\eta}{p})^2 + |z^{k+1,p} - z^{k,p}|^2 < \delta^2$ then

$$|z^p(t) - z^p(t_k^p) - \int_{t_k^p}^t f(s, z^p(s))ds|$$

$$= |\int_{t_k^p}^t (f(t_k^p, z^{k,p}) - f(s, z^p(s)))ds| < \varepsilon(t - t_k^p), \quad t \in [t_k^p, t_{k+1}^p], \quad k = -p, ..., p-1.$$

(h) Likewise, if $k > 0$,

$$|z^p(t_k^p) - z^p(t_{k-1}^p) - \int_{t_{k-1}^p}^{t_k^p} f(s, z^p(s))ds| < \varepsilon(t_k^p - t_{k-1}^p), .$$

This can be continued down to t_0.

(i) Taking $p = p_j$ and passing to the limit as $j \to \infty$ we get

$$y(t) - y_0 = \int_{t_0}^t f(s, y(s))ds, \quad t \in I.$$

(j) Thus, $y(t)$ satisfies (***). □

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- REMARK: As was seen in Summary 2 (Examples), the solution is not necessarily unique.

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- **NEXT EXISTENCE AND UNIQUENESS THEOREM**
- We use the same notation as above, but make the following *additional assumption* on f.
- ASSUMPTION (**Lipschitz continuity**): Let $K \subset D$ be compact. Then there exists a constant $L_K > 0$ such that for any two points $(t, \xi^1), (t, \xi^2) \in K$,

$$|f(t, \xi^1) - f(t, \xi^2)| < L_K |\xi^1 - \xi^2|.$$

- Note: We take the *same* t.
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Theorem. *(Picard) Let the closed set ("cylinder")* $\Gamma = [t_0 - \eta, t_0 + \eta] \times \overline{B(y_0, \alpha)} \subseteq D$.

Let $M = \max_{(t,x) \in \Gamma} |f(t, x)|$ *and assume that* $M\eta < \alpha$.

Assume further that f *satisfies the above Lipschitz assumption.*

Then the initial value problem (*)-(**) has a unique solution in $I = (t_0 - \eta, t_0 + \eta)$.

- *Proof.* (a) Define on the space $C(\bar{I}, \mathbb{R}^m)$ a map by:

$$\mathcal{G}\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds.$$

(b) Let $Y \subseteq C(\bar{I}, \mathbb{R}^m)$ be the closed set of functions that assume values in $\overline{B(y_0, \alpha)}$. From the assumption $M\eta < \alpha$ it is clear that if $\phi \in Y$ then also $\mathcal{G}\phi \in Y$.

(c) So in what follows all functions considered are assumed to be in Y.

The norm on such functions is $\|\phi\| = \max_{t \in \bar{I}} |\phi(t)|$.

(d) By the Lipschitz assumption

$$|\mathcal{G}\phi(t) - \mathcal{G}\psi(t)| \leq L_\Gamma \int_{t_0}^{t} |\phi(s) - \psi(s)|ds \leq L_\Gamma \|\phi - \psi\| |t - t_0|.$$

(e) Repeating this we have

$$|\mathcal{G}^2\phi(t) - \mathcal{G}^2\psi(t)| \leq L_\Gamma \int_{t_0}^{t} |\mathcal{G}\phi(s) - \mathcal{G}\psi(s)|ds \leq L_\Gamma^2 \|\phi - \psi\| \int_{t_0}^{t} |s - t_0|ds.$$

(f) By induction therefore, for any integer p,

$$|\mathcal{G}^p\phi(t) - \mathcal{G}^p\psi(t)| \leq \frac{L_\Gamma^p}{p!} \|\phi - \psi\| |t - t_0|^p, \quad t \in \bar{I}.$$

(g) It follows that

$$\|\mathcal{G}^p\phi - \mathcal{G}^p\psi\| \leq \frac{L_\Gamma^p}{p!} \|\phi - \psi\| \eta^p.$$

(h) Choose p so large that $\frac{L_\Gamma^p}{p!}\eta^p < 1$. Then $\mathcal{G}^p$ is a *contraction* on the closed set Y.

(i) Using the contraction map theorem, let $z \in Y$ be the unique fixed point of $\mathcal{G}^p$.

(j) Let

$$y(t) = \mathcal{G}z(t).$$

Then $y(t) \in Y$ and $\mathcal{G}^p y = \mathcal{G}^p(\mathcal{G}z) = \mathcal{G}z = y$.

It follows that y is a fixed point of $\mathcal{G}^p$ in Y so by uniqueness $y = z$.

(k) Thus, $y = \mathcal{G}y$ and it follows that $y(t)$ satisfies (***).

(l) **Uniqueness**: Suppose two functions, y^1, y^2 satisfy (***) in some interval $\bar{I} = [t_0 - \eta, t_0 + \eta]$. Suppose that their values are contained in a closed ball $\overline{B(y_0, \alpha)}$ and let L_Γ be the Lipschitz constant for $\Gamma = \bar{I} \times \overline{B(y_0, \alpha)}$. By taking a smaller η, if needed, we can assume $L_\Gamma \eta < 1$.

(m) As in (d) it follows that

$$\|y^1 - y^2\| = \|\mathcal{G}y^1 - \mathcal{G}y^2| \leq L_\Gamma \eta \|y^1 - y^2\| < \|y^1 - y^2\|,$$

which is a contradiction. Thus $y^1 \equiv y^2$ in $[t_0 - \eta, t_0 + \eta]$ and the process can be continued to intervals centered at the endpoints (if the condition $L_\Gamma \eta < 1$ forced the reduction of η). □

- DEFINITION (**SUCCESSIVE APPROXIMATION**): The method of proof of the last theorem is known as *"successive approximation"*.
- Indeed, starting from *any* function $\phi \in Y$ the sequence $\left\{\mathcal{G}\phi, \mathcal{G}^2\phi, ..., \mathcal{G}^l\phi, ...\right\}$ converges to the unique solution $y(t)$.
- EXERCISE: Show this! (hint: The sequence is Cauchy, see (d)).

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