BASIC EXAMPLES AND DEFINITIONS

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Notation

- Euclidean norm $|x|^2 = \sum_{i=1}^{n} x_i^2$ in $\mathbb{R}^n$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- An open BOX in $\mathbb{R}^n$ is $Q = \prod_{i=1}^{n}(a_i, b_i)$. The corresponding closed box is $\overline{Q} = \prod_{i=1}^{n}[a_i, b_i]$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.

  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.

  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m$ which are continuously differentiable up to (including) order k.

  (d) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.

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- **A "DIFFERENTIAL EQUATION" MEANS FINDING AN "UNKNOWN FUNCTION"**
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- **SOME EXAMPLES**
- Find a function $y(t) \in C^1(\mathbb{R})$ such that $y'(t) = ky(t)$ for some $k \in \mathbb{R}$.
- Let $I = (a,b) \subseteq \mathbb{R}$ and $a(t), b(t) \in C(I)$. Find a function $y(t) \in C^1(I)$ such that

$$y'(t) = a(t)y(t) + b(t).$$

SOLUTION: Let $t_0 \in I$ and define $u(t) = \exp(-\int_{t_0}^{t} a(s)ds)$. Then

$$(u(t)y(t))' = u(t)b(t)$$

so,

$$y(t) = [y(t_0) + \int_{t_0}^{t} b(s)\exp(-\int_{t_0}^{s} a(r)dr)ds]\exp(\int_{t_0}^{t} a(s)ds).$$

- REMARK: These are all the solutions (why?) and they exist in *all of* I.
- Find a function $y(t) \in C^1(\mathbb{R})$ such that $y' = y^2$.
  SOLUTION: $y(t) \equiv 0$ is certainly a solution.
  Otherwise, take $u(t) = \frac{1}{y(t)}$, so $u'(t) = -1 \Rightarrow u(t) = \alpha - t \Rightarrow y(t) = \frac{1}{\alpha - t}$ for some real α.
- Show there are no other solutions. Conclude that only the "trivial" solution is defined on the whole line.
- Find a function $y(t) \in C^1(\mathbb{R})$ such that $y' = -y^2$.
- Find a function $0 \le y(t) \in C^1(\mathbb{R})$ such that $y' = \sqrt{y}$.
  SOLUTION: If $v(t) = \sqrt{y}$ then $v'(t) = \frac{1}{2} \Rightarrow v(t) - v(t_0) = \frac{1}{2}(t - t_0), \quad t > t_0$, since the computation assumes $v > 0$. Define $k = 2v(t_0) - t_0$ so that

$$y(t) = \frac{1}{4}(k+t)^2, \quad t \ge -k, \quad \forall k \in \mathbb{R}.$$

However, also $y \equiv 0$ is a solution and also

$$y(t) = \begin{cases} 0, & t \le t_0, \\ \frac{1}{4}(t-t_0)^2, & t \ge t_0. \end{cases}$$

- Conclude that there are *infinitely many solutions* through any point $(t_0, 0)$.
- Find a function $y(t) \in C^1(\mathbb{R})$ such that $y' = (4t^3 - 2t)y^2, \quad y(t_0) = y_0$.
  SOLUTION: Verify that $y(t) = \frac{y_0}{1 + y_0(t^2 - t^4 - t_0^2 + t_0^4)}$.
  Is this the only solution? Are there solutions defined on the whole line?

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- **BASIC DEFINITIONS**
- DEFINITION: A scalar differential equation of order n is an algebraic equation

$$F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0,$$

for an unknown function $y(t)$ defined in a certain interval $t \in (a, b) \subseteq \mathbb{R}$.

- DEFINITION: The equation is **regular** if it can be "solved" for the highest-order derivative, namely, if it can be written as

$$y^{(n)}(t) = G(t, y(t), y'(t), ..., y^{(n-1)}(t))$$

where the dependence of G on its first variable (t) is regular.
- DEFINITION: The equation is **linear** if F (first definition above) is *linear* in its second,...,last variables (namely, $y(t), y'(t), ..., y^{(n)}(t)$).
- DEFINITION: Let $f : (a, b) \times D \to \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$ is open. A **first-order system** is a system of equations of the form

$$z'(t) = f(t, z(t))$$

for an unknown *vector function* $z : (a, b) \subseteq \mathbb{R} \to D$.
- DEFINITION: The **initial value problem** for any of the above consists of finding a solution $y(t)$ (resp. $z(t)$) such that, for some $t_0 \in (a, b)$ and given values $y_0, ..., y_0^{(n-1)}$ (resp. $z_0 \in \mathbb{R}^n$),

$$y(t_0) = y_0, y'(t_0) = y_0^1, , ..., y^{(n-1)}(t_0) = y_0^{(n-1)}$$

(resp. $z(t_0) = z_0$).
- CLAIM: Any regular $n-$th order equation can be reduced to a first-order system.

  PROOF: Simply define a vector $z(t) \in \mathbb{R}^n$ by

$$(z_1(t), z_2(t), ..., z_n(t)) = (y(t), y'(t), ..., y^{(n-1)}(t)).$$

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- **FIRST-ORDER EQUATIONS IN TWO VARIABLES**

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  NOTATION: Following common use, we use "symmetric" notation x, y.

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- $$(*) \qquad M(x, y)dx + N(x, y)dy = 0.$$

  M, N are continuous (real) functions defined in some open domain $D \subseteq \mathbb{R}^2$, and such that $M^2 + N^2 > 0$.
- The meaning of the equation is that in a neighborhood of any point $(x_0, y_0) \in D$ we look for a function $y(x)$ satisfying $M(x, y(x)) + N(x, y(x))y'(x) = 0$ or a function $x(y)$ satisfying $M(x(y), y)x'(y) + N(x(y), y) = 0$.
- **EXACT EQUATIONS**: If in some open set $U \subseteq D$ there exists a function $\phi \in C^1(U)$ such that $\nabla\phi = (M, N)$ in U, we say that the equation (*) is *exact*.

- This is equivalent to saying that the (vector)field (M, N) is *conservative* in U, with *potential function* ϕ.
- In this case every solution in U is given implicitly by $\phi(x, y) = const$.
- REMINDER: If U is an open disk and $M, N \in C^1(U)$ then a *necessary and sufficient condition* for the equation to be exact in U is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- **SEPARABLE EQUATIONS**: This is a *special case* of the previous one, where $M(x, y) = A(x), N(x, y) = B(y)$. The potential is then $\phi(x, y) = \alpha(x)+\beta(y)$ where $\alpha'(x) = A(x), \beta'(y) = B(y)$.
- **INTEGRATING FACTOR**: Let $\mu(x, y) \neq 0$ and multiply (*) by μ to get

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0.$$

DEFINITION: If this equation is exact, we say that $\mu(x, y)$ is an **integrating factor** of (*).
- EXAMPLE: The equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0.$$

An integrating factor is $\mu(x, y) = x$ so that the new equation has a potential function $\phi(x, y) = x^3y + \frac{1}{2}x^2y^2$.
- **HOMOGENEOUS EQUATIONS**: The equation (*) is *homogeneous* in an open set $U \subseteq D$ if, say, $N(x, y) > 0$ in U and the quotient $\frac{M(x,y)}{N(x,y)} = -F(\frac{y}{x})$, where $F(t)$ is a continuous function of a *single* (real) variable t.
- In this case the equation (*) becomes

$$y'(x) = F(\frac{y}{x}).$$

- For the solution, introduce a new *unknown* function $v = \frac{y}{x}$, which satisfies the equation

$$xv' + v = F(v)$$

which can be rewritten as a separable equation (for $v = v(x)$).
- EXAMPLE: Solve the equation (see above, the example for integrating factor)

$$(3xy + y^2)dx + (x^2 + xy)dy = 0,$$

using the method of homogeneous equations.
- **THE RICCATI EQUATION**:

$$y'(x) = q_1(x) + q_2(x)y + q_3(x)y^2,$$

in some open interval $I \subseteq \mathbb{R}$, where the coefficients are continuous functions.

If *some* solution $y_1(x)$ is known, then defining a new unknown function $v(x)$ by

$$v(x) = \frac{1}{y(x) - y_1(x)},$$

we get a *linear* equation

$$v' = -(q_2(x) + 2q_3(x)y_1(x))v - q_3(x).$$

- **THE BRACHISTOCHRONE PROBLEM**:

  Find the curve along which a particle will slide without friction in the minimum time from one given point P to another Q, the latter being lower than the former but not directly beneath it.

  (One of the most famous problems in the history of mathematics, posed by Johann Bernoulli, 1696. Read the full story in:

  D. E. Smith, "A Source Book in Mathematics, Vol. 2", pp. 644-655).
- SOLUTION (W.E. Boyce and R. C. DiPrima, "Elementary Differential Equations and Boundary Value Problems", 3-rd Ed. Ch. 2.10, p. 69):

  (a) Take $P = (0, 0)$ and the $y-$axis directed down, so $Q = Q(x_0, y_0)$ where $x_0, y_0 > 0$.

  (b) The equation for the unknown curve $y(x)$ is

$$[1 + y'(x)^2]y(x) = k^2,$$

where $k > 0$ is a constant (determined by physical constants).

We look for monotone increasing solutions.

(c) Introduce a new unknown function u by

$$y = k^2 \sin^2 u,$$

and the equation is transformed to

$$dx - 2k^2 \sin^2 u du = 0.$$

(d) The solution is then

$$x = \frac{k^2}{2}(2u - \sin(2u)),$$

and from the definition of u,

$$y(x) = \frac{k^2}{2}(1 - \cos(2u)).$$

(d) The solution is then a graph, *parametrized* by u, namely, $x = x(u), y = y(u)$.

(e) This graph is the **cycloid**.

- **THE LOGISTIC EQUATION**(Verhulst, 1838):

$$y'(t) = y(a - by), \quad y(0) = y_0, \quad t \in [0, \infty).$$

$a, b > 0$ are (known) parameters.

- This is a model for *population evolution* where a represents the "growth rate without environmental influence" while b represents the "decrease of growth rate due to increasing population density".
- SOLUTION: (a) If for some $t_0 \geq 0$ we have $a - by(t_0) = 0$ then $y(t) \equiv y(t_0)$ (consequence of a uniqueness theorem to be proved later).

(b) So $sgn(a - by(t)) = sgn(a - by_0)$ for all $t \in [0, \infty)$. By separation of variables

$$\frac{y(t)}{y_0} \left| \frac{a - by_0}{a - by(t)} \right| = e^{at}, \quad t \in [0, \infty),$$

or

$$y(t) = \frac{\frac{a}{b}}{1 + \frac{a - by_0}{by_0} e^{-at}}.$$

(c) The asymptotic value, as $t \to \infty$, is always $\frac{a}{b}$.

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