BASIC COMPACTNESS THEOREMS

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- **REMINDER: TOPOLOGICAL CONCEPTS IN $\mathbb{R}^n$.**
- Definition: Euclidean norm $|x|^2 = \sum_{i=1}^{n} x_i^2$ in $\mathbb{R}^n$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- An open BOX in $\mathbb{R}^n$ is $Q = \prod_{i=1}^{n}(a_i, b_i)$. The corresponding closed box is $\overline{Q} = \prod_{i=1}^{n}[a_i, b_i]$.
- Definition: A BOUNDED set is a set that's contained in a ball centered at 0.
- If C is bounded, we define its DIAMETER $\delta(C)$ by

$$\delta(C) = \sup\{|x - y|, \quad x, y \in C\}.$$

- Definition: An OPEN set is a set such that each of its points has a closed/open ball (of positive radius) around it, contained in the set.
- Definition: A CLOSED set is such that its complement is an open set.
- Claim : The union of open sets is open, intersection of closed sets is closed.
- Claim: The intersection of finitely many open sets is open and the union of finitely many closed sets is closed.
- Definition: The CLOSURE of a set A (notation $\overline{A}$) is the minimal closed set containing A.
- Definition: The INTERIOR of a set A (notation $\mathring{A}$) is the union of all open balls contained in A.
- Definition: A point x is the limit of a sequence $\{x^{(k)}\}$ if every ball around x contains all $x^{(k)}$ for $k > K$, where K depends on the ball.
- Definition: The set $K \subseteq \mathbb{R}^n$ is COMPACT if every sequence in K has a subsequence converging to a point in K.
- THEOREM: The following statements are all equivalent (i.e., imply each other).

(a) $K \subseteq \mathbb{R}^n$ is compact.
(b) K is closed and bounded.
(c) Every open cover of K has a finite subcover.

- **Notation**:
  (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.
  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.
  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m)$ which are continuously differentiable up to (including) order k.
  (d) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.
- THEOREM: If a sequence in $C(D, \mathbb{R}^m)$ converges uniformly then the limit function is again in $C(D, \mathbb{R}^m)$.
- **BASIC COMPACTNESS THEOREMS**
- CONVENTION: $K \subseteq \mathbb{R}^n$ is ALWAYS a COMPACT set.
- Definition: Let $K \subseteq \mathbb{R}^n$ be compact. We define a **norm** on the vector space $C(K, \mathbb{R}^m) \equiv C_b(K, \mathbb{R}^m)$ by

$$\|f\| = \max_{x \in K} |f(x)|.$$

  ($|f(x)|$ is the Euclidean norm in $\mathbb{R}^m$).
- DEFINITION (Convergence in $C(K, \mathbb{R}^m)$): A sequence $\left\{f^k\right\}_{k=1}^{\infty} \subseteq C(K, \mathbb{R}^m)$ converges to $f \in C(K, \mathbb{R}^m)$ if

$$\|f - f^k\| \xrightarrow[k \to \infty]{} 0.$$

- OBSERVE: This is just UNIFORM CONVERGENCE.
- DEFINITION (**Closed set**): A subset $G \subseteq C(K, \mathbb{R}^m)$ is **closed** if all limits of (converging) sequences contained in G are in G. Briefly, it contains all its "limit points".
- REMINDER: A function $f \in C(K, \mathbb{R}^m)$ is **uniformly continuous**.
- DEFINITION (**equicontinuity**): A sequence $\left\{f^k\right\}_{k=1}^{\infty} \subseteq C(K, \mathbb{R}^m)$ is **equicontinuous** if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x, y \in K, \quad |x - y| < \delta \Rightarrow \quad |f^k(x) - f^k(y)| < \varepsilon, \quad k = 1, 2, ...$$

- **Theorem.** *(Arzela-Ascoli): A sequence* $\left\{f^k\right\}_{k=1}^{\infty} \subseteq C(K, \mathbb{R}^m)$ *which is uniformly bounded and equicontinuous contains a convergent subsequence.*

Proof. (a) For every integer l let $\varepsilon_l = \frac{1}{l}$ and let $\delta_l > 0$ be the corresponding number as in the definition of the equicontinuity above. We can assume that $\delta_l \downarrow 0$.

(b) For every integer l, with δ_l as in (a), there exists a finite subset $\left\{x^{l,1}, ..., x^{l,s_l}\right\} \subseteq K$ such that the balls $\left\{B(x^{l,1}, \delta_l), ..., B(x^{l,s_l}, \delta_l)\right\}$ cover K. (why?).

(c) Denote by $\Omega = \{x^1, x^2, ..., x^p, ...\}$ the sequence consisting of the union of all the $\left\{x^{l,1}, ..., x^{l,s_l}\right\}$.

(d) Using a diagonal process, there exists a subsequence $\left\{f^{k_\beta}\right\}_{\beta=1}^{\infty}$ such that the sequence $\left\{f^{k_\beta}(x^p)\right\}_{\beta=1}^{\infty}$ converges for all $x^p \in \Omega$.

Denote the limit by $f(x^p)$.

(e) Let l be an integer and let $\left\{B(x^{l,1}, \delta_l), ..., B(x^{l,s_l}, \delta_l)\right\}$ be balls as in (b) above.

(f) There exists an index Λ such that if $\beta > \Lambda$ then

$$|f^{k_\beta}(x^{l,\alpha}) - f(x^{l,\alpha})| < \frac{1}{l}, \quad \alpha \in \{1, 2, ..., s_l\}.$$

(g) Let $x \in K$. There exists $r \in \{1, 2, ..., s_l\}$ such that $x \in B(x^{l,r}, \delta_l)$.

(h) By equicontinuity, if $\beta_2 > \beta_1 > \Lambda$,

$$|f^{k_{\beta_2}}(x) - f^{k_{\beta_1}}(x)| < \frac{4}{l}, \quad x \in K.$$

(i) It follows that the sequence $\left\{f^{k_\beta}(x)\right\}_{\beta=1}^{\infty}$ is a Cauchy sequence, hence convergent, for all $x \in K$. We denote the limit by $f(x)$.

(j) In view of (h) the sequence $\left\{f^{k_\beta}\right\}_{\beta=1}^{\infty}$ is "uniformly Cauchy", hence converges uniformly to f. It follows that $f \in C(K, \mathbb{R}^m)$.

(k) Remark: The last statement can be reformulated as: the sequence $\left\{f^{k_\beta}\right\}_{\beta=1}^{\infty}$ is Cauchy with respect to the norm $\|\cdot\|$. □

- DEFINITION (**Contraction mapping**): Let $X \subseteq C(K, \mathbb{R}^m)$. A mapping $T : X \to X$ is a **contraction** if there exists a constant $0 \leq C < 1$ such that for all $f, g \in X$ we have $\|Tf - Tg\| \leq C\|f - g\|$.
- EXAMPLE: Take $n = m = 1$, $K = [0, 1]$ and let $X = \{f \in C[0, 1], \quad \|f\| \leq 1\}$.

  Take $Tf(x) = \frac{1}{4}\int\limits_0^x f^2(\xi)d\xi$. Then T is a contraction on X.
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Theorem. *(Contraction mapping theorem) Let $X \subseteq C(K, \mathbb{R}^m)$ be closed and let $T : X \to X$ be a contraction mapping. Then T*

*has exactly one **fixed point**. That is, there exists exactly one function $g \in X$ such that $Tg = g$.*

Proof. (a) Fix a function $h \in X$ and consider the "orbit" $\{h, Th, T^2h, ...\}$.

(b) The sequence is Cauchy (by contraction), hence converges to a limit, denoted by g.

OBSERVE: The sequence is "uniform Cauchy", hence converges in the sense of "uniform convergence" of functions on K.

(c) Since T is continuous (why?) we have $Tg = g$.

(d) If we have $g_1 \neq g_2$ as "fixed points then $\|g_1 - g_2\| = \|Tg_1 - Tg_2\| \leq C\|g_1 - g_2\| < \|g_1 - g_2\|$, which is a contradiction. □

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