AUTONOMOUS SYSTEMS, HYPERBOLIC CASE: THE HARTMAN-GROBMAN THEOREM (FOLLOWING [1, CHAPTER 4])

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June 2008

Notation

- The scalar product in $\mathbb{R}^m$ is denoted by $(\cdot,\cdot)$.
- Euclidean norm $|x|^2 = \sum_{i=1}^{m} x_i^2$ in $\mathbb{R}^m$.
- For every $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ we denote by $\|A\|$ its (operator) norm with respect to $|\cdot|$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- For every multi-index $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$ we denote

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}}$$

  and $|\alpha| = \alpha_1 + ... + \alpha_m$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.

  (b) We denote by $C_b(D, \mathbb{R}^m) \subseteq C(D, \mathbb{R}^m)$ the set of BOUNDED continuous functions on D.

  The norm is defined by:

$$\|\phi\|_{C_b} = \sup_{x \in D} |\phi(x)|.$$

  (c) We denote by $C^k(D, \mathbb{R}^m)$ the subset of functions in $C(D, \mathbb{R}^m)$ which are continuously differentiable up to (including) order k.

  (d) We denote by $C_b^k(D, \mathbb{R}^m)$ the subset of $C^k(D, \mathbb{R}^m)$ such that all derivatives are bounded up to (including) order k.

  The norm is defined by:

$$\|\phi\|_{C_b^k} = \sup_{x \in D} \sum_{|\alpha| \le k} |D^\alpha \phi(x)|.$$

  (e) If $m = 1$ we simplify to $C(D), \quad C_b(D), \quad C^k(D)$.

  (f) $\quad C_0^\infty(U)$ is the space of smooth functions supported in an open set $U \subseteq \mathbb{R}^m$.

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- **BASIC DEFINITIONS**
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- We consider (**AN AUTONOMOUS SYSTEM**)

$$(A) \qquad\qquad y'(t) = f(y(t)), \quad t \in I \subseteq \mathbb{R},$$

  $f(y) \in C^1(D, \mathbb{R}^m)$, where $D \subseteq \mathbb{R}^m$ is an open set.
- We denothe by $y(t; P)$ the (unique) solution such that $y(0; P) = P$, $P \in D$.
- DEFINITION (**critical point**): A point $Q \in D$ is said to be **critical** (also **equilibrium**) for (A) if $f(Q) = 0$.

  NOTE: The unique solution passing through Q is $y(t; Q) \equiv Q$.
- DEFINITION (**hyperbolic critical point**): A critical point $Q \in D$ for (A) is said to be **hyperbolic** if $\Re\lambda \neq 0$ for every eigenvalue λ of the Jacobian $f'(Q) = Df(Q)$.
- DEFINITION (**infinitesimally hyperbolic matrix**): A matrix $A \in Hom(\mathcal{C}^m, \mathcal{C}^m)$ is said to be **infinitesimally hyperbolic** if $\Re\lambda \neq 0$ for every eigenvalue λ of A.
- DEFINITION (**hyperbolic matrix**): A matrix $A \in Hom(\mathcal{C}^m, \mathcal{C}^m)$ is said to be **hyperbolic** if $|\lambda| \neq 1$ for every eigenvalue λ of A.
- REMARK. Note that if $B \in Hom(\mathcal{C}^m, \mathcal{C}^m)$ is infinitesimally hyperbolic then $A = e^{tB}$ is hyperbolic for any $t \neq 0$.

  In particular, the evolution matrix $e^{tDf(0)}$ of the linearized system

$$(AL) \qquad\qquad y'(t) = Df(0)y(t), \quad t \in \mathbb{R}$$

  is hyperbolic (for any $t \neq 0$) if $y = 0$ is a hyperbolic critical point.

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- **HYPERBOLIC MATRICES and MATRIX NORMS**
- **LEMMA.** Let $A \in Hom(\mathcal{C}^m, \mathcal{C}^m)$. Let

$$\rho(A) := \max\{|\lambda_1|, ..., |\lambda_m|, \quad \lambda_j \text{ is an eigenvalue of } A,\ 1 \leq j \leq m\}.$$

  Then for every $\varepsilon > 0$ there exists a norm $\|\cdot\|$ on $\mathcal{C}^m$ such that

$$\|A\| < \rho(A) + \varepsilon,$$

  where $\|A\|$ is the operator norm induced by $\|\cdot\|$.

  Note that for *every* norm $\|\cdot\|$ on $\mathcal{C}^m$, we have $\|A\| \geq \rho$.

  $\rho(A)$ is called the *spectral radius* of A.

Proof. Suppose the claim holds for A upper diagonal (i.e., all elements below the diagonal = 0). Then, for any A, let $P \in Hom(\mathcal{C}^m, \mathcal{C}^m)$ such that PAP^{-1} is upper diagonal, and let $\|\cdot\|$ be the corresponding norm. We then define a new norm by $\|z\|_1 = \|Pz\|$, $z \in \mathcal{C}^m$, and get

$$\|Az\|_1 = \|PAz\| = \|PAP^{-1}Pz\|$$
$$\leq (\rho(PAP^{-1}) + \varepsilon)\|Pz\| = (\rho(A) + \varepsilon)\|Pz\| = (\rho(A) + \varepsilon)\|z\|_1.$$

Thus, it suffices to deal with a Jordan block of the form

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda \end{pmatrix}.$$

Let

$$Q = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu^2 & 0 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mu^m \end{pmatrix}.$$

Then

$$(QAQ^{-1}) = \begin{pmatrix} \lambda & \mu^{-1} & 0 & 0 \\ 0 & \lambda & \mu^{-1} & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda \end{pmatrix}.$$

The claim is now proved by taking the norm $\|z\| = \max\{|z_1|, ..., |z_m|\}$ and $\mu > \varepsilon^{-1}$. □

- REMARK. The Lemma is true also for $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$, simply by regarding it as a complex matrix and restricting the norm to $\mathbb{R}^m$.

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- **THE HARTMAN-GROBMAN THEOREM for DIFFEOMORPHISMS.**

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- ASSUMPTION: $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ hyperbolic, nonsingular.
- We assume (changing coordinates) that A is in "block-diagonal" form (with $m = m_1 + m_2$, and one of them can be 0)

$$(BD) \qquad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1 \in Hom(\mathbb{R}^{m_1}, \mathbb{R}^{m_1})$ has eigenvalues *inside* the unit circle and $A_2 \in Hom(\mathbb{R}^{m_2}, \mathbb{R}^{m_2})$ has eigenvalues *outside* the unit circle.

- We use the Lemma above to get norms $\| \cdot \|_i$ on $\mathbb{R}^{m_i},\ i = 1, 2$ such that, with some $0 < \nu < 1$,

$$(*) \qquad \|A_1\|_1 < \nu, \quad \|A_2^{-1}\|_2 < \nu.$$

The norm on $\mathbb{R}^m$ is taken as the sum of these norms.

- **CLAIM:** Define a linear transformation L on $C_b(\mathbb{R}^m, \mathbb{R}^m)$ (into itself) by

$$L(h)(x) = h(Ax) - Ah(x), \quad h \in C_b(\mathbb{R}^m, \mathbb{R}^m).$$

Then L is invertible.

REMARK: The fact that A is hyperbolic is essential in this claim.

Proof. We decompose $h = (h^1, h^2)$, where $h^i \in C_b(\mathbb{R}^m, \mathbb{R}^{m_i})$, $i = 1, 2$. Thus L can be decomposed as

$$Lh(x) = (L_1 h^1(x), L_2 h^2(x))$$
$$= (h^1(Ax) - A_1 h^1(x), h^2(Ax) - A_2 h^2(x)), \quad x \in \mathbb{R}^m,$$

where L_i is a linear map on $C_b(\mathbb{R}^m, \mathbb{R}^{m_i})$ (into itself).

We use the norms introduced above, so that (*) holds. Note that the linear map $T_i = h^i(Ax)$, $i = 1, 2$, on $C_b(\mathbb{R}^m, \mathbb{R}^{m_i})$ is invertible with norm 1. Thus, it suffices to prove the invertibility of the map $\widetilde{L}h = ((I_1 - T_1^{-1}A_1)h^1, (A_2^{-1}T_2 - I_2)h^2)$, where I_i is the identity on $C_b(\mathbb{R}^m, \mathbb{R}^{m_i})$, $i = 1, 2$. But the invertibility of $\widetilde{L}$ follows clearly from (*), in view of Neumann's series. Furthermore, $\|\widetilde{L_i}^{-1}\| \le \frac{1}{1-\nu}$, $i = 1, 2$, where the norm is the operator norm on $C_b(\mathbb{R}^m, \mathbb{R}^{m_i})$. We conclude that $\|L_1^{-1}\| = \|\widetilde{L_1}^{-1} T_1^{-1}\| \le \frac{1}{1-\nu}$ and $\|L_2^{-1}\| = \|\widetilde{L_1}^{-1} A_2^{-1}\| \le \frac{\nu}{1-\nu} \le \frac{1}{1-\nu}$, so that

$$\|L^{-1}\| \le \frac{2}{1-\nu}.$$

□

- **A FUNCTIONAL EQUATION.**
- Consider the equation

$$(FE) \qquad h(Ax) - Ah(x) = p(x + h(x)).$$

The function $p \in C_b^1(\mathbb{R}^m, \mathbb{R}^m)$ is given (with $p(0) = 0$) and we look for a solution $h \in C_b(\mathbb{R}^m, \mathbb{R}^m)$ such that $h(0) = 0$.

REMARK: The solution to this equation is the main ingredient in the proof of the Hartman-Grobman Theorem below.

- **PROPOSITION:** Fix $0 < \alpha < 1$. There exists $\varepsilon > 0$ with the following property:

  For every $p \in C_b^1(\mathbb{R}^m, \mathbb{R}^m)$ such that $p(0) = 0$ and $\|p\|_{C_b^1} < \varepsilon$ there exists a unique function $h \in C_b(\mathbb{R}^m, \mathbb{R}^m)$ such that $\|h\|_{C_b} < \alpha$, $h(0) = 0$ and (FE) is satisfied.

Proof. Define a map on $C_b(\mathbb{R}^m, \mathbb{R}^m)$ by $\Phi(h)(x) = p(x+h(x)) - p(x)$. The equation (FE) can then be rewritten as $Lh(x) = \Phi(h)(x) + p(x)$ or

$$(**) \qquad h = L^{-1}\Phi(h) + L^{-1}p,$$

where L is the linear operator in the above Claim. From the estimate at the end of the proof of that Claim we get

$$\|L^{-1}\Phi(h) + L^{-1}p\|_{C_b} \leq \frac{2}{1-\nu}\|\Phi(h) + p\|_{C_b}.$$

Using the definition of the norm in $C_b^1(\mathbb{R}^m, \mathbb{R}^m)$ we get

$$\|\Phi(h)\|_{C_b} \leq \|p\|_{C_b^1}\|h\|_{C_b}.$$

Taking $\varepsilon > 0$ such that

$$\frac{2\varepsilon}{1-\nu}(1+\alpha) < \alpha,$$

we see that for any p such that $\|p\|_{C_b^1} < \varepsilon$ the right-hand side of (**) maps the ball of radius α in $C_b(\mathbb{R}^m, \mathbb{R}^m)$ into itself.

Furthermore, for any $h, \tilde{h}$ in this ball

$$\begin{aligned}\|L^{-1}\Phi(h) - L^{-1}\Phi(\tilde{h})\|_{C_b} &\leq \frac{2}{1-\nu}\|p\|_{C_b^1}\|h-\tilde{h}\|_{C_b} \\ &\leq \frac{2\varepsilon}{1-\nu}\|h-\tilde{h}\|_{C_b} \leq \alpha\|h-\tilde{h}\|_{C_b},\end{aligned}$$

so that the map in the right-hand side of (**) is a contraction on the ball of radius α in $C_b(\mathbb{R}^m, \mathbb{R}^m)$.

We conclude that in the ball there is exactly one fixed point of the map (for every $\|p\|_{C_b^1} < \varepsilon$), which is the unique solution of (**). □

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- DEFINITION (**hyperbolic fixed point of a smooth map**): Let $\Omega \subseteq \mathbb{R}^m$ be open and $0 \in \Omega$. Let $\Psi \in C^1(\Omega, \mathbb{R}^m)$ and assume that $\Psi(0) = 0$. We say that 0 is a **hyperbolic fixed point** if

$\Psi'(0) = D\Psi(0)$ is hyperbolic (i.e., has no eigenvalues on the unit circle).

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Theorem (THE HARTMAN-GROBMAN THEOREM–diffeomorphism). *Let $F : B(0,a) \subseteq \mathbb{R}^m \hookrightarrow \mathbb{R}^m$ for some $a > 0$. Suppose that,*

(i) $F \in C^1(B(0,a), \mathbb{R}^m)$.

(ii) *$x = 0$ is a hyperbolic fixed point of F and the Jacobian $DF(0)$ is nonsingular.*

Then there are $0 < c < b < a$ and an open set $U \subseteq \mathbb{R}^m$, such that,

(1) *There exists a homeomorphism H mapping $B(0,b)$ onto U, and $H(0) = 0 \in U$.*

(2) *Both H and $DF(0)$ map $B(0,c)$ into $B(0,b)$.*

(3) $F(H(x)) = H(DF(0)x), \quad x \in B(0,c)$.

REMARK: Writing $H^{-1}(F(H(x))) = DF(0)x, \quad x \in B(0,c)$, we see that the restriction of F to the image of $B(0,c)$ by H is "similar" to the action of $DF(0)$ on this ball.

Proof. (1) We denote $A = DF(0)$. Fix $0 < \alpha < 1$, and let $\varepsilon > 0$ be given by the above Proposition. Replacing $a > 0$ by a smaller radius if necessary, we can assume that

$$\|F(x) - Ax\|_{C_b^1(B(0,a))} < \frac{\varepsilon}{3}.$$

In particular,

$$\|F(x) - Ax\|_{C_b(B(0,a))} < \frac{\varepsilon}{3}a.$$

(2) Let $\phi \in C_0^\infty(B(0,a))$, be such that , for a suitable $b < a$,

$$0 \leq \phi(x) \leq 1,\ x \in B(0,a),\ \phi(x) = 1,\ x \in B(0,b),\ \|\phi\|_{C_b^1(B(0,a))} < \frac{2}{a}.$$

(3) Define $p(x) = \phi(x)(F(x) - Ax)$. Then p is supported in $B(0,a)$ and

$$\|p\|_{C_b^1(\mathbb{R}^m)} \leq \frac{2}{a}\frac{\varepsilon}{3}a + \frac{\varepsilon}{3} = \varepsilon.$$

(4) In addition, we take $\varepsilon > 0$ sufficiently small (this might involve a smaller radius a) so that

$$\|A^{-1}p\|_{C_b^1(\mathbb{R}^m)} < 1.$$

This means that

$$\|(I + A^{-1}p)(x) - (I + A^{-1}p)(y)\| \geq (1 - \|A^{-1}p\|_{C_b^1(\mathbb{R}^m)})\|x - y\|,$$

so that,

$$(***) \qquad (A+p)(x) \neq (A+p)(y) \quad \text{if } x \neq y.$$

(Note: $(A+p)(x) = Ax + p(x)$).

(5) The Proposition implies that there exists $h \in C_b(\mathbb{R}^m, \mathbb{R}^m)$, with $\|h\|_{C_b} < \alpha$, $h(0) = 0$, such that (see (FE)),

$$h(Ax) - Ah(x) = p(x + h(x)), \quad x \in \mathbb{R}^m.$$

Defining $H(x) = x + h(x)$, this can be written as

$$(****) \qquad (A+p)(H(x)) = H(Ax).$$

(6) There exists $0 < c < b$ such that

$$x \in B(0,c) \Rightarrow x + h(x) \in B(0,b) \Rightarrow (A+p)(H(x)) = F(H(x)).$$

We can decrease (if needed) c so that also $x \in B(0,c) \Rightarrow Ax \in B(0,b)$, and we obtain assertions (2) and (3) of the theorem.

(7) It remains to check that H is a homeomorphism on $B(0,b)$. Suppose first that for some $x \neq y \in \mathbb{R}^m$, we have $H(x) = H(y)$. By (****) we have $H(Ax) = H(Ay)$. Continuing like that we get

$$H(A^k x) = H(A^k y), \quad k = 1, 2, ...$$

Replacing x by $A^{-1}x$ in (****) and noting (***) we have also $H(A^{-1}x) = H(A^{-1}y)$ and continuing like that we get

$$H(A^k x) = H(A^k y), \quad k = -1, -2, ...$$

(8) The definition of H now yields

$$\|A^k x - A^k y\| = \|h(A^k x) - h(A^k y)\| \leq 2\|h\|_{C_b(\mathbb{R}^m)},$$

where the norm is as introduced above (before the Claim).

(9) Now we use the *hyperbolic structure* of A (which was also essential for the Claim above). Using the block-diagonal form (BD), we decompose

$$x - y = (x-y)_1 + (x-y)_2 \in \mathbb{R}^m = \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_1},$$

and note that

$$\|(x-y)_1\| = \|A_1^k A_1^{-k}(x-y)_1\| \leq 2\nu^k \|h\|_{C_b(\mathbb{R}^m)} \xrightarrow[k\to\infty]{} 0,$$

with a similar argument for $(x-y)_2$ (with $k \to -\infty$). We conclude that $x = y$.

(10) We know that H is one-to-one on $\overline{B}(x,b)$ and hence (since the closed ball is compact) a homeomorphism onto its image. In particular, the restriction of H^{-1} to the image $U = H(B(0,b))$ is continuous. Finally, the set U is open by the theorem on Invariance of Domain [2, Chapter 18].
□

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- **THE HARTMAN-GROBMAN THEOREM for DIFFERENTIAL EQUATIONS.**
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- We turn back to the autonomous system

$$(A) \qquad y'(t) = f(y(t)), \quad t \in I \subseteq \mathbb{R},$$

$f(y) \in C^1(D, \mathbb{R}^m)$, where $D \subseteq \mathbb{R}^m$ is an open set, and its linearization

$$(AL) \qquad y'(t) = Df(0)y(t), \quad t \in \mathbb{R}.$$

- ASSUMPTION: $y = 0 \in D$ is a hyperbolic critical point, namely, $Df(0)$ has no pure imaginary eigenvalues.
  We also assume that $Df(0)$ is **regular**.
- Our goal is to construct a local homeomorphism that "transforms" the flow of (A) to the flow of (AL) near the critical point.
- As in the case of diffeomorphisms above, we start with a certain functional equation, related to a hyperbolic matrix A.
- ASSUMPTION: $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ hyperbolic, nonsingular.
- We assume (changing coordinates) that A is in "block-diagonal" form (with $m = m_1 + m_2$, and one of them can be 0)

$$(BD) \qquad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1 \in Hom(\mathbb{R}^{m_1}, \mathbb{R}^{m_1})$ has eigenvalues *inside* the unit circle and $A_2 \in Hom(\mathbb{R}^{m_2}, \mathbb{R}^{m_2})$ has eigenvalues *outside* the unit circle.
- **CLAIM:** Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism (i.e., continuous onto , with continuous inverse).
  Define a linear transformation M on $C_b(\mathbb{R}^m, \mathbb{R}^m)$ (into itself) by

$$M(h)(x) = Ah(x) - h(F(x)), \quad h \in C_b(\mathbb{R}^m, \mathbb{R}^m).$$

Then M is invertible.

- REMARK. The operator M is slightly more general than the operator L in the previous Claim (for diffeomorphisms), which corresponds here to $F(x) = x$. However, the linear operator $h \to h(F(x))$ is of unit norm and the proof is identical.
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Theorem (THE HARTMAN-GROBMAN THEOREM–differential equations)**.** *Consider the system (A), where $y = 0$ is a hyperbolic fixed point and $Df(0)$ is regular.*

Let Φ_t be the flow of (A) and Ψ_t the flow of the linear system (AL).

Then there is an open neighborhood $0 \in U \subseteq D$, and a homeomorphism $G : U \to G(U) \subseteq D$, so that

$$G(\Phi_t(x)) = \Psi_t(G(x)),$$

for all $(t, x) \in \mathbb{R} \times U$, such that $\Phi_t(x) \in U$.

Proof. In this proof we refer to the above theorem (HARTMAN-GROBMAN for diffeomorphisms) as the "previous theoerm" (and its proof).

(1) Let $B(0, b) \subseteq B(0, a) \subseteq D$ and a cutoff function

$$0 \leq \phi(x) \leq 1,\ x \in B(0, a),\ \phi(x) = 1,\ x \in B(0, b),\ \|\phi\|_{C^1_b(B(0,a))} < \frac{2}{a}.$$

We now define

$$f^*(x) = \phi(x)f(x) + (1 - \phi(x))Df(0)(x),$$

and replace the system (A) by

$$(A^*) \qquad\qquad y'(t) = f^*(y(t)), \quad t \in \mathbb{R}.$$

We denote its (global) flow by Φ^*_t.

(2) We denote

$$F(x) = \Phi^*_1(x), \quad A = e^{Df(0)} = \Psi_1.$$

and set

$$p(x) = F(x) - Ax,$$

so that

$$p(x) = \int_0^1 [f^*(\Phi^*_t(x)) - Df(0)\Psi_t(x)]dt.$$

Note that

$$\Phi_t^*(x) - \Psi_t(x) = \int_0^t [f^*(\Phi_s^*(x)) - Df(0)\Psi_s(x)]ds =$$

$$\int_0^t [f^*(\Phi_s^*(x)) - f^*(\Psi_s(x))]ds + \int_0^t [f^*(\Psi_s(x)) - Df(0)\Psi_s(x)]ds.$$

For any $\varepsilon > 0$ we can choose $0 < a < 1$ sufficiently small, so that $|f^*(z) - Df(0)z| < \varepsilon \min(1, |z|)$ for all $z \in \mathbb{R}$. Thus

$$|\Phi_t^*(x) - \Psi_t(x)| \leq \beta \int_0^t |\Phi_s^*(x)) - \Psi_s(x)|ds + \varepsilon e^{|t|\|Df(0)\|}, \quad |t| \leq 1,$$

where $\beta > 0$ is a Lipschitz constant for f^*. Gronwall's inequality now yields

$$(*****) \qquad |\Phi_t^*(x) - \Psi_t(x)| \leq \varepsilon e^{|t|(\|Df(0)\|+\beta)}.$$

(3) In particular it follows that $p = \Phi_1^*(x) - \Psi_1(x) \in C_b(\mathbb{R}^m, \mathbb{R}^m)$, so by the above Claim we have a unique solution $g \in C_b(\mathbb{R}^m, \mathbb{R}^m)$ to the equation $Mg = p$, i.e.,

$$Ag(x) - g(F(x)) = p(x), \quad x \in \mathbb{R}^m.$$

Denoting $G(x)x + g(x)$ we can rewrite the last equation as

$$G(F(x)) = AG(x), \quad x \in \mathbb{R}^m.$$

(4) Define

$$\mathcal{G}(x) = \int_0^1 \Psi_{-s}(G(\Phi_s^*(x)))ds, \quad x \in \mathbb{R}^m.$$

Using the linearity of Ψ_t we have

$$\Psi_{-t}(\mathcal{G}(\Phi_t^*(x))) = \int_0^1 \Psi_{-t-s}(G(\Phi_{t+s}^*(x)))ds = \int_{t-1}^{t} \Psi_{-\tau}(\Psi_{-1}(G(\Phi_1^*(\Phi_\tau^* x))))d\tau$$

$$= \int_{t-1}^{t} \Psi_{-\tau}(A^{-1}(G(F(\Phi_\tau^* x))))d\tau = \int_{t-1}^{t} \Psi_{-\tau}(G(\Phi_\tau^* x))d\tau =$$

$$\int_{t-1}^{0} \Psi_{-\tau}(G(\Phi_\tau^* x))d\tau + \int_0^t \Psi_{-\tau}(G(\Phi_\tau^* x))d\tau =$$

$$\int_t^1 \Psi_{1-\tau}(G(\Phi_{\tau-1}^* x))d\tau + \int_0^t \Psi_{-\tau}(G(\Phi_\tau^* x))d\tau =$$

$$\int_t^1 \Psi_{-\tau}(AG(F^{-1}(\Phi_\tau^* x)))d\tau + \int_0^t \Psi_{-\tau}(G(\Phi_\tau^* x))d\tau =$$

$$\int_0^1 \Psi_{-\tau}(G(\Phi_\tau^* x))d\tau = \mathcal{G}(x)$$

(5) We conclude

$$\Psi_{-t}(\mathcal{G}(\Phi_t^*(x))) = \mathcal{G}(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^m,$$

and in particular (with $t = 1$),

$$\mathcal{G}(F(x)) = A\mathcal{G}(x).$$

This is the same equation satisfied by $G(x) = x + g(x)$ and by the uniqueness of g we conclude that $\mathcal{G}(x) = G(x)$ if

$$\mathcal{G}(x) - x \in C_b(\mathbb{R}^m, \mathbb{R}^m).$$

(6) To prove it we write

$$\mathcal{G}(x) - x = \int_0^1 \Psi_{-s}(G(\Phi_s^*(x)))ds - x = \int_0^1 \Psi_{-s}(G(\Phi_s^*(x)) - \Psi_s(x))ds =$$

$$\int_0^1 \Psi_{-s}(G(\Phi_s^*(x)) - \Phi_s^*(x) + \Phi_s^*(x) - \Psi_s(x))ds,$$

so that, using (*****) above (see (2)),

$$|\mathcal{G}(x)-x| \leq e^{\|Df(0)\|} \sup_{0\leq s\leq 1}[|G(\Phi_s^*(x))-\Phi_s^*(x)|+|\Phi_s^*(x)-\Psi_s(x)|]$$
$$\leq e^{\|Df(0)\|}[\|g\|_{C_b}+\sup_{0\leq s\leq 1}|\Phi_s^*(x)-\Psi_s(x)|]$$
$$\leq e^{\|Df(0)\|}[\|g\|_{C_b}+\varepsilon e^{\|Df(0)\|}e^{\beta}].$$

Thus $\mathcal{G}=G$.

(7) Next we note that (*****) (see (2) above) implies that for any $\varepsilon>0$ we can choose $a>0$ sufficiently small so that $\|p\|_{C_b(\mathbb{R}^m)}\leq\varepsilon$.
We also have

$$D\Phi_t^*(x)-D\Psi_t(x)=\int_0^t[Df^*(D\Phi_s^*(x))-Df(0)(D\Psi_s(x))]ds=$$
$$\int_0^t[Df^*(D\Phi_s^*(x))-Df^*(D\Psi_s(x))]ds+\int_0^t[Df^*(D\Psi_s(x))-Df(0)(D\Psi_s(x))]ds.$$

For any $\varepsilon>0$ we can choose $0<a<1$ sufficiently small, so that $\|Df^*(z)-Df(0)\|<\varepsilon$ for all $z\in\mathbb{R}$. Also note that $D\Psi_s(x)=\Psi_s(x)$. Let γ be a Lipschitz constant for Df^*.
We obtain, for $|t|\leq 1$,

$$|D\Phi_t^*(x)-D\Psi_t(x)|\leq\gamma\int_0^t|D\Phi_s^*(x))-D\Psi_s(x)|ds+\varepsilon e^{|t|\|Df(0)\|},$$

Gronwall's inequality now yields, as in (*****),

$$|D\Phi_t^*(x)-D\Psi_t(x)|\leq\varepsilon e^{|t|(\|Df(0)\|+\gamma)},\quad |t|\leq 1.$$

Combined with the above estimate for $p=\Phi_1^*(x)-\Psi_1(x)$ we conclude that for any $\varepsilon>0$ we can choose $a>0$ sufficiently small so that $\|p\|_{C_b^1(\mathbb{R}^m)}\leq\varepsilon$.

(8) In particular, by the above Proposition , we have a unique function $h\in C_b(\mathbb{R}^m,\mathbb{R}^m)$ such that $\|h\|_{C_b}<1,\ h(0)=0$ and (FE) is satisfied:

$$(FE)\qquad h(Ax)-Ah(x)=p(x+h(x)).$$

Letting $H(x)=x+h(x)$ it can be rewritten as

$$F(H(x))=H(Ax).$$

H is a *GLOBAL* homeomorphism on $\mathbb{R}^m$.

(Note the difference between this and (****) in the proof of the theorem for diffeomorphisms, where $A+p=F$ only for "small" x.).

(9) We get
$$G(H(Ax)) = G(F(H(x))) = AG(H(x)),$$
and denoting $K = G \circ H$ it gives $K(Ax) = AK(x)$. But $K(x) = x + h(x) + g(H(x)) = x + h(x) + g(x + h(x))$ so, with $r(x) = h(x) + g(x + h(x)) \in C_b(\mathbb{R}^m, \mathbb{R}^m)$ we have
$$0 = K(Ax) - AK(x) = r(Ax) - Ar(x)$$
and the Claim above (with $M(r)(x) = Ar(x) - r(A(x))$) yields $r(x) \equiv 0$. So $K = G \circ H = I$, the identity, and $G = H^{-1}$ is a homeomorphism on $\mathbb{R}^m$.

(10) From $\mathcal{G} = G$ proved above and the result of (4)
$$\Psi_{-t}(\mathcal{G}(\Phi_t^*(x))) = \mathcal{G}(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^m,$$
we obtain,
$$G(\Phi_t^*(x)) = \Psi_t(G(x)), \quad x \in \mathbb{R}^m.$$

(11) Finally, if $x \in B(0,b)$ (see the beginning of the proof), we have the equality of the flows $\Phi_t^*(x) = \Phi_t(x)$ in some interval $t \in (-\delta(x), \delta(x))$.

□

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