REMARKS ON PERIODIC SOLUTIONS

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Notation

- The scalar product in $\mathbb{R}^m$ is denoted by $(\cdot,\cdot)$.
- Euclidean norm $|x|^2 = \sum_{i=1}^{m} x_i^2$ in $\mathbb{R}^m$.
- For every $A \in Hom(\mathbb{R}^m, \mathbb{R}^m)$ we denote by $\|A\|$ its (operator) norm with respect to $|\cdot|$.
- Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- (a) If $D \subseteq \mathbb{R}^n$ we denote by $C(D,\mathbb{R}^m)$ the set of continuous (vector) functions on D into $\mathbb{R}^m$.

  (b) We denote by $C_b(D,\mathbb{R}^m) \subseteq C(D,\mathbb{R}^m)$ the set of BOUNDED continuous functions on D.

  (c) We denote by $C^k(D,\mathbb{R}^m)$ the subset of functions in $C(D,\mathbb{R}^m)$ which are continuously differentiable up to (including) order k.

  (d) If $m = 1$ we simplify to $C(D)$, $C_b(D)$, $C^k(D)$.

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- REMARK: In the previous Summary (#6) we have seen a few examples of periodic solutions for **AUTONOMOUS SYSTEMS** (nonlinear pendulum, Lotka-Volterra system). Here we discuss examples of periodic solutions for **NON-AUTONOMOUS SYSTEMS**.

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- Let $f(t,y) \in C(D)$, where $D = \mathbb{R} \times I$ and $I \subseteq \mathbb{R}$ is an open (finite or infinite) interval.
- We always assume that:

  $f(t,y)$ is Lipschitz in y.
- We consider the (scalar) equation

$$(P) \qquad\qquad y'(t) = f(t,y(t)), \quad t \in \mathbb{R},$$

  subject to the initial condition $y(t_0) = y_0 \in I$.
- We know that there exists a *unique* solution defined for t in some open maximal interval (t_{min}, t_{max}) containing t_0.
- NOTATION: This solution is denoted by $y(t;t_0,y_0)$.
- **LEMMA**: Suppose that for some $p \in I$ we have $f(t,p) > 0$ for all $t \in \mathbb{R}$. Then $y_0 > p \Rightarrow y(t;t_0,y_0) > p$ for all $t \in [t_0, t_{max})$. In other words, the solution "stays above" the line $y = p$.

  PROOF: Otherwise there is a first point $\tau \in (t_0, t_{max})$ such that $y(\tau;t_0,y_0) = p$. Clearly at this point $y'(\tau;t_0,y_0) \le 0$, but $y'(\tau;t_0,y_0) = f(\tau,p) > 0$, a contradiction.

  Q.E.D.

- **COROLLARY**: Suppose in addition to the assumption of the Lemma that for some $p < q \in I$, we have $f(t,q) < 0$ for all $t \in \mathbb{R}$. Then , if $y_0 \in [p,q]$, the solution $y(t;t_0,y_0)$ exists for all $t \geq t_0$ (i.e.,$t_{max} = \infty$) and $y(t;t_0,y_0) \in (p,q)$ for all $t \in (t_0,\infty)$.
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- **PERIODIC SOLUTIONS**
- **ASSUME** $f(t,p) > 0, f(t,q) < 0, \quad [p,q] \subseteq I$ **and ADD THE ASSUMPTION:**
  $f(t,y)$ is **periodic** in t, i.e., there exists $T > 0$ such that
  $$f(t+T,y) = f(t,y), \quad (t,y) \in D.$$
- By the Corollary, $y_0 \in [p,q] \Rightarrow y(t;t_0,y_0) \in [p,q]$, $t \geq t_0$.
- Define the map $\Phi : [p,q] \hookrightarrow [p,q]$ by
  $$[p,q] \ni y_0 \to \Phi(y_0) = y(t_0+T;t_0,y_0).$$
- CLAIM: The function ϕ is (strictly) monotone increasing.
  PROOF: Uniqueness!
- CLAIM: The solution $y(t;t_0,y_0)$ is periodic (with period T) if and only if
  $$\Phi(y_0) = y_0.$$
- **Theorem.** *Under all the assumptions on f above (including periodicity), the equation (P) has at least one periodic solution $y(t;t_0,\xi_0) = y(t+T;t_0,\xi_0)$, for some $\xi \in [p,q]$ and all $t \in \mathbb{R}$.*
  PROOF: The map $\Phi : [p,q] \hookrightarrow [p,q]$ must have a fixed point.
- **Theorem.** *: Suppose that $f \in C^2(D)$ and that $\frac{\partial^2}{\partial y^2} f(t,y) < 0$ in D. Then the periodic solution of the previous theorem is* unique *in $\mathbb{R} \times [p,q]$.*
  Furthermore, for any initial value $y_0 \in [p,q]$, the solution $y(t;t_0,y_0)$ approaches the periodic solution $y(t;t_0,\xi_0)$ in the sense that
  $$\lim_{t\to\infty} |y(t;t_0,y_0) - y(t;t_0,\xi_0)| = 0.$$

PROOF: The equation satisfied by the derivative of the solution with respect to the initial data (see Summary 4, the section on "regularity of the solution") is:

$$\frac{\partial}{\partial t}(\frac{\partial}{\partial y_0} y(t;t_0,y_0)) = \frac{\partial}{\partial y} f(t,y(t;t_0,y_0)) \frac{\partial}{\partial y_0} y(t;t_0,y_0), \quad \frac{\partial}{\partial y_0} y(t_0;t_0,y_0) = 1,$$

so that

$$\frac{\partial}{\partial y_0} y(t_0+T;t_0,y_0) = \Phi'(y_0) = \exp\,(\int_{t_0}^{t_0+T} \frac{\partial}{\partial y} f(s,y(s;t_0,y_0))ds) > 0.$$

(It was already observed above that $\Phi(y_0)$ is strictly monotone increasing even without second-order differentiability of f).

Now the second-order derivative $w(t;t_0,y_0) = \frac{\partial^2 y(t;t_0,y_0)}{\partial y_0^2}$ satisfies (see Summary 4, the section on "regularity of the solution"):

$$\frac{\partial}{\partial t}w(t;t_0,y_0)=\frac{\partial^2}{\partial y^2}f(t,y(t;t_0,y_0))(\frac{\partial y(t;t_0,y_0)}{\partial y_0})^2+\frac{\partial}{\partial y}f(t,y(t;t_0,y_0))w(t;t_0,y_0),$$
$$w(t_0;t_0,y_0)=0.$$

By the assumption,

$$\frac{\partial}{\partial t}w(t;t_0,y_0)<\frac{\partial}{\partial y}f(t,y(t;t_0,y_0))w(t;t_0,y_0),$$

and since $w(t_0;t_0,y_0)=0$

$$w(t;t_0,y_0)\exp(\int_{t_0}^{t}\frac{\partial}{\partial y}f(s,y(s;t_0,y_0))ds)<0\Rightarrow w(t;t_0,y_0)<0,\quad t>t_0.$$

In particular

$$\frac{\partial^2}{\partial y_0^2}y(t_0+T;t_0,y_0)=\Phi''(y_0)<0.$$

Thus, $\Phi(y_0)$ cannot have two fixed points (it is a *concave, increasing* function that can intersect the diagonal only once–give an analytic proof!).

Finally, we prove the convergence of any solution $y(t;t_0,y_0)$, $y_0\in[p,q]$, to the periodic one $y(t;t_0,\xi_0)$.

Note that $\lim_{k\to\infty}\Phi^k(y_0)=\xi_0$ (Prove this!).

By the theorem on continuous dependence on initial data, given $\varepsilon>0$, we can find $\eta>0$ such that

$$|v_0-\xi_0|<\eta\Rightarrow|y(t;t_0,v_0)-y(t;t_0,\xi_0)|<\varepsilon,\quad t\in[t_0,t_0+T].$$

For this $\eta>0$, there is a K such that

$$k>K\Rightarrow|\Phi^k(y_0)-\xi_0|<\eta.$$

Hence, by the periodicity,

$$|y(t;t_0+kT,\Phi^k(y_0))-y(t;t_0+kT,\xi_0)|<\varepsilon,\quad t\in[t_0+kT,t_0+(k+1)T],\quad k>K,$$

that is

$$|y(t;t_0,y_0)-y(t;t_0,\xi_0)|<\varepsilon,\quad t\in[t_0+kT,t_0+(k+1)T],\quad k>K.$$

Q.E.D.

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- **A TIME PERIODIC LOGISTIC EQUATION**
  (See Summary 2 for the *logistic equation* with constant coefficients).
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- We consider

$$(LP)\qquad y'(t)=ay(t)(b(t)-y(t)),\quad (t,y)\in\mathbb{R}\times\mathbb{R}.$$

where $a>0$ is a constant and $b(t)>0$ is a continuous periodic function , $b(t+T)=b(t)$.

All the assumptions above (including the fact that the second-order derivative of $f(t,y)$ with respect to y is negative) are satisfied here with $f(t,y)=ay(b(t)-y)$. In addition, if

$$0<p<\min_{t\in\mathbb{R}}b(t)<q,$$

then $f(t,q)<0<f(t,p)$. We have therefore:

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Theorem. *Equation (LP) has a unique periodic solution* $\phi(t)$*. For any* $t_0, y_0) \in \mathbb{R}\times(0,\infty)$*, the solution* $y(t;t_0,y_0)$ *converges to the periodic solution*

$$\lim_{t\to\infty} |y(t;t_0,y_0)-\phi(t)| = 0.$$

PROOF: We can apply the previous theorem with any $y_0 > 0$, since we can take very small p and very large q. For some t_0, let $y(t;t_0,\xi_0)$ be the periodic solution given by the previous theorem. Since *all* solutions (starting with $y_0 > 0$) converge to it, it is independent of the choice of t_0 and we are justified in calling it $\phi(t)$. Q.E.D.

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EXAMPLE OF NONUNIQUE PERIODIC SOLUTIONS

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- Consider the equation

$$(NUP) \qquad y'(t) = y(t) - y(t)^3 + b(t),$$

Where $b(t)$ is continuous, periodic ($b(t) = b(t+T)$) and

$$|b(t)| < \frac{2}{3\sqrt{3}}, \quad t \in \mathbb{R}.$$

- CLAIM: Equation (NUP) has at least three different periodic (with period T) solutions, $\phi^{\pm}(t), \phi^0(t)$, such that

$$\phi^+(t) > \frac{1}{\sqrt{3}}, \quad \phi^-(t) < -\frac{1}{\sqrt{3}}, \quad |\phi^0(t)| < \frac{1}{\sqrt{3}}, \quad t \in \mathbb{R}.$$

PROOF: Set $f(t,y) = y - y^3 + b(t)$. Take $p = \frac{1}{\sqrt{3}}, q = 2$. Then

$$f(t,p) = \frac{2}{3\sqrt{3}} + b(t) > 0, \quad f(t,q) < 0, \quad t \in \mathbb{R}.$$

Thus, by the general theorem, we have the existence of a periodic solution $\phi^+(t) > p = \frac{1}{\sqrt{3}}$. Similarly,

$$f(t,-p) = -\frac{2}{3\sqrt{3}} + b(t) < 0, \quad f(t,-q) > 0, \quad t \in \mathbb{R}.$$

Thus, by the general theorem, we have the existence of a periodic solution $\phi^-(t) < -p = -\frac{1}{\sqrt{3}}$.

Finally, the function $\tilde{y}(t) = y(-t)$ satisfies the equation

$$\tilde{y}'(t) = -\tilde{y}(t) + \tilde{y}(t)^3 - b(-t).$$

Set $\tilde{f}(t,\tilde{y}) = -\tilde{y} + \tilde{y}^3 - b(-t)$.

We have

$$\tilde{f}(t,-p) > 0, \quad \tilde{f}(t,p) < 0, \quad t \in \mathbb{R},$$

so by the general theorem there exists a periodic solution $\psi(t)$ to this equation , with $|\psi(t)| < p$. Then $\phi^0(t) = \psi(-t)$ is a third periodic solution to (NUP).

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