VECTOR-VALUED FUNCTIONS ON $\mathbb{R}^n$.

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Functions here are defined on a subset $D \subseteq \mathbb{R}^n$ and take values in $\mathbb{R}^m$, where m can be smaller, equal or greater than n.

The (open) ball of radius r, centered at x, is denoted by $B(x, r)$. It will be clear from the context what is the dimension (as in the continuity definition below).

Usually, we use the Euclidean norm. The ball depends on the norm, but all the concepts are "norm independent".

- `NOTATION`: For $C \subseteq D$ we denote $f(C) = \{f(y), \quad y \in C\}$.
- <u>DEFINITION</u>: A function $f(x)$ is CONTINUOUS at $x_0 \in D$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $f(B(x_0, \delta) \cap D) \subseteq B(f(x_0), \varepsilon)$.
- <u>DEFINITION</u>: A function f is CONTINUOUS on D if it is continuous at any point of D.
- **THEOREM**: A function f is continuous at $x_0 \in D$ iff the following condition is satisfied:
  If a sequence $\{x^{(k)}\}_{k=1}^{\infty} \subseteq D$ converges to x_0 then $f(x^{(k)}) \to f(x_0)$.
- The components of f are given by $f(x) = (f_1(x), ..., f_m(x))$.
- **THEOREM**: $f : D \hookrightarrow \mathbb{R}^m$ is continuous at $x_0 \in D$ iff every component $f_j(x), \quad j = 1, ..., m$, is continuous at x_0 as a real function.
- **THEOREM**: Let $C \subseteq \mathbb{R}^n$ be compact and f a continuous function on C. Then the image $f(C) \subseteq \mathbb{R}^m$ is compact.
- **THEOREM**: Let $C \subseteq \mathbb{R}^n$ be connected and f a continuous function on C. Then the image $f(C) \subseteq \mathbb{R}^m$ is connected.
- **THEOREM**: Let $C \subseteq \mathbb{R}^n$ be a compact set, and let $f : C \to \mathbb{R}^m$ be continuous. Then f is UNIFORMLY CONTINUOUS, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x, y \in C, \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

- `NOTATION`: Usually , a point $x \in \mathbb{R}^n$ will be considered also as a COLUMN vector of length n, namely, $x = (x_1, ..., x_n)^T$, as in the following definition.
- <u>DEFINITION</u>: A LINEAR FUNCTION from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by

$$f(x) = Ax,$$

  where A is an $m \times n$ matrix (of real constants).
- **CLAIM**: A linear function is continuous.
- `NOTATION`: $M_{m \times n}$ is the space of (real) matrices with m rows and n columns.
- REMARK: $M_{m \times n}$ is isomorphic to $\mathbb{R}^{mn}$.

- DEFINITION (**OPERATOR NORM**): Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $f(x) = Ax$ be a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then the "OPERATOR NORM" of the matrix A (with respect to the two given norms) is defined by:
$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1=1} \|Ax\|_2.$$
- REMARK: An operator norm $\|A\|$ induces a norm on $\mathbb{R}^{mn}$.
- If we take $m = n$ and require that $\|\cdot\|_1 \equiv \|\cdot\|_2$ then NOT EVERY NORM on $\mathbb{R}^{n^2}$ is an operator norm (on $M_{n\times n}$).
- EXAMPLE: Take $m = n$. Let $A \in M_{n\times n}$. Take identical norms $\|\cdot\|_1 \equiv \|\cdot\|_2$.
  CLAIM: $\|A\| \geq |\lambda|$ for every eigenvalue λ of A.
- The situation is not different even if we are given a "higher degree of liberty".
- **THEOREM**: Let $\|\cdot\|$ be a norm on $M_{m\times n}$, and suppose that the norm is induced by a scalar product, namely, there is a scalar product on $M_{m\times n}$, such that its norm is $\|\cdot\|$.
  Then, if $m > 1$ and $n > 1$ there are **no norms** $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that the given norm $\|\cdot\|$ is the operator norm with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$.
- PROOF: See
  http://math.stackexchange.com/questions/1245591/is-the-norm-operator-between-normed-spaces-ever-induced-from-an-inner-product

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