SURFACES IN THREE-DIMENSIONAL EUCLIDEAN SPACE $\mathbb{R}^3$

MATANIA BEN-ARTZI

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NOTATION:

- If $a \in \mathbb{R}^3$ then $|a|$ is its Euclidean norm.
- For $a, b \in \mathbb{R}^3$, the scalar product is denoted by $a \cdot b$.
- For $a, b \in \mathbb{R}^3$, the vector product is denoted by $a \times b$.
- For $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ denote $\Psi_{u_i} = \frac{\partial}{\partial u_i}\Psi$.

DEFINITION: Let $D \subseteq \mathbb{R}^2$ be an open, connected domain, and denote by $u = (u_1, u_2)$ the coordinates in D.

Let $\Psi : D \to \mathbb{R}^3$ be a smooth (at least C^3) map such that:

- Ψ is one-to-one.
- $\Psi_{u_1} \times \Psi_{u_i} \neq 0$ at any point $u \in D$.

Then $S = \Psi(D)$ is called a **simple (or elementary) surface**. The domain D is called the **parameter domain** of S and the map Ψ is the **parametrization** of S.

Special case–Functional Graph. Let $\psi : D \to \mathbb{R}$ be a smooth scalar function. Then the map

$$\Psi(u_1, u_2) = (u_1, u_2, \psi(u_1, u_2)), \quad (u_1, u_2) \in D, \tag{1}$$

defines a simple surface called the **graph of ψ (over D).**

REMARK. For any simple surface, the parameters (u_1, u_2) form a system of **coordinates** on the surface.

As in the case of curves, we discuss first the issue of "change of parameters."

DEFINITION: Let $\widetilde{D} \subseteq \mathbb{R}^2$ be an open, connected domain, and denote the coordinates there by $v = (v_1, v_2)$. A smooth map $\Phi : \widetilde{D} \to D$ is called an **admissible change of parameters** if:

- $\Phi = (\Phi_1(v), \Phi_2(v))$ is one-to-one and onto.
- The Jacobian $J(\Phi) = \begin{pmatrix} \frac{\partial}{\partial v_1}\Phi_1 & \frac{\partial}{\partial v_2}\Phi_1 \\ \frac{\partial}{\partial v_1}\Phi_2 & \frac{\partial}{\partial v_2}\Phi_2 \end{pmatrix}$ is regular and $det(J(\Phi)) > 0$.

CLAIM 1. Let Φ be an admissible change of parameters as above. Then

$$(\Psi \circ \Phi)_{v_1} \times (\Psi \circ \Phi)_{v_2} = det(J(\Phi))(\Psi_{u_1} \circ \Phi) \times (\Psi_{u_2} \circ \Phi).$$

Proof. By the chain rule

$$(\Psi \circ \Phi)_{v_i} = \sum_{j=1}^{2} \frac{\partial \Phi_j}{\partial v_i} (\Psi_{u_j} \circ \Phi),$$

so the claim follows from the distributive rule of vector products and $\Psi_{u_j} \times \Psi_{u_j} = 0$. $\square$

Notational Convention. In what follows we denote by $P = P(u)$ a point in D with coordinates u. Without risking confusion, we use "P" both for $P = P(u) \in D$ and $P = \Psi(P(u)) \in S$.

DEFINITION: The vector $n(P) = \frac{\Psi_{u_1}(P) \times \Psi_{u_2}(P)}{|\Psi_{u_1}(P) \times \Psi_{u_2}(P)|}$ is called the **unit normal** to S at P.

COROLLARY to the CLAIM. The unit normal is invariant under an admissible change of parameters.

Example 1. In the case of a graph (1),

$$n = \frac{(-\psi_{u_1}, -\psi_{u_1}, 1)}{\sqrt{1 + \psi_{u_1}^2 + \psi_{u_2}^2}}.$$

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SURFACE AREA and TANGENT PLANE

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DEFINITION: Let $E \subseteq D$ be an open domain so that $\overline{E} \Subset D$.

The **surface area** of $\Psi(E) \subseteq S$ is defined as

$$Area(\Psi(E)) = \int_E |\Psi_{u_1} \times \Psi_{u_2}| du_1 du_2.$$

REMARK. In view of Claim 1 and the formula for the change of variables in integration, if $\widetilde{E} = \Phi^{-1}(E)$, then

$$Area(\Psi \circ \Phi(\widetilde{E})) = Area(\Psi(E)).$$

Thus, the area is well defined under admissible change of parameters.

Example 1-revisited. In the case of a graph

$$Area(\Psi(E)) = \int_E \sqrt{1 + \psi_{u_1}^2 + \psi_{u_2}^2} du_1 du_2.$$

The vectors Ψ_{u_1}, Ψ_{u_2} are linearly independent at every point $P(u_1, u_2) \in D$.

DEFINITION: Let $P = (u_1, u_2) \in D$. The plane spanned by $\Psi_{u_1}(P)$, $\Psi_{u_2}(P)$ is called the **tangent plane to the surface at** P and denoted by $T_P S$.

REMARK. By our convention, $P \in S$ is identified with its coordinates $P(u) \in D$.

DEFINITION: Let $\{\alpha(t),\, t \in (a, b)\} \subseteq D \subseteq \mathbb{R}^2$ be a smooth curve in the parameter domain. Then the curve $\gamma(t) = \Psi(\alpha(t))$ is called **a curve on the surface** S.

We refer to $\alpha(t)$ as the **coordinates** of the curve.

CLAIM 2. Let $P \in S$. Then the tangent plane $T_P S$ is equal to the plane spanned by all tangents of curves $\gamma(t)$ on S passing through P.

$$T_P S = \{\gamma'(0),\ \gamma(0) = P\}.$$

Proof. If $\gamma(t) = \Psi(\alpha(t))$ is a curve on S, with $\gamma(0) = P$, then

$$\gamma'(0) = \alpha_1'(0)\Psi_{u_1}(P) + \alpha_2'(0)\Psi_{u_2}(P) \in T_P S.$$

Conversely, let $X = \lambda_1\Psi_{u_1}(P) + \lambda_2\Psi_{u_2}(P) \in T_P S$. Take the straight line segment $\alpha(t) = \alpha(0) + t(\lambda_1, \lambda_2) \subseteq D$, with $\alpha(0) = P$ (recall that we identify $P \in S$ with its coordinates $P(\alpha(0)) \in D$) . Then $\gamma(t) = \Psi(\alpha(t))$ satisfies $\gamma'(0) = \lambda_1\Psi_{u_1}(P) + \lambda_2\Psi_{u_2}(P) = X$. □

COROLLARY. The tangent plane $T_P S$ is the two-dimensional space spanned by the (linearly independent) vectors $\Psi_{u_1}(P)$, $\Psi_{u_2}(P)$.

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THE FIRST FUNDAMENTAL FORM = THE METRIC

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Let $P \in S$ and let $X, Y \in T_P S$. We express both of them in terms of the "basis vectors" $\Psi_{u_1}(P), \Psi_{u_2}(P)$. Then

$$X \cdot Y = \sum_{i,j=1}^{2} X_i Y_j\, \Psi_{u_i}(P) \cdot \Psi_{u_j}(P). \tag{2}$$

DEFINITION: The four numbers

$$\left\{g_{ij}(P) = \Psi_{u_i}(P) \cdot \Psi_{u_j}(P), \quad 1 \le i, j \le 2,\ P \in S\right\}$$

are called the **metric coefficients** on S.

CLAIM 3. The symmetric matrix $\{g_{ij}(P)\}_{i,j=1}^{2}$ is positive definite, for every $P \in S$.

In particular

$$det\,\{g_{ij}(P)\} = |\Psi_{u_1}(P) \times \Psi_{u_2}(P)|^2. \tag{3}$$

Proof. It is the matrix of the scalar product on $T_P S$.

If θ is the angle between $\Psi_{u_1}(P)$ and $\Psi_{u_2}(P)$ then

$$|\Psi_{u_1}(P) \times \Psi_{u_2}(P)|^2 = |\Psi_{u_1}(P)|^2 |\Psi_{u_2}(P)|^2 \sin^2(\theta)$$
$$= |\Psi_{u_1}(P)|^2 |\Psi_{u_2}(P)|^2 - |\Psi_{u_1}(P) \cdot \Psi_{u_2}(P)|^2 = det(g_{ij}(P)).$$

□

DEFINITION: The positive bilinear form

$$X \cdot Y = \sum_{i,j=1}^{2} g_{ij}(P) X_i Y_j, \quad X, Y \in T_P S$$

is called the **first fundamental form.**

NOTATION

- The inverse matrix to $\{g_{ij}(P)\}_{i,j=1}^{2}$ is denoted by $\left\{g^{ij}(P)\right\}_{i,j=1}^{2}$.
- $g(P) = det\,\{g_{ij}(P)\}$.

Example 1-re-revisited. In the case of a graph

$$\Psi_{u_1}(P) = (1, 0, \psi_{u_1}), \ \ \Psi_{u_1}(P) = (0, 1, \psi_{u_2}),$$

hence, at the point $P = (u_1, u_2, \psi(u_1, u_2))$,

$$g_{11} = 1 + \psi_{u_1}^2,\ g_{22} = 1 + \psi_{u_2}^2,$$
$$g_{12} = g_{21} = \psi_{u_1}\psi_{u_2}.$$

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THE SECOND FUNDAMENTAL FORM and the CHRISTOFFEL SYMBOLS

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By assumption the triple $\{\Psi_{u_1}(P), \Psi_{u_2}(P), n(P)\}$ is a basis to $\mathbb{R}^3$ at every point $P \in S$. In particular, we can write

$$\Psi_{u_i u_j}(P) = L_{ij}(P)n(P) + \sum_{k=1}^{2} \Gamma_{ij}^k(P)\Psi_{u_k}(P). \tag{4}$$

THEOREM. In Equation (4), known as the **Gauss Formula,** we have

$$L_{ij}(P) = \Psi_{u_i u_j}(P) \cdot n(P).$$
$$\Gamma_{ij}^k(P) = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(P)\Big(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l}\Big)(P).$$

Proof. The first equation is clear, since $n(P)$ is orthogonal to $\Psi_{u_1}(P)$, $\Psi_{u_2}(P)$.

Concerning the second equation, we show first that (we omit the "P" for simplicity)

$$\Gamma_{ij}^k = \sum_{l=1}^{2} g^{lk}\Psi_{u_i u_j} \cdot \Psi_{u_l}.$$

Indeed,

$$\Psi_{u_i u_j} \cdot \Psi_{u_l} = \sum_{m=1}^{2} \Gamma_{ij}^m g_{ml},$$

so multiplying by g^{lk} and summing over l,

$$\sum_{l=1}^{2} \Psi_{u_i u_j} \cdot \Psi_{u_l} g^{lk} = \sum_{m=1}^{2} \Gamma_{ij}^m \delta_{km} = \Gamma_{ij}^k.$$

We therefore need to compute $\Psi_{u_i u_j} \cdot \Psi_{u_l}$ in terms of the metric g.

To do this we write

$$\frac{\partial g_{il}}{\partial u_j} = \frac{\partial}{\partial u_j}\Psi_{u_i} \cdot \Psi_{u_l} = \Psi_{u_i u_j} \cdot \Psi_{u_l} + \Psi_{u_l u_j} \cdot \Psi_{u_i},$$

and similarly

$$\frac{\partial g_{jl}}{\partial u_i} = \Psi_{u_i u_j} \cdot \Psi_{u_l} + \Psi_{u_l u_i} \cdot \Psi_{u_j},$$
$$\frac{\partial g_{ji}}{\partial u_l} = \Psi_{u_l u_j} \cdot \Psi_{u_i} + \Psi_{u_l u_i} \cdot \Psi_{u_j}.$$

Thus

$$\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ji}}{\partial u_l} = 2\Psi_{u_i u_j} \cdot \Psi_{u_l}.$$

$\square$

DEFINITION: The matrix $\{L_{ij}(P)\}_{i,j=1}^2$ is symmetric. Its elements are called the **coefficients of the second fundamental form.**

REMARK. In the classical notation the coefficients (at a point $P \in S$) of the first and second fundamental forms are designated as

$$\boxed{\begin{array}{r} E = g_{11},\ \ F = g_{12} = g_{21},\ G = g_{22}, \\ L = L_{11},\ M = L_{12} = L_{21},\ N = L_{22}, \end{array}}$$

DEFINITION: The coefficients $\left\{\Gamma_{ij}^k,\ 1 \le i, j, k \le 2\right\}$ are called the **Christoffel symbols (of the second kind).**

DEFINITION: We say that a quantity is **intrinsic (to the surface)** if it depends (in addition to the local coordinates) only on the metric (g_{ij}) (as function of the coordinates u).

COROLLARY. The Christoffel symbols $\left\{\Gamma_{ij}^k,\ 1 \le i, j, k \le 2\right\}$ are intrinsic.

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CURVATURE OF A CURVE ON THE SURFACE
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Length parameter of a curve. Let $\{\alpha(t),\ t \in (a,b)\} \subseteq D \subseteq \mathbb{R}^2$ be a smooth curve in the parameter domain , and let $\gamma(t) = \Psi(\alpha(t))$ be the corresponding curve on the surface S (so that $\alpha(t)$ are its coordinates). Then, with $P = \gamma(t)$,

$$\gamma'(t) = \alpha_1'(t)\Psi_{u_1}(P) + \alpha_2'(t)\Psi_{u_2}(P) \in T_P S,$$

Length of arc of γ from t_0 to t :

$$s(t) = \int_{t_0}^{t} |\gamma'(\sigma)| d\sigma = \int_{t_0}^{t} \sqrt{\sum_{i,j=1}^{2} g_{ij}(\gamma(\sigma))\alpha_i'(\sigma)\alpha_j'(\sigma)}\ d\sigma. \tag{5}$$

COROLLARY. The arc length is an intrinsic property of the curve.

We can use the arc length parameter s also for the coordinates $\alpha(s)$.

Let $\gamma(s)$ be a unit speed curve on the surface S. We denote by $\alpha(s) = (\alpha_1(s), \alpha_2(s))$ its coordinates; $\gamma(s) = \Psi(\alpha(s))$.

$T(s)$, $N(s)$, $B(s)$, $\kappa(s)$, $\tau(s)$, is the **Frenet-Serret apparatus** of the curve.

Note that $T(s) \in T_{\gamma(s)}S$.

DEFINITION: Let $n(s)$ be the unit normal to S at $\gamma(s)$. Then the **intrinsic normal** to the curve is defined by

$$m(s) = n(s) \times T(s).$$

Note. Since $m(s)$ is orthogonal to $n(s)$, it follows that $m(s) \in T_{\gamma(s)}S$, and $m(s)$, $n(s)$ form a basis to the plane orthogonal to $T(s)$ at $\gamma(s)$.

CLAIM 4. The derivative of the tangent $T(s)$ along the curve can be written as a linear combination of $n(s)$ and $m(s)$:

$$\gamma''(s) = T'(s) = \kappa_n(s)n(s) + \kappa_g(s)m(s). \tag{6}$$

Also, in terms of the coordinates $\alpha(s)$,

$$T'(s) = \sum_{i,j=1}^{2} \alpha_i'(s)\alpha_j'(s)\Psi_{u_iu_j}(\alpha(s)) + \sum_{k=1}^{2} \alpha_k''(s)\Psi_{u_k}(\alpha(s)). \tag{7}$$

Proof. Since $|T(s)| \equiv 1$, the derivative $T'(s)$ is orthogonal to $T(s)$, hence a linear combination of $n(s)$ and $m(s)$.

To prove Equation (7) we differentiate (with respect to s) the equation

$$T(s) = \gamma'(s) = \sum_{i=1}^{2} \alpha_i'(s)\Psi_{u_i}(\alpha(s)).$$

□

Using the Frenet-Serret system, $T'(s) = \kappa(s)N(s)$, so

$$\gamma''(s) = \kappa(s)N(s) = \kappa_n(s)n(s) + \kappa_g(s)m(s).$$

DEFINITION: $\kappa_n(s)$ is the **normal curvature** of the curve.
$\kappa_g(s)$ is the **geodesic curvature** of the curve.

CLAIM 5. The tangential part of $T'(s)$ is given by

$$\kappa_g(s)m(s) = \sum_{k=1}^{2} \Big(\alpha_k''(s) + \sum_{i,j=1}^{2} \alpha_i'(s)\alpha_j'(s)\Gamma_{ij}^k(\alpha(s))\Big)\Psi_{u_k}(\alpha(s)).$$

Proof. In Equation (7) take the tangential part of $\Psi_{u_iu_j}(\alpha(s))$ according to Equation (4). □

THEOREM. The geodesic curvature $\kappa_g(s)$ is intrinsic.

Proof. By definition

$$\kappa_g(s) = \kappa_g(s)m(s) \cdot (n(s) \times T(s)).$$

From Claim 5 we get

$$\kappa_g(s) = \sum_{k=1}^{2}\Big(\alpha''_k(s) + \sum_{i,j=1}^{2}\alpha'_i(s)\alpha'_j(s)\Gamma^k_{ij}(\alpha(s))\Big)\Psi_{u_k}(\alpha(s))\cdot(n(s)\times T(s))$$

$$= \sum_{k=1}^{2}\Big(\alpha''_k(s) + \sum_{i,j=1}^{2}\alpha'_i(s)\alpha'_j(s)\Gamma^k_{ij}(\alpha(s))\Big)\Psi_{u_k}(\alpha(s))\cdot\Big(n(s)\times\sum_{l=1}^{2}\alpha'_l(s)\Psi_{u_l}(\alpha(s)\Big)$$

$$= \sum_{l=1}^{2}\sum_{k=1}^{2}\Big(\alpha''_k(s) + \sum_{i,j=1}^{2}\alpha'_i(s)\alpha'_j(s)\Gamma^k_{ij}(\alpha(s))\Big)\alpha'_l(s)\Psi_{u_k}(\alpha(s))\cdot(n(s)\times\Psi_{u_l}(\alpha(s)).$$

But

$$\Psi_{u_k}(\alpha(s))\cdot(n(s)\times\Psi_{u_l}(\alpha(s)) = n(s)\cdot(\Psi_{u_l}(\alpha(s))\times\Psi_{u_k}(\alpha(s)) = \begin{cases}\sqrt{g(\alpha(s)}, \ l=1,k=2,\\ -\sqrt{g(\alpha(s)}, \ l=2,k=1,\\ 0, \ l=k.\end{cases}$$

Thus, all the terms in the last expression for $\kappa_g(s)$ are intrinsic. □

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GEODESIC CURVES

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DEFINITION: Let $\gamma(s) \subseteq S$ be a unit speed curve. It is called **geodesic** if its geodesic curvature $\kappa_g(s)$ is identically equal to zero.

COROLLARY. A necessary and sufficient condition for $\gamma(s)$ to be geodesic is that $\gamma''(s)$ is orthogonal to the surface at every point (i.e., a scalar multiple of $n(\gamma(s))$.

Proof. See Claim 4, Equation (6). □

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NORMAL CURVATURE and WEINGARTEN MAP

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We continue to consider unit speed curves $\gamma(s)$ on the surface S (so that s is the arc length parameter).

We turn to the normal curvature $\kappa_n(s)$ of a curve $\gamma(s)$ on the surface S. Recall that the unit speed curve $\gamma(s))$ is expressed as $\gamma(s) = \Psi(\alpha(s))$, with coordinates $\alpha(s) \in D$).

CLAIM 6. The normal curvature satisfies, in terms of the second fundamental form, the equation

$$\kappa_n(s) = \sum_{i,j=1}^{2} L_{ij}(\gamma(s))\alpha'_i(s)\alpha'_j(s), \tag{8}$$

where

$$|\gamma'(s)|^2 = \sum_{i,j=1}^{2} g_{ij}(\gamma(s))\alpha'_i(s)\alpha'_j(s) \equiv 1.$$

Proof. In Equation (7) we put the normal (to the surface) part of $\Psi_{u_i u_j}(\alpha(s))$, which is by definition L_{ij} (see Equation (4)).

$\square$

CLAIM 7. Fix $P \in S$. Then the set of **all possible normal curvatures** of curves passing through P is equal to the following set of values, computed by the second funadamental form:

$$\left\{ \sum_{i,j=1}^{2} L_{ij} X_i X_j, \quad X = \sum_{i=1}^{2} X_i \Psi_{u_i}(P) \in T_P S,\ |X| = 1 \right\}.$$

Proof. Any unit vector $X = X_1 \Psi_{u_1}(P) + X_2 \Psi_{u_2}(P) \in T_P S$ is a tangent vector $X = \gamma'(0)$ of some unit speed curve $\gamma(s)$ so that $\gamma(0) = P$.

Indeed, if $P = \Psi(u^0)$, take $\widetilde{\alpha}(t) = u^0 + t(X_1, X_2)$, and let $\widetilde{\gamma}(t) = \Psi(\widetilde{\alpha}(t))$ be the corresponding curve on the surface.

Its length (from P) is given by

$$s(t) = \int_0^t \sqrt{\sum_{i,j=1}^{2} g_{ij}(\widetilde{\gamma}(\sigma)) \widetilde{\alpha}_i'(\sigma) \widetilde{\alpha}_j'(\sigma)}\ d\sigma.$$

Express $t = t(s)$ and take the unit speed curve

$$\gamma(s) = \widetilde{\gamma}(t(s)).$$

then $\left.\frac{ds}{dt}\right|_{t=0} = 1$ and

$$\gamma'(s)_{s=0} = \widetilde{\gamma}'(t)_{t=0} = X_1 \Psi_{u_1}(P) + X_2 \Psi_{u_2}(P).$$

$\square$

From linear algebra we know that every bilinear form defines a linear map. In the case of the second fundamental form we have:

DEFINITION: Let $P \in S$. The **Weingarten map** $\mathcal{L}_P : T_P S \to T_P S$ is given by

$$\mathcal{L}_P X \cdot Y = \sum_{i,j=1}^{2} L_{ij}(P) X_i Y_j, \quad X, Y \in T_P S,$$

$$X = \sum_{i=1}^{2} X_i \Psi_{u_i}(P), \quad Y = \sum_{i=1}^{2} Y_i \Psi_{u_i}(P).$$

From Claim 7 we get

THEOREM. For any $P \in S$ there are two orthonormal vectors $Z^1(P)$, $Z^2(P) \in T_P S$ so that

- $Z^1(P)$, $Z^2(P)$ are eigenvectors of $\mathcal{L}_P$, with corresponding eigenvalues $\kappa_1(P) \leq \kappa_2(P)$.
- The eigenvalues $\kappa_1(P)$, $\kappa_2(P)$ are the minimal and maximal normal curvatures at P.

DEFINITION: The eigenvalues $\kappa_1(P)$, $\kappa_2(P)$ are called the **principal curvatures** of the surface S at P.

CLAIM 8. Let

$$\mathcal{L}_P \Psi_{u_j}(P) = \sum_{l=1}^{2} L_j^l \Psi_{u_l}(P), \quad j = 1, 2,$$

then

$$L_j^i = \sum_{k=1}^{2} L_{jk} g^{ki},$$

where all the above quantities are computed at P.

Proof. Taking the scalar product with $\Psi_{u_k}(P)$ we have

$$L_{jk} = \mathcal{L}_P \Psi_{u_j}(P) \cdot \Psi_{u_k}(P) = \sum_{l=1}^{2} L_j^l g_{lk}, \tag{9}$$

so the result is obtained by multiplying by g^{ki} and summing over k. $\square$

COROLLARY. At every point $P \in S$ the product $K(P) = \kappa_1(P)\kappa_2(P)$ of the principal curvatures satisfies

(10)

$$K(P) = det \left\{ L_j^l \right\}_{l,j=1} = det \left\{ L_{ij} \right\}_{i,j=1} det \left\{ g^{ij} \right\}_{i,j=1} = \frac{L_{11}(P)L_{22}(P) - L_{12}(P)L_{21}(P)}{g(P)}.$$

REMARK. In the classical notation the coefficients (at a point $P \in S$) of the first and second fundamental forms are designated as

$$E = g_{11}, \ F = g_{12} = g_{21}, \ G = g_{22},$$
$$L = L_{11}, \ M = L_{12} = L_{21}, \ N = L_{22},$$

so that the formula for $K(P)$ can be rewritten as

$$\boxed{K(P) = \tfrac{LN - M^2}{EG - F^2}.}$$

If $X \in T_P S$ is a unit vector, the normal curvature in the direction of X is given by $\kappa_X(P) = \mathcal{L}_P X \cdot X$. Expressing X in terms of the basis vectors in the direction of the principal curvatures we obtain:

EULER's THEOREM. If $X = \cos(\theta) Z^1(P) + \sin(\theta) Z^2(P)$, then

$$\kappa_X(P) = \mathcal{L}_P X \cdot X = \cos^2(\theta)\kappa_1(P) + \sin^2(\theta)\kappa_2(P).$$

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GAUSS MAP and the GEOMETRIC MEANING of the WEINGARTEN MAP.

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Now note that the unit normal $n(P)$, as **function of** $P \in S$,) can be viewed as valued in S^2, the unit sphere.

<u>DEFINITION:</u> The map $S \ni P \to n(P) \in S^2$ is called the **Gauss Map.**

The *tangent plane* to S^2 at $n(P)$ is denoted by $T_{n(P)}S^2$, and is normal to $n(P)$. However, by definition, also the tangent plane $T_P S$ (to S at P) is perpendicular to $n(P)$. Therefore, **the tangent planes $T_{n(P)}S^2$ and $T_P S$ are parallel and can be identified.**

COROLLARY. The differential $Dn(P)$ is a linear map of $T_P S$ into itself.

CLAIM 9. The Weingarten map, in $T_P S$, satisfies:

$$\mathcal{L}_P = -Dn(P).$$

Proof. Note first that, by definition of the directional derivative,

$$Dn(p)\Psi_{u_i}(P) = \frac{\partial n}{\partial u_i}(P), \quad i = 1, 2, \ P \in S.$$

On the other hand, for $1 \le i, j \le 2$ we have, by definition of the coefficients of the second fundamental form,

$$0 = \frac{\partial}{\partial u_i}(n \cdot \Psi_{u_j})(P) = \frac{\partial n}{\partial u_i}(P) \cdot \Psi_{u_j}(P) + n(P) \cdot \Psi_{u_j u_i}(P) = \frac{\partial n}{\partial u_i}(P) \cdot \Psi_{u_j}(P) + L_{ij}(P).$$

By definition of the Weingarten map,

$$\mathcal{L}_P \Psi_{u_i}(P) \cdot \Psi_{u_j}(P) = L_{ij}(P), \quad 1 \le i, j \le 2,$$

so the previous equality can be written as

$$0 = \frac{\partial n}{\partial u_i}(P) \cdot \Psi_{u_j}(P) + \mathcal{L}_P \Psi_{u_i}(P) \cdot \Psi_{u_j}(P) = \Big(\frac{\partial n}{\partial u_i}(P) + \mathcal{L}_P \Psi_{u_i}(P)\Big) \cdot \Psi_{u_j}(P), \quad 1 \le i, j \le 2.$$

It follows that

$$\mathcal{L}_P \Psi_{u_i}(P) = -\frac{\partial n}{\partial u_i}(P), \quad i = 1, 2, \ P \in S. \tag{11}$$

□

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GAUSS CURVATURE and THEOREMA EGREGIUM

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<u>DEFINITION:</u> The product $K(P) = \kappa_1(P)\kappa_2(P)$ of the principal curvatures at a point $P \in S$ is called the **Gaussian curvature** of the surface at P.

We introduce

<u>DEFINITION:</u> (the **Riemann curvature tensor**). At every point $P \in S$ we define:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u_j} - \frac{\partial \Gamma^l_{ij}}{\partial u_k} + \sum_{m=1}^{2} (\Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}), \quad 1 \le i, j, k, l \le 2. \tag{12}$$

REMARK. Note that by its definition the Riemann tensor is **intrinsic.**

We now express the tensor in terms of the coefficients of the second fundamental form (and their related coefficients of the Weingarten map).

CLAIM 10. Let $\{L_{ij},\, L^l_k\}$ be, respectively, the coefficients of the second fundamental form (see Equation (4)) and the Weingarten map (see Claim 8). Then we have the **Gauss equation**:

$$R^l_{ijk} = L_{ik}L^l_j - L_{ij}L^l_k, \quad 1 \le i, j, k, l \le 2. \tag{13}$$

Proof. We differentiate Equation (4) with respect to u_k :

$$\Psi_{u_i u_j u_k}(P) = \frac{\partial}{\partial u_k}\Big[L_{ij}(P)n(P) + \sum_{m=1}^{2}\Gamma_{ij}^{m}(P)\Psi_{u_m}(P)\Big]$$

$$= \Big(\frac{\partial}{\partial u_k}L_{ij}\Big)n + L_{ij}\frac{\partial}{\partial u_k}n + \sum_{l=1}^{2}\Big(\frac{\partial}{\partial u_k}\Gamma_{ij}^{l}\Big)\Psi_{u_l} + \sum_{l,m=1}^{2}\Gamma_{ij}^{m}\Gamma_{mk}^{l}\Psi_{u_l} + \sum_{l=1}^{2}\Gamma_{ij}^{l}L_{lk}n,$$

where in the second line we have omitted the point P.

Now we use Equation (11) and the expression in Claim 8 to write

$$\frac{\partial}{\partial u_k}n = -\mathcal{L}_P\Psi_{u_k} = -\sum_{l=1}^{2}L_k^l\Psi_{u_l},$$

so that

(14)

$$\Psi_{u_i u_j u_k} = \Big[\frac{\partial}{\partial u_k}L_{ij} + \sum_{l=1}^{2}\Gamma_{ij}^{l}L_{lk}\Big]n + \sum_{l=1}^{2}\Big\{\Big(\frac{\partial}{\partial u_k}\Gamma_{ij}^{l}\Big) - L_{ij}L_k^l + \sum_{m=1}^{2}\Gamma_{ij}^{m}\Gamma_{mk}^{l}\Big\}\Psi_{u_l}.$$

Now we interchange the indices j, k in this equality and obtain

(15)

$$\Psi_{u_i u_k u_j} = \Big[\frac{\partial}{\partial u_j}L_{ik} + \sum_{l=1}^{2}\Gamma_{ik}^{l}L_{lj}\Big]n + \sum_{l=1}^{2}\Big\{\Big(\frac{\partial}{\partial u_j}\Gamma_{ik}^{l}\Big) - L_{ik}L_j^l + \sum_{m=1}^{2}\Gamma_{ik}^{m}\Gamma_{mj}^{l}\Big\}\Psi_{u_l}.$$

Since the mixed derivatives are equal, we can compare the coefficients of Ψ_{u_l},

$$\frac{\partial}{\partial u_k}\Gamma_{ij}^{l} - L_{ij}L_k^l + \sum_{m=1}^{2}\Gamma_{ij}^{m}\Gamma_{mk}^{l} = \frac{\partial}{\partial u_j}\Gamma_{ik}^{l} - L_{ik}L_j^l + \sum_{m=1}^{2}\Gamma_{ik}^{m}\Gamma_{mj}^{l},$$

which is exactly the required equality (13), if we take into account the definition (12) of the Riemann tensor.

□

THEOREMA EGREGIUM of GAUSS:

The Gaussian curvature $K(P)$ of a surface is intrinsic.

Proof. By the Remark following Equation (12) we know that R_{ijk}^l is intrinsic.

We now multiply Equation (13) by g_{lm} and sum over l :

(16)

$$\sum_{l=1}^{2}R_{ijk}^{l}g_{lm} = \sum_{l=1}^{2}\Big[L_{ik}L_j^l - L_{ij}L_k^l\Big]g_{lm} = L_{ik}L_{jm} - L_{ij}L_{km}, \quad 1 \le i, j, k, m \le 2,$$

where we used Equation (9).

Since the left-hand side is intrinsic, this is true also for the right-hand side.

In particular, if we take $i = k = 1$ and $j = m = 2$ we conclude that $L_{11}L_{22} - L_{12}L_{21}$ is intrinsic. In view of the formula (10) the proof is complete (note that of course $g(P)$ is intrinsic!). □

REMARK¿ In the proof of the Gauss Equation (13) we compared the coefficients of Ψ_{u_l} in Equations (14), (15). If we compare the coefficient of n in the two equations we obtain the

Codazzi-Mainardi Equations:

$$\frac{\partial}{\partial u_k} L_{ij} + \sum_{l=1}^{2} \Gamma_{ij}^{l} L_{lk} = \frac{\partial}{\partial u_j} L_{ik} + \sum_{l=1}^{2} \Gamma_{ik}^{l} L_{lj}, \quad 1 \leq i, j, k \leq 2. \tag{17}$$

Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel
E-mail address: mbartzi@math.huji.ac.il