CURVES IN THREE-DIMENSIONAL EUCLIDEAN SPACE $\mathbb{R}^3$

MATANIA BEN-ARTZI

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NOTATION:

- If $a \in \mathbb{R}^3$ then $|a|$ is its Euclidean norm.
- For $a, b \in \mathbb{R}^3$, the scalar product is denoted by $a \cdot b$.
- For $a, b \in \mathbb{R}^3$, the vector product is denoted by $a \times b$.

We assume here that $\gamma(t) : (a, b) \to \mathbb{R}^3$ is a C^3 curve (in particular, no discontinuities of the derivatives) and $\gamma'(t) \neq 0,\ t \in (a, b)$. We refer to such a curve as a **regular curve.**

Example 1. The curve

$$\gamma(t) = \begin{cases} (t, \exp(-\frac{1}{t^2}), 0), & t < 0, \\ (0, 0, 0), & t = 0, \\ (t, 0, \exp(-\frac{1}{t^2})), & t > 0, \end{cases}$$

is regular.

Note that it "bends" from the $x - y$ plane ($t < 0$) to the $x - z$ plane ($t > 0$).

DEFINITION: Let $t_0 \in (a, b)$. The function

$$s(t) = \int_{t_0}^{t} |\gamma'(u)| du, \quad t \in (a, b),$$

is an admissible change of parameter, $s \in (\int_{t_0}^{a} |\gamma'(u)| du, \int_{t_0}^{b} |\gamma'(u)| du)$.

The new parameter s is called the **arc length parameter.**

Note that

$$s(t) = \text{signed length of the curve segment } \{\gamma(u),\ u \in [t_0, t]\}.$$

Its inverse is denoted by $t(s)$.

CLAIM. The equivalent curve $\gamma^*(s) = \gamma(t(s))$ satisfies

$$\left| \frac{d}{ds} \gamma^*(s) \right| = 1, \quad s \in (\int_{t_0}^{a} |\gamma'(u)| du, \int_{t_0}^{b} |\gamma'(u)| du).$$

We simplify the notation and write $\gamma^*(s)$ as $\gamma(s)$.

DEFINITION: We refer to $\gamma(s)$ as a *unit speed curve.*

Note. If a regular curve $\gamma(t)$ satisfies $|\gamma'(t)| \equiv 1$, then actually $t = s$ is the length parameter (up to a constant).

Example 2. The **unit speed circular helix** is

$$\gamma(s) = (r\cos(\omega s), r\sin(\omega s), h\omega s), \ \ s \in \mathbb{R},$$

where $\omega^2 = (r^2 + h^2)^{-1}$.

In what follows, curves are parametrized by arc length s, unless explicitly stated otherwise.

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The FRENET-SERRET System

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To simplify notation in what follows, we do not specify explicitly the domain of the parameter.

DEFINITION: Let $\gamma(s)$ be a unit speed curve. Then

$T(s) = \gamma'(s)$ is called the **unit tangent vector along the curve**.

The function $\kappa(s) = |T'(s)|$ is called the **curvature along the curve**.

CLAIM 1. The curvature $\kappa(s) \equiv 0$ if and only if $\gamma(s)$ is a (segment of a) straight line.

Proof. A straight line is of the form $\gamma(s) = \alpha s + \beta$, where $\alpha, \beta \in \mathbb{R}^3$ and $|\alpha| = 1$. Thus $T(s) = \alpha = constant$.

Conversely, $\kappa(s) \equiv 0 \Rightarrow T'(s) \equiv 0 \Rightarrow T(s) = constant = \alpha$, hence $\gamma(s) = \alpha s + \beta$. □

The following example complements this claim by giving a typical case where the curvature *never vanishes.*

Example 3. Let $\gamma(s)$ be a regular unit speed curve, and assume that it lies on a sphere $|x - x_0| = r$. Then $\kappa(s) \neq 0$ for all s.

Indeed, the condition means that $|\gamma(s) - x_0|^2 \equiv r^2$, and by two successive differentiations

$$T(s) \cdot (\gamma(s) - x_0) \equiv 0 \ \Rightarrow T'(s) \cdot (\gamma(s) - x_0) + |T(s)|^2 = T'(s) \cdot (\gamma(s) - x_0) + 1 \equiv 0.$$

From now on we assume that $\kappa(s) \neq 0$.

We regard $s \to T(s) \in \mathbb{R}^3$ as a curve . Note that it lies on the unit sphere, but s **is not the arc length parameter** of this curve.

CLAIM 2 (generalizing Example 3). Let $\delta(t) \subseteq \mathbb{R}^3$ be a smooth curve with $|\delta(t)| \equiv 1$. Then $\delta'(t) \cdot \delta(t) \equiv 0$, namely, $\delta'(t) \perp \delta(t)$ for all t.

(Observe that here t is in general not the arc length parameter).

Proof. Differentiate the equation $\delta(t) \cdot \delta(t) \equiv 0$. □

DEFINITION: The unit vector $N(s) = \kappa(s)^{-1}T'(s)$ is called the **normal along the curve**.

By Claim 2, $N(s) \perp T(s)$ for all s.

DEFINITION: The unit vector $B(s) = T(s) \times N(s)$ is called the **binormal along the curve**.

DEFINITION: The orthonormal system $T(s)$, $N(s)$, $B(s)$ is called **The Frenet-Serret** frame. It is clearly a basis for $\mathbb{R}^3$ that varies along the curve, following the "twists" of the curve in space.

DEFINITION: The scalar function $\tau(s) = -B'(s) \cdot N(s)$ is called the **torsion of the curve**.

DEFINITION: The Frenet-Serret system $T(s)$, $N(s)$, $B(s)$ combined with the two scalar functions $\kappa(s)$, $\tau(s)$, is called the **Frenet-Serret apparatus** of the curve.

THEOREM (The Frenet-Serret Equations). Given a regular unit speed curve $\gamma(s)$, with $\kappa(s) \neq 0$, the following equations are satisfied:

$$\begin{aligned} T'(s) =& \kappa(s)N(s), \\ N'(s) =& -\kappa(s)T(s) + \tau(s)B(s), \\ B'(s) =& -\tau(s)N(s). \end{aligned}$$

Proof. The first equation follows from the definition of $N(s)$.

For the second equation, note first that by Claim 2, $N'(s) \perp N(s)$.

For the other coefficients of $N'(s)$ in the T, N, B system we have:

$$0 = \frac{d}{ds}(T(s) \cdot N(s)) = T'(s) \cdot N(s) + T(s) \cdot N'(s) = \kappa(s) + T(s) \cdot N'(s),$$

and

$$0 = \frac{d}{ds}(B(s) \cdot N(s)) = B'(s) \cdot N(s) + B(s) \cdot N'(s) = -\tau(s) + B(s) \cdot N'(s).$$

For the third equation, similarly,

$$\frac{d}{ds}(T(s) \cdot B(s)) = T'(s) \cdot B(s) + T(s) \cdot B'(s) = 0 + T(s) \cdot B'(s),$$

since $T'(s) \cdot B(s) = \kappa(s)N(s) \cdot B(s) = 0$.

Finally, $N(s) \cdot B'(s) = -\tau(s)$ by definition. □

The geometric meaning of $\tau(s)$ is made clear by the following claim, which complements Claim 1.

CLAIM 3. Let $\gamma(s)$ be a regular unit speed curve. Then the following three conditions are equivalent:

(1) It is a planar curve (namely, there exists a plane containing it).
(2) $B(s) \equiv B_0$, where $B_0 \in \mathbb{R}^3$ is a constant vector.
(3) $\tau(s) \equiv 0$.

REMARK. Note that in view of Claim 1, if $\kappa(s) \equiv 0$, then the curve is a straight segment, hence planar. Thus, we can assume $\kappa(s) \neq 0$.

Proof of Claim 3.

- (1) $\Rightarrow$ (2). In this case $T(s)$, hence also $N(s)$, are in this plane so $B(s) =$(constant) unit normal to the plane, which is B_0.
- (2) $\Leftrightarrow$ (3). Both directions follow from the third F-S equation.
- (2) $\Rightarrow$ (1). Both $T(s)$, $N(s)$ are in the plane P perpendicular to B_0, and we can assume it contains some point on the curve, say $\gamma(0)$. Then

$$\gamma(s) - \gamma(0) = \int_0^s T(\tau)d\tau \in P.$$

□

The following claim asserts that the two scalar functions $\kappa(s)$, $\tau(s)$, determine the curve $\gamma(s)$.

CLAIM 4. Let $\gamma^1(s)$, $\gamma^2(s)$, $s \in (l, L)$, be two regular curves with Frenet-Serret apparatus $T^1(s)$, $N^1(s)$, $B^1(s)$, $\kappa^1(s)$, $\tau^1(s)$, and $T^2(s)$, $N^2(s)$, $B^2(s)$, $\kappa^2(s)$, $\tau^2(s)$, respectively, and such that:

- $$\kappa^1(s) = \kappa^2(s), \quad \tau^1(s) = \tau^2(s), \; s \in (l, L),$$
- $$\gamma^1(s_0) = \gamma^2(s_0), \; T^1(s_0) = T^2(s_0), \; N^1(s_0) = N^2(s_0) \, B^1(s_0) = B^2(s_0)$$

  for some $s_0 \in (l, L)$.

Then $\gamma^1(s) = \gamma^2(s), \; s \in (l, L)$.

Proof. Define a scalar function

$$f(s) = |T^1(s) - T^2(s)|^2 + |N^1(s) - N^2(s)|^2 + |B^1(s) - B^2(s)|^2.$$

Differentiating and using the F-S equations (note that curvatures and torsions are equal) it is easy to see that

$$\frac{d}{ds} f(s) = 0, \; s \in (l, L).$$

Thus $f(s) = const$, and by assumption $f(s_0) = 0$, so $f(s) \equiv 0$.

It follows that $T^1(s) \equiv T^2(s)$, so

$$\gamma^1(s) = \gamma^1(s_0) + \int_{s_0}^{s} T^1(\xi) d\xi = \gamma^2(s_0) + \int_{s_0}^{s} T^2(\xi) d\xi = \gamma^2(s).$$

□

REMARK. The claim says that if two curves intersect at some point s_0, and they have there the same Frenet-Serret frame $T(s_0)$, $N(s_0)$, $B(s_0)$, and if their curvatures and torsions are identical (for all s), then they coincide.

Note that if $\gamma^1(s)$, $\gamma^2(s)$, are two regular curves, then by rigid translation and rotation we can "align" them at a given point, namely, they intersect at this point and their Frenet-Serret systems are equal there. Such transformations do not change the curvature and torsion (why?), so the claim can be rephrased:

CLAIM 4'. The curvature and torsion of a curve determine it uniquely, up to translation and rotation.

Note. The question if, for a given pair of continuous functions $\kappa(s)$, $\tau(s)$, there exists a curve $\gamma(s)$ so that these functions are, respectively, its curvature and torsion, is a question of *existence* of a solution to the linear system of the Frenet-Serret differential equations. The claim says that we can find *at most* one such curve.

Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel
E-mail address: `mbartzi@math.huji.ac.il`