RIEMANN INTEGRATION ON A MULTIDIMENSIONAL RECTANGULAR BOX

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Functions here are **real, bounded** , defined on a closed rectangular box

$$Q = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n].$$

All rectangular boxes have sides parallel to the axes.

The (open) ball of radius r, centered at x, is denoted by $B(x, r)$. It will be clear from the context what is the dimension .

We use the Euclidean norm.

- DEFINITION(**Partition of a real interval**): A partition P of $[a, b] \subseteq \mathbb{R}$ is a finite set of points:

$$P: \quad a = t_0 < t_1 < ... < t_m = b.$$

- DEFINITION: The intervals $[t_i, t_{i+1}], \quad 0 \le i \le m-1$ are called the **partition intervals**.
- DEFINITION (**Multidimensional partition**): A partition P of Q is a product $P = P_1 \times ... \times P_n$, where P_j is a partition of $[a_j, b_j], \quad 1 \le j \le n$.
- DEFINITION (**Partition boxes**): A rectangular box (closed) such that its side on the $j-$th axis is a P_j partition interval $(1 \le j \le n)$ is called a **partition box** (of P).
- NOTATION: We use the simplified notation $S \in P$ for a partition box S.
- DEFINITION (**Volume of a box**): The volume $v(S)$ is the product of its (n) sides.
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- DEFINITION:Let f be a real, bounded function on Q. Given a partition P of Q, we define for every $S \in P$,

$$M_S(f) = \sup_S f, \quad m_S(f) = \inf_S f.$$

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- DEFINITION (**Upper and Lower Sums**): Given the function f and partition P as above, we define:

$$L(f, P) = \sum_{S \in P} m_S(f) v(S), \quad U(f, P) = \sum_{S \in P} M_S(f) v(S).$$

- DEFINITION (**Refinement of a partition**): The partition $P' = P'_1 \times ... \times P'_n$ is a **refinement** of $P = P_1 \times ... \times P_n$ if $P'_i \supseteq P_i, \quad 1 \le i \le n$.

- CLAIM: If f is a real bounded function and P' is a refinement of P, then
$$L(f,P) \le L(f,P'), \quad U(f,P) \ge U(f,P').$$
- CLAIM: For **any two partitions** (not necessarily refinements of each other) P, P',
$$L(f,P) \le U(f,P').$$
- DEFINITION (**Integral and Integrability**): f is real, bounded on Q. We say that f is (Riemann-) **integrable** on Q if
$$\sup_P L(f,P) = \inf_P U(f,P) = I.$$
The number I is called the **integral** of f (on Q). It is denoted by $I(f) = \int f$.
- REMARK: If the "domain of integration" needs to be specified, we write $I_Q(f) = \int_Q f$.
- **LEMMA**: f is integrable iff for every ε there exists a partition P such that
$$U(f,P) - L(f,P) \le \varepsilon.$$
- **BASIC PROPERTIES OF THE INTEGRAL**
- The set of integrable functions is a linear space. On this space I is a linear functional.
- The integral is **order preserving.** If $f \ge g$ (and both are integrable) then $I(f) \ge I(g)$.
- If f, g are integrable then also the product fg, as well as $\max(f,g)$ and $\min(f,g)$ are integrable.
- If f is integrable then so are $|f|$, $f^{\pm} = \frac{1}{2}(|f| \pm f)$.
- **LEMMA**: (a) Let Q_0 be a rectangular box contained in Q. If f is integrable on Q then it is integrable on Q_0.

  (b) Let $Q = \bigcup_{l=1}^{m} Q_l$ be a union of rectangular boxes with no common interior points ($\overset{\circ}{Q}_j \cap \overset{\circ}{Q}_k = \emptyset$, $j \ne k$). Suppose that f is integrable over each Q_l. Then it is integrable over Q and
$$I_Q(f) = \sum_{l=1}^{m} I_{Q_l}(f).$$
- **THEOREM**: If f is continuous on Q then it is integrable on Q.
- **THE BASIC NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY**

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- **THE INTERIOR LEMMA** *Let P be a partition of a box Q. Let $\{Q_l\}_{l=1}^m$ be a finite collection of rectangular boxes with no common interior points, such that , for every $1 \le l \le m$,*
$$Q_l \subseteq S, \quad \textit{for some } S \in P.$$
Then
$$\sum_{l=1}^{m} (M_{Q_l}(f) - m_{Q_l}(f)) v(Q_l) \le U(f,P) - L(f,P).$$

- DEFINITION (**Parameter of a partition**): If P is a partition then its **parameter** is defined by:
$$\delta(P) = \sup_{S \in P} diam(S).$$
- Fix a partition P of Q.
- If P' is another partition, we say that $S' \in P'$ is a **boundary box** (with respect to P) if there is NO $S \in P$ such that $S' \subseteq S$.
- **THE BOUNDARY LEMMA**: *Fix a partition P of Q and let $\varepsilon > 0$ be given. Then there exists a $\eta > 0$ such that for any partition P' with $\delta(P') < \eta$ we have*
$$\sum_{S' \in P',\, S' \text{ boundary box}} (M_{S'}(f) - m_{S'}(f))v(S') \leq \varepsilon.$$
- **THEOREM:** *Let f be a bounded function defined in a box Q. The following condition is necessary and sufficient for f to be integrable in Q.*

  For any $\varepsilon > 0$ there exists an $\eta > 0$ so that, for every partition P' of Q, with $\delta(P') < \eta$, we have
$$U(f, P') - L(f, P') < \varepsilon.$$
- **PROOF.** It is clear that the condition is sufficient, by the definition of integrability.

  To prove it is necessary, suppose f is integrable and let P be a partition so that
$$U(f, P) - L(f, P) < \frac{\varepsilon}{2}.$$
  Now take any partition P' with $\delta(P') < \eta$ so that by the boundary lemma,
$$\sum_{S' \in P',\, S' \text{ boundary box}} (M_{S'}(f) - m_{S'}(f))v(S') \leq \frac{\varepsilon}{2}.$$
  For the interior boxes of P' use the interior lemma.

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- **RIEMANN SUMS**

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- DEFINITION (**RIEMANN SUM**): Given f real and bounded and P a partition of Q, we choose a point $x_S \in S$ for every $S \in P$. The *Riemann sum* of f, relative to P and the choice of points is defined by:
$$R(f; P, x_S) = \sum_{S \in P} f(x_S)v(S).$$
- OBSERVE:
$$L(f, P) \leq R(f; P, x_S) \leq U(f, P).$$
- DEFINITION (**THE RIEMANN CONDITION**): Let f be real and bounded in Q. There exists a number $\bar{I}$ so that for any $\varepsilon > 0$, there exists $\eta > 0$ such that
$$|\bar{I} - R(f; P, x_S)| < \varepsilon, \quad \forall P \quad \text{such that} \quad \delta(P) < \eta,$$
  and for *all selections of points* $x_S \in S \in P$.

- **THEOREM (The Riemann Condition is Necessary and Sufficient for Integrability)**: *A real bounded function f is integrable iff it satisfies the Riemann condition. In this case $\bar{I} = I$.*
- **PROOF**: (a) If the Riemann condition is satisfied, then

$$\delta(P) < \eta \Rightarrow U(f,P) - L(f,P) \leq \varepsilon.$$

(b) If f is integrable and $\varepsilon > 0$ is given, take $\eta > 0$ such that if P' is any partition with

$$\delta(P') < \eta \Rightarrow U(f,P') - L(f,P') \leq \varepsilon,$$

Then clearly

$$|R(f;P',x_{S'}) - I| < \varepsilon.$$

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- **NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY IN TERMS OF CONTINUITY PROPERTY**

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- DEFINITION (**Oscillation in a set**): Let $x \in Q$. The *oscillation of* f (real, bounded) in $D \subseteq Q$ is

$$osc(f,D) = \sup_D f - \inf_D f.$$

- DEFINITION (**Oscillation at a point**): The *oscillation of* f at $x \in Q$ is defined by:

$$o(f,x) = \inf_{\delta>0} osc(f, B(x,\delta) \cap Q) = \lim_{\delta \to 0} osc(f, B(x,\delta) \cap Q).$$

- OBSERVE: The function f is **continuous** at x iff $o(f,x) = 0$.
- REMARK: Clearly, these definitions can be applied to functions defined in *any set*, not necessarily a box.
- LEMMA: Let f be real, bounded in Q. Then for every $\eta > 0$ the set

$$A_\eta = \{x \in Q, \quad o(f,x) \geq \eta\}$$

is closed.
- PROOF: If a sequence $\{x^k\} \subseteq A_\eta$ converges to $x \in Q$, then clearly for every $\delta > 0$

$$osc(f, B(x,\delta)) \geq \eta,$$

since there is a point $x^{k_0} \in B(x,\delta)$ and hence $\delta_1 > 0$ such that $B(x^{k_0},\delta_1) \subseteq B(x,\delta)$.
- REMARK: This lemma means that $o(f,x)$ is *upper semicontinuous* in Q.
- DEFINITION(**Zero content of a subset**): A subset $D \subseteq Q$ is of *zero content* if , for any $\varepsilon > 0$, it can be covered by a **finite** number of open boxes with total volume less than ε.
- **THEOREM**: The (real, bounded) function f on Q is integrable *if and only if* for every $\eta > 0$ the set

$$A_\eta = \{x \in Q, \quad o(f,x) \geq \eta\}$$

is of zero content.

- PROOF: (a) Suppose f is integrable, let $\varepsilon, \eta > 0$ be given.

  Find a partition P such that $U(f,P) - L(f,P) < \varepsilon\eta$. Let $\{S_1, ..., S_r\} \subseteq P$ be the boxes in the partition P satisfying
$$osc(f, S_j) \geq \eta, \quad 1 \leq j \leq r.$$
  We have
$$\eta \sum_{j=1}^{r} v(S_j) \leq \varepsilon\eta \Rightarrow \sum_{j=1}^{r} v(S_j) \leq \varepsilon.$$
  Let $\Lambda = \underset{S \in P}{\cup} \partial S \subseteq Q$ be the union of all the boundaries of boxes in the partition P.

  Clearly
$$A_\eta = \{x, \quad o(f,x) \geq \eta\} \subseteq \underset{1 \leq j \leq r}{\cup} S_j \cup \Lambda,$$
  since if x is interior to some $S \in P$, then $S = S_j$ for some $1 \leq j \leq r$.

  Now take, for every $j = 1, ..., r$, an open box $\widetilde{S_j}$ containing S_j and such that $v(\widetilde{S_j}) < 2v(S_j)$. Thus
$$\sum_{j=1}^{r} v(\widetilde{S_j}) \leq 2\varepsilon.$$
  Also, clearly Λ has content zero, so can be covered by a finite number of open boxes with total volume less than ε.

  Thus A_η is covered by a finite number of open boxes of total volume $< 3\varepsilon$.

  (b) Conversely, suppose that A_η is of zero content for every $\eta > 0$. Let $\varepsilon > 0$ be given and take $\eta = \frac{\varepsilon}{2v(Q)}$. Cover $A_\eta \cup \partial Q$ by a finite number of *open* boxes S_α of total volume smaller than $\frac{\varepsilon}{4M}$, where $M = \sup_Q |f|$.

  The set $G = Q \setminus \underset{\alpha}{\cup} S_\alpha \subseteq \overset{\circ}{Q}$ is a compact set and $o(f,x) < \eta$ for all $x \in G$.

  Hence G can be covered by a *finite* set of open boxes T_β such that $osc(f, \widetilde{T}_\beta) < \eta$, where $\widetilde{T}_\beta$ is T_β whose sides are expanded by a factor of 2 (and same center).

  By removing common interior points we can assume that $T_{\beta_1} \cap T_{\beta_2} = \emptyset$ if $\beta_1 \neq \beta_2$.

  It follows that
$$\sum_{S_\alpha} (M_{\overline{S}_\alpha} - m_{\overline{S}_\alpha})(f) v(S_\alpha) + \sum_{T_\beta} (M_{\overline{T}_\beta} - m_{\overline{T}_\beta})(f) v(T_\beta)$$
$$\leq 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2v(Q)} v(Q) = \varepsilon.$$
  Let P be a partition such that every $S \in P$ is contained in some $\overline{S}_\alpha$ or some $\overline{T}_\beta$. Then
$$U(f,P) - L(f,P) < \varepsilon.$$
- DEFINITION (**Zero measure**): A set $D \subseteq Q$ is of *zero measure* if, for every $\varepsilon > 0$, it can be covered by a **countable** number of boxes of total volume less than ε.
- LEMMA: A closed (hence compact) subset of Q is of zero measure iff it is of zero content.

- LEMMA: A subset of a set of zero measure is itself of zero measure.
- LEMMA: A countable union of sets of zero measure is of zero measure.
- OBSERVE: It follows from the above theorem that f is integrable iff $A_{\frac{1}{k}}$ is of zero measure, for any integer k.
- The UNION $\bigcup_{k=1}^{\infty} A_{\frac{1}{k}}$ is exactly **the points of discontinuity** of f.

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- **THEOREM**: *A NECESSARY AND SUFFICIENT CONDITION FOR THE INTEGRABILITY OF f IS THAT ITS SET OF POINTS OF DISCONTINUITY IS OF ZERO MEASURE.*
- COROLLARY: If $f \geq 0$ is integrable on Q and $\int_Q f = 0$ then the set $\{x, \quad f(x) > 0\}$ is of zero measure.

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- **FUBINI'S THEOREM**

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- Let $Q' \subseteq \mathbb{R}^n, \quad Q'' \subseteq \mathbb{R}^m$ be boxes. Then $Q = Q' \times Q''$ is a box in $\mathbb{R}^{n+m}$.
  A point $x \in Q$ is $x = (x', x''), \quad x' \in Q', \, x'' \in Q''$.
- We denote partitions in Q', Q'' by P', P'', respectively. Every partition P of Q is of the form $P = P' \times P''$.
- Let $f(x', x'')$ be a real bounded function on Q. For every $x' \in Q'$ define:

$$\phi(x') = \sup_{P''} L(f(x', \cdot), P''), \quad \psi(x') = \inf_{P''} U(f(x', \cdot), P'').$$

- Clearly: $\phi(x') \leq \psi(x'), \quad x' \in Q'$.
- THEOREM (**Fubini's theorem**): Let f be integrable on Q. Then $\phi(x')$, $\psi(x')$ are integrable on Q' and

$$\int_Q f = \int_{Q'} \phi = \int_{Q'} \psi.$$

- PROOF: (a) Given $\varepsilon > 0$ let $\eta > 0$ be such that

$$|R(f; P, x_S) - \int_Q f| < \varepsilon \quad \text{if} \quad \delta(P) < \eta.$$

  (b) Let $P = P' \times P''$. For every $S'_i \in P'$ choose $x'_i \in S'_i$ and take any $x''_{j,i} \in S''_j \in P''$.
  If $S = S'_i \times S''_j$ take $x_S = (x'_i, x''_{j,i})$.
  (c) By varying the $x''_{j,i}$ we get

$$\sum_i U(f(x'_i, \cdot), P'')v(S'_i) \leq \int_Q f + \varepsilon,$$
$$\sum_i L(f(x'_i, \cdot), P'')v(S'_i) \geq \int_Q f - \varepsilon.$$

  (d) It follows that

$$\int_Q f - \varepsilon \leq \sum_i \phi(x'_i)v(S'_i) \leq \sum_i \psi(x'_i)v(S'_i) \leq \int_Q f + \varepsilon.$$

(e) So by definition both ϕ and ψ are integrable on Q' and

$$\int\limits_{Q} f = \int\limits_{Q'} \phi = \int\limits_{Q'} \psi.$$

- REMARK: It follows that $\phi(x') = \psi(x')$ except (possibly) for a zero measure set of $x' \in S'$.

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